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S-LATTICES

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Abstract. In this paper we study S-elements in C-lattices and characterize almost principal element lattices and principal element lattices in terms of S-lattices.

An element e of a multiplicative lattice L is said to be principal if it satisfies the dual identities (i) $a \wedge be = ((a : e) \wedge b)e$ and (ii) $(a \vee be) : e = (a : e) \vee b$. Elements satisfying (i) are called meet principal and elements satisfying (ii) are called join principal. By a C-lattice is meant a (not necessarily modular) complete multiplicative lattice, with least element 0 and compact greatest element 1 (a multiplicative identity), which is generated under joins by a multiplicatively closed subset C of compact elements. In a principally generated C-lattice, principal elements are compact ([1], Theorem 1.3) and a finite product of principal elements is again a principal element [5].

Throughout in this paper, we assume that L is a C-lattice generated by compact join principal elements. L_* denotes the set of all compact elements of L. C-lattices can be localized. For any prime element p of L, L_p denotes the localization at $F = \{x \in C | x \not\leq p\}$. For basic properties of localization, the reader is referred to [10]. A prime element p of L is said to be an ℓ-prime if the set of all p-primary elements of L is linearly ordered. For any prime p of L, p^{Δ} denotes the meet of all p-primary elements of L. A prime element p of L is said to be branched (unbranched) if $p > p^{\Delta} (p = p^{\Delta})$. Let p, m be two prime elements of L. We say m covers p if $m > p$ and there is no prime element p_1 of L such that $m > p_1 > p$. Following [11], a prime element p of L is said to be a d-prime if L_p is a discrete valuation lattice (i.e., consists just of the elements 0,1, and the powers of p all of which are distinct). L is said to be an almost discrete valuation lattice if L_m is a discrete valuation lattice (i.e., m is a d-prime) for every maximal prime m of L [11].

L is said to be *reduced* if 0 is the only nilpotent element of L. Principal elements were introduced into multiplicative lattices by R. P. Dilworth [5]. A multiplicative lattice L in which every element is principal is called a *principal element lattice*. Similarly, L is said to be an *almost principal element lattice* if L_m is a principal element lattice for every maximal element m of L. For various characterizations of almost principal element lattices and principal element lattices, the reader is referred to [4], [8] and [9]. L is said to be a Prüfer lattice if every compact element is principal. It is well known that a principally generated C-lattice L is a Prüfer lattice if and only if L_p is totally ordered

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for every prime p of L. For more information on Prüfer lattices, the reader is refereed to ([1, Theorem 3.4] and [13]).

In this paper, we introduce S-elements and study S-lattices. Next we prove that a principally generated reduced C-lattice L is a Prüfer lattice satisfying the a.c.c (ascending chain condition) for prime elements if and only if L is an S-lattice in which every non minimal prime is branched if and only if every prime is an ℓ-prime and every non minimal prime is minimal over some $x \in L_*$. Using these results, almost principal element lattices are characterised. Finally, it is established that if L is principally generated, then L is a finite direct product of principal element domains if and only if every maximal prime element is a strong compact S-element. For general back-ground and terminology, the reader may consult [1], [3] and [10].

We shall begin with the following definitions.

Definition 1. An element p of L is said to be an S-element if p satisfies the following conditions:

(i) p is an l-prime.
(ii) p^{Δ} is a prime element.
(iii) p^{Δ} contains each prime element properly contained in p.

Definition 2. L is said to be an S-lattice if every prime element is an S-element.

Observe that by definition 1, every S-element is an l-prime. Note that almost discrete valuation lattices ([11], Theorem 3) and regular lattices [2, Theorem 7] are examples of S-lattices. Also complemented prime elements and non minimal principal prime elements [4, Lemma 1.4] are examples of S-elements. If R is an integral domain, then an ideal I of R is an S-ideal (in the sense of [6]) if and only if I is an S-element of $L(R)(L(R)$ is the lattice of ideals of R). Therefore R is an S-domain (in the sense of [6]) if and only if $L(R)$ is an S-domain.

Lemma 1. *Let m be an l-prime element of L and let m be minimal over $m_0 \vee x$ for some prime element $m_0 < m$ and for some compact element $x \in L$. Then $m_0 \leq m^{\Delta}$, m^{Δ} is prime and m covers m^{Δ}.*

Proof. Let $\Psi = \{p \in L | m_0 \leq p < m$ and p is a prime element$\}$. By Zorn's lemma, Ψ contains a maximal element say p. We show that $p = m^{\Delta}$. Let $y \leq m^{\Delta}$ be any compact join principal element. If $y \not\leq p$, then $(p \vee y^2)_m$ is m-primary [10, Property 0.5] and $y \leq (p \vee y^2)_m$, a contradiction (see the proof of Lemma 12 of [9]) and therefore $m^{\Delta} \leq p$. Suppose $m^{\Delta} < p$. Then there exists an m-primary element q such that $p \not\leq q$. We claim that $q \leq p$. If $q \not\leq p$, then $((p \vee z^2))_m$ is m-primary for some compact join principal element $z \leq q$ and $z \not\leq p$. As m is an l-prime and $(p \vee z^2)_m \not\leq q$, it follows that $z \leq q \leq (p \vee z^2)_m$, a contradiction. Therefore $q \leq p$ and so $m = \sqrt{q} \leq p$, which is again a contradiction. This shows that $p = m^{\Delta}$. Obviously m covers m^{Δ}. This completes the proof of the lemma.

Theorem 1. *Let m be a non minimal l-prime element of L. Then m is a branched S-element if and only if m is a minimal prime over some $x \in L_*$.*

Proof. Suppose m is a minimal prime over some $x \in L_*$. By Lemma 1, m^Δ is prime and m covers m^Δ, so m is branched. Again if $p < m$ is a prime element, then m is a minimal prime over $p \vee x$, so by Lemma 1, $p \leq m^\Delta$ and hence m is a branched S-element. The converse part is obvious.

Theorem 2. *L is an S-lattice if and only if every prime is an ℓ-prime and every branched prime element is non minimal and minimal over x for some $x \in L_*$.*

Proof. Note that every unbanched prime element is an S-element. Now the result follows from Theorem 1.

Lemma 2. *If p is an unbranched non minimal prime element of L, then p is the join of a chain of branched prime elements properly contained in p.*

Proof. Let p be an unbranched non minimal prime element of L. By lemma 12 of [9], p contains properly a branched prime element of L. Let Q be a maximal chain of branched prime elements properly contained in p (its existence follows from Zorn's lemma). By Lemma 12 of [9], $p = \vee\{p_\alpha | p_\alpha \in Q\}$.

Theorem 3. *L is an S-lattice if and only if L satisfies the following conditions:*

(i) *The minimal prime elements of L are unbranched.*

(ii) *Every prime is an ℓ-prime.*

(iii) *Every non minimal prime element p of L is unbranched if and only if it is the join of a chain of prime elements properly contained in p.*

Proof. Suppose L is an S-lattice. Clearly, L satisfies the conditons (i) and (ii). By Lemma 2 and Theorem 2, L satisfies the conditon (iii).

Conversly, assume that L satisfies the conditons (i), (ii) and (iii). Note that every unbranched prime element is an S-element. Let p be a branched prime element of L. Let B be a maximal chain of prime elements properly contained in p (by Zorn's lemma) and let $p_0 = \vee p_\alpha$, $p_\alpha \in B$. Chose any compact element $x \leq p$ such that $x \nleq p_0$. Then p is a minimal prime over x and so by Theorem 1, p is an S-element and hence L is an S-lattice. This completes the proof of the theorem.

Lemma 3. *Let L be a reduced Prüfer lattice. Then L is an S-lattice.*

Proof. As L is reduced it follows that the minimal prime elements of L are unbranched [11, Lemma 3] and hence every minimal prime is an S-element. Obviously, every unbranched prime element is an S-element. Let p be a non minimal branched prime element. Choose any compact element $x \leq p$ such that $x \nleq p^\Delta$. Then p is minimal over x. Therefore by Theorem 1, p is an S-element. This shows that L is an S-lattice.

Lemma 4. *Suppose L is a quasi-local lattice such that for any non minimal prime element p of L, there exists a prime element $p_0 < p$ such that if $p_1 < p$ is a prime* *element, then* $p_1 \leq p_0$. *Then* L *satisfies the a.c.c. for prime elements and the prime elements are linearly ordered* (*and conversely*).

Proof. The proof of the lemma is similar to the proof of [7, Lemma 3.4].

For any $a \in L$, we denote $a^\omega = \wedge_{n=1}^{\infty} a^n$. We say a is a strong compact element if both a and a^ω are compact elements. Strong compact elements have been studied in [12] to characterize almost principal element lattices and principal element lattices.

Lemma 5. *Let* L *be a principally generated quasi-local lattice and let* m *be the non minimal maximal prime element. If* m *is strong compact* S*-element, then* L *is a discrete valuation lattice.*

Proof. By Lemma 1, m is a branched S-element. As m is compact, we have $m \neq m^2$. Choose any principal element $x \leq m$ such that $x \not\leq m^2$. Then x is m-primary. Let $y \leq m$ be any principal element. If $y \not\leq m^2$, then y is m-primary, so x and y are comparable and hence $x = y$. If $y \leq m^2$, then $y \leq m^2 \leq x$, so $m = x$ as L is principally generated. Therefore m is principal. Again by Lemma 1.4 of [4], $m^\omega = m^\Delta$ and $mm^\omega = m^\omega$, so by Theorem 1.4 of [1], $m^\omega = 0$. Again by [12, Lemma 3(iii)], L is a discrete valuation lattice and the proof is complete.

Theorem 4. *Let* L *be a principally generated reduced lattice. Then the following statements on* L *are equivalent:*

(i) *L is a Prüfer lattice satisfying the ascending chain condition for prime elements.*

(ii) *L is an S-lattice in which every non minimal prime element is branched.*

(iii) *Every prime is an ℓ-prime and every non minimal prime element is minimal over some $x \in L_*$.*

Proof. (i)⇒(ii). Suppose (i) holds. By (i) and Lemma 3, L is an S-lattice. Let p be a non minimal prime element. Let $p_0 < p$ be a prime element maximal with respect to properly contained in p (by the a.c.c for prime elements). Choose any compact element $x \leq p$ such that $x \not\leq p_0$. Then p is minimal over $x \vee p_0$ and hence by Lemma 12 of [9], p is branched. Therefore (ii) holds.

(ii)⇔(iii) follows from Theorem 1.

(ii)⇒(i). Suppose (ii) holds. As L is reduced, by Lemma 4, L_p is a domain for every prime element p of L. By localising if necessary, we may assume that L is a quasi-local domain in which the a.c.c for prime elements is valid and the prime elements are linearly ordered (by Lemma 4). It is enough if we show that any two principal elements are comparable. Suppose there exist non comparable principal elements. Let $\Psi = \{p \in L | p$ is the radical of two non comparable principal elements$\}$. By our assumption, $\Psi \neq \emptyset$. As the prime elements are linearly ordered, it follows that every $p \in \Psi$ is a prime element. Again by the a.c.c for prime elements, Ψ contains a maximal elements say m. Let $m = \sqrt{x \vee y}$, where x and y are non comparable principal elements. As m is branched, $m > m^\Delta$. Again since m is an S-element, it follows that either $x \not\leq m^\Delta$ or $y \not\leq m^\Delta$, so either x_m or y_m is m-primary and hence x_m and y_m are comparable. Since L is quasi-local, every principal element is (completely) join irreducible. Using this fact, it can be easily shown that there exist two principal elements z, $z_1 \in L$ such that $xz = yz_1$, and either $z \not\leq m$ or $z_1 \not\leq m$. (see also [3, Theorem 9]). Note that z and z_1, are non comparable as x and y are non-comparable. Let $m_0 = \sqrt{z \vee z_1}$. Then $m_0 \in \Psi$ and $m < m_0$, a contradiction. Therefore L is a Prüfer lattice and the proof is complete.

Theorem 5. *Suppose L is a principally generated reduced lattice. Then L is an almost principal element lattice if and only if every prime is an S-element and locally compact.*

Proof. The "only if" part is clear [8, Theorem 2.]. Conversely, assume that every prime is a locally compact S-element. We claim that L satisfies the a.c.c for prime elements. Let $p_1 \leq p_2 \leq \cdots \leq p_n \leq \cdots$ be a chain of prime elements. Then $p = \vee_{i=1}^{\infty} p_i$ is a prime element. Let m be any maximal prime element such that $p \leq m$. Note that $p_{1_m} \leq p_{2_m} \leq \cdots$ is a chain of prime elements in L_m whose join is p_m. As p is locally compact, it follows that $p = p_i$ for some i. Therefore L satisfies the a.c.c for prime elements. Let p be a non minimal prime element. Again by the a.c.c for prime elements, there exists a prime element q such that p covers q. Therefore p is minimal over $q \vee x$ for any principal element $x \leq p$ such that $x \not\leq q$. Consequently, by lemma 12 of [9], p is branched. By Theorem 4, L is a Prüfer lattice. Now the result follows from [1, Theroem 2.5] and [8, Theorem 2].

Theorem 6. *Suppose L is principally generated. Then L is a finite direct product of principal element domains if and only if every maximal prime element is a strong compact S-element.*

Proof. The "only if" part is clear. Now we prove the "if" part. Let m be a maximal prime element. If m is unbranched, then $m = m^2$, so by Theorem 1.4 of [1], L_m is a two element chain. If m is branched, then by Lemma 5, L_m is a discrete valuation lattice and hence L is a reduced almost principal element lattice. Therefore $\dim L \leq 1$ [9, Lemma 2 and Theorem 1]. Again if $p < m$ are prime elements, then by [9, Theorem 1] and [10, Lemma 5], $p = m^{\omega}$. So by hypothesis, every prime element is compact. As L is a Prüfer lattice, it follows that every prime element is principal and hence every element principal. Now the result follows from [11, Theorem 10]. This completes the proof of the theorem.

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