



Optimality conditions using convexifactors for a multiobjective fractional bilevel programming problem

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Abstract. In this paper, a multiobjective fractional bilevel programming problem is considered and optimality conditions using the concept of convexifactors are established for it. For this purpose, a suitable constraint qualification in terms of convexifactors is introduced for the problem. Further in the paper, notions of asymptotic pseudoconvexity and asymptotic quasiconvexity in terms of convexifactors are given and using them sufficient optimality conditions are derived.

Keywords. Multiobjective fractional bilevel programming problem, convexifactors, value function, constraint qualifications

1 Introduction

Bilevel programming problem (BLPP) consists of two optimization problems where variables of the first (or upper-level or leader's) problem are the parameters of the second (or lower-level or follower's) problem and the optimal solution of the second is needed to calculate the objective function value of the first problem. They play an important role not only theoretically but also practically. During past many years, researchers like Bard [1, 2], Dempe [12, 13], Outrata [31], Suneja and Kohli [32] and many others have worked for the development of optimality conditions for this problem. For applications and recent developments on bilevel programming problems one can see Bard [3], Dempe [14].

Multiobjective bilevel programming problem, where one of the objective functions of the upper-level problem and/or the lower-level problem is a vector function, needs to be investigated intensively as it is demanded from the point-of-view of applications. They have been studied in the literature by authors such as Bouibed et al. [8], Ye [36], Bonnel [5], Bonnel and Morgan [6, 7], Tung [35], Luu and Mai [29] and others. Recently, Luu and Mai [29] developed necessary and sufficient efficiency conditions for multiobjective bilevel programming problem via convexifactors.

Fractional programming problems are of great importance in transportation, production, information theory and numerical analysis. Fractional bilevel programming problems have been studied in the literature by Calvete and Gale [9, 10, 11]. Very recently, Luu and Linh [30] obtained optimality and duality results for multiobjective fractional problems using convexifactors. In this paper, we have combined these two important problems, multiobjective fractional programming

problem and bilevel programming problem to develop optimality conditions for multiobjective fractional bilevel programming problem where the upper-level problem is a multiobjective fractional programming problem and the lower-level problem is a scalar optimization problem.

Bilevel programming problems do not satisfy the usual constraint qualifications (CQs) such as Slater CQ, Mangasarian-Fromovitz CQ etc. It has been a subject of great challenge towards researchers to find a suitable CQ so as to develop Karush-Kuhn-Tucker (KKT) type necessary optimality conditions. Work has been done in this direction. Ye [37] introduced CQs in terms of Michel-Penot subdifferential and used them to obtain optimality conditions for bilevel programming problem. Later, Ye [38] introduced nonsmooth constraint qualifications for bilevel programming problem and used them to derive KKT type necessary optimality conditions. Dempe et al. [15] obtained KKT type optimality conditions for optimistic bilevel programming problem under the assumption of partial calmness CQ. Kohli [25] introduced two versions of Abadie CQ in terms of convexifactors and Clarke subdifferential and employed the weaker one to develop new necessary KKT type optimality conditions for optimistic bilevel programming problem using an upper estimate of Clarke subdifferential of value function and the concept of convexifactors.

In this paper, our aim is to prove KKT type optimality conditions for multiobjective fractional bilevel programming problem (MOFBLPP). For that, we have introduced a nonsmooth CQ namely ∂^* -MOFBLPP CQ in terms of convexifactors on the lines of Jourani [21]. Using weighted sum scalarization method, first we transform multiobjective fractional bilevel programming problem to a scalar optimization problem and after that we derive necessary optimality conditions. Further in the paper, we define asymptotic pseudoconvex and asymptotic quasiconvex functions in terms of convexifactors. Finally, under the assumption of these functions, we derive sufficient optimality conditions for the problem. Optimality conditions developed in terms of generalized subdifferential, convexifactor are more general since in the form of a convexifactor we get a smaller set. These convexifactors are tools of nonsmooth analysis and were given by Demyanov [16]. They were further studied by Jeyakumar and Luc [20], Dutta and Chandra [17, 18], Li and Zhang [27], Luu [28], Suneja and Kohli [32, 33, 34], Kohli [26], Kabgani and Soleimani-damaneh [22, 23], Kabgani et al. [24] and others. Since for locally Lipschitz functions, many well known subdifferentials such as Clarke subdifferential, Michel-Penot subdifferential etc may often contain convex hull of a convexifactor (Example 2.1 [17, 20]), hence using convexifactors, we obtain sharp optimality conditions than those using Clarke, Michel-Penot subdifferentials etc.

The paper is organized as follows. In Section 2, we give definitions of convexifactors along with some basic results which will be used in the proof of main results. MOFBLPP has been studied in Section 3. In Section 4, we define asymptotic pseudoconvex and asymptotic quasiconvex functions in terms of convexifactors and also we give a constraint qualification using the concept of convexifactors. Section 5 is devoted to the study of necessary optimality conditions for MOFBLPP. Finally, in Section 6, we give sufficient optimality conditions for the problem.

2 Convexifactors

We begin by defining upper and lower Dini derivatives as follows:

Definition 1. Let $F : \mathbb{R}^{n_1} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be an extended real valued function and let $x \in \mathbb{R}^{n_1}$ where $F(x)$ is finite. Then the upper and lower Dini derivatives of F at x in the direction v are defined respectively by

$$(F)_d^+(x, v) := \limsup_{t \rightarrow 0^+} \frac{F(x + tv) - F(x)}{t} \quad \text{and} \quad (F)_d^-(x, v) := \liminf_{t \rightarrow 0^+} \frac{F(x + tv) - F(x)}{t}.$$

Dini derivatives may be finite as well as infinite. In particular, if F is locally Lipschitz, both the upper and lower Dini derivatives are finite. For any set $A \subset \mathbb{R}^{n_1}$, the closure, convex hull and the closed convex hull of A are denoted respectively by $\text{cl } A$, $\text{conv } A$ and $\text{clconv } A$. We now give the definitions of convexifactors as given by Dutta and Chandra [17].

Definition 2. Let $F : \mathbb{R}^{n_1} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ be an extended real valued function and let $x \in \mathbb{R}^{n_1}$ where $F(x)$ is finite.

- (i) F is said to admit an upper convexifactor (UCF) $\partial^u F(x)$ at x iff $\partial^u F(x) \subseteq \mathbb{R}^{n_1}$ is a closed set and

$$(F)_d^-(x, v) \leq \sup_{x^* \in \partial^u F(x)} \langle x^*, v \rangle, \quad \text{for all } v \in \mathbb{R}^{n_1}.$$

- (ii) F is said to admit a lower convexifactor (LCF) $\partial_l F(x)$ at x iff $\partial_l F(x) \subseteq \mathbb{R}^{n_1}$ is a closed set and

$$(F)_d^+(x, v) \geq \inf_{x^* \in \partial_l F(x)} \langle x^*, v \rangle, \quad \text{for all } v \in \mathbb{R}^{n_1}.$$

- (iii) F is said to admit a convexifactor (CF) $\partial^* F(x)$ at x iff $\partial^* F(x)$ is both an (UCF) and (LCF) of F at x .

- (iv) F is said to admit an upper semiregular convexifactor (USRCF) $\partial^{us} F(x)$ at x iff $\partial^{us} F(x) \subseteq \mathbb{R}^{n_1}$ is a closed set and

$$(F)_d^+(x, v) \leq \sup_{x^* \in \partial^{us} F(x)} \langle x^*, v \rangle, \quad \text{for all } v \in \mathbb{R}^{n_1}.$$

In particular, if equality holds in above, then, $\partial^{us} F(x)$ is called an upper regular convexifactor (URCF) of F at x .

- (v) F is said to admit a lower semiregular convexifactor (LSRCF) $\partial_{ls} F(x)$ at x iff $\partial_{ls} F(x) \subseteq \mathbb{R}^{n_1}$ is a closed set and

$$(F)_d^-(x, v) \geq \inf_{x^* \in \partial_{ls} F(x)} \langle x^*, v \rangle, \quad \text{for all } v \in \mathbb{R}^{n_1}.$$

In particular, if equality holds in above, then, $\partial_{ls} F(x)$ is called a lower regular convexifactor (LRCF) of F at x .

It may be noted that convexifactors are always closed sets but not necessarily convex or compact (Jeyakumar and Luc [20], Dutta and Chandra [17, 18]) though the most well known subdifferentials such as Clarke, Michel-Penot etc are always convex and compact. Because of these relaxations, convexifactors can be easily applied to a large class of nonsmooth problems.

We now give one of the calculus rules given by Jeyakumar and Luc [20] for convexifactors in the form of the following remark.

Remark 1. Assume that the functions $f, g : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$ admit UCFs $\partial^u f(x)$ and $\partial^u g(x)$ at x , respectively, and that one of the CFs is upper regular at x , then $\partial^u f(x) + \partial^u g(x)$ is an UCF of $f + g$ at x .

Similarly, if the functions $f, g : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$ admit LCFs $\partial_l f(x)$ and $\partial_l g(x)$ at x , respectively, and that one of the CFs is lower regular at x , then $\partial_l f(x) + \partial_l g(x)$ is a LCF of $f + g$ at x .

The following lemma is crucial for our investigations.

Lemma 2.1 (Lemma 2.1 (b) and (c), Li and Zhang [27]). *Let $\partial^u f(x)$, $\partial_l f(x)$ and $\partial^* f(x)$ respectively be UCF, LCF and CF of f at x . Then,*

- (i) $\forall \lambda < 0$, $\lambda \partial^u f(x)$ and $\lambda \partial_l f(x)$ respectively are LCF and UCF of λf at x .
- (ii) $\forall \lambda \in \mathbb{R}$, $\lambda \partial^* f(x)$ is a CF of λf at x .

Lemma 2.1(i) is valid for semiregular and regular CFs also.

Remark 2. Let $\partial^{us} f(x)$ and $\partial_{ls} f(x)$ respectively be USRCF and LSRCF of f at x . Then, $\forall \lambda < 0$, $\lambda \partial^{us} f(x)$ and $\lambda \partial_{ls} f(x)$ respectively are LSRCF and USRCF of λf at x .

Same holds for URCF and LRCF of λf at x also.

Proof. Using definitions of upper (lower) dini derivatives and of limit superior (limit inferior), we can see that $\forall \lambda < 0$,

- (i) $(\lambda f)_d^-(x, v) = \lambda (f)_d^+(x, v)$, and
- (ii) $(\lambda f)_d^+(x, v) = \lambda (f)_d^-(x, v)$, for all $v \in \mathbb{R}^{n_1}$.

Since $\partial^{us} f(x)$ is an USRCF of f at x , we have

$$\begin{aligned} (f)_d^+(x, v) &\leq \sup_{x^* \in \partial^{us} f(x)} \langle x^*, v \rangle, \quad \text{for all } v \in \mathbb{R}^{n_1} \\ \Rightarrow \lambda (f)_d^+(x, v) &\geq \lambda \sup_{x^* \in \partial^{us} f(x)} \langle x^*, v \rangle, \quad \text{as } \lambda < 0 \end{aligned}$$

Using (i), we get

$$\begin{aligned} (\lambda f)_d^-(x, v) &\geq -\mu \sup_{x^* \in \partial^{us} f(x)} \langle x^*, v \rangle, \quad \text{as } \lambda = -\mu, \mu > 0 \\ &= - \sup_{x^* \in \partial^{us} f(x)} \langle \mu x^*, v \rangle \\ &= \inf_{x^* \in \partial^{us} f(x)} -\langle \mu x^*, v \rangle \\ &= \inf_{\xi \in \lambda \partial^{us} f(x)} \langle \xi, v \rangle. \end{aligned}$$

That is,

$$(\lambda f)_d^-(x, v) \geq \inf_{\xi \in \lambda \partial^{us} f(x)} \langle \xi, v \rangle, \quad \text{for all } v \in \mathbb{R}^{n_1}.$$

Thus $\lambda \partial^{us} f(x)$ is a LSRCF of λf at x .

Similarly, using (ii) we can prove that $\lambda \partial_{ls} f(x)$ is an USRCF of λf at x . □

Lemma 2.2 (Proposition 4.1, Jeyakumar and Luc [20]). *Suppose that the function $f : \mathbb{R}^{n_1} \rightarrow \mathbb{R}$ admits a CF $\partial^* f(x)$ at $x \in \mathbb{R}^{n_1}$. If f attains its extremum at x , then*

$$0 \in cl \ conv(\partial^* f(x))$$

3 Multiobjective fractional bilevel programming problem

In this section, we study the multiobjective fractional bilevel programming problem (MOFBLPP) which is defined as follows:

$$\begin{aligned} \text{(MOFBLPP)} \quad & \text{Minimize}_{x,y} \left(\frac{F_1(x,y)}{F'_1(x,y)}, \dots, \frac{F_p(x,y)}{F'_p(x,y)} \right) \\ & \text{subject to } G_j(x,y) \leq 0, \quad j \in J, \quad y \in \psi(x), \end{aligned}$$

where, for each $x \in \mathbb{R}^{n_1}$, $\psi(x)$ is the set of the optimal solutions to the following optimization problem

$$\begin{aligned} & \text{Minimize}_y f(x,y) \\ & \text{subject to } g_i(x,y) \leq 0, \quad i \in I, \end{aligned}$$

where $F_k, F'_k : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$, $k = 1, \dots, p$, $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$, $G_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$, $j \in J := \{1, 2, \dots, m_2\}$ and $g_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$, $i \in I := \{1, 2, \dots, m_1\}$; n_i and m_i , $i := 1, 2$ are integers with $n_i \geq 1$ and $m_i \geq 1$. $F_k(x,y) \geq 0$, $F'_k(x,y) > 0$ for all $k = 1, \dots, p$ and $(x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, f and g_i , $i \in I$ are continuous, convex and

$$\psi(x) := \underset{y}{\operatorname{argmin}} \{f(x,y) : g_i(x,y) \leq 0, \quad i \in I\}.$$

So, the idea is that, first, the leader fixes his choice x , and after that, the follower minimizes his objective function based on this choice and returns the solution $y = y(x)$ to the leader, who then uses it to minimize his objective function. If the lower-level problem has a unique optimal solution for all $x \in \mathbb{R}^{n_1}$, MOFBLPP is well defined, but if it has multiple solutions for a given x , then the upper-level objective becomes a set-valued map. To overcome this difficulty, two solution concepts namely the optimistic solution and the pessimistic solution have been discussed in the literature. In this paper, we follow optimistic approach. According to this approach, the leader presupposes cooperation of the follower in the sense that the latter will choose, each time that solution in $\psi(x)$ which is best suited with respect to the leader's objective function.

To convert BLPP into a single level programming problem, commonly three approaches have been discussed in the literature. Here, we adopt the optimal value reformulation initiated by Outrata [31] according to which multiobjective fractional bilevel programming problem can be converted into a single level mathematical programming problem with the help of the value function of the lower-level problem given by

$$V(x) := \min_y \{f(x,y) : g_i(x,y) \leq 0, \quad i \in I, \quad y \in \mathbb{R}^{n_2}\}.$$

Then the reformulated multiobjective fractional bilevel programming problem (RMOFBLPP) is given as:

$$\begin{aligned} \text{(RMOFBLPP)} \quad & \text{Minimize}_{x,y} \left(\frac{F_1(x,y)}{F'_1(x,y)}, \dots, \frac{F_p(x,y)}{F'_p(x,y)} \right) \\ & \text{subject to } f(x,y) - V(x) \leq 0, \\ & \quad g_i(x,y) \leq 0, \quad i \in I, \\ & \quad G_j(x,y) \leq 0, \quad j \in J, \\ & \quad (x,y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \end{aligned}$$

Let $\phi(x, y) := \left(\frac{F_1(x, y)}{F'_1(x, y)}, \dots, \frac{F_p(x, y)}{F'_p(x, y)} \right)$.

Let $E \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ denote the feasible set for RMOFBLPP, that is,

$$E := \{(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid f(x, y) - V(x) \leq 0, g_i(x, y) \leq 0, i \in I, G_j(x, y) \leq 0, j \in J\}.$$

Remark 3. On the lines of the remark made by Gadhi and Dempe [19], we can say that the problems MOFBLPP and RMOFBLPP are fully equivalent both with respect to local and global weak efficient solutions.

Now we define the notions of weak efficient and local weak efficient solutions of MOFBLPP.

Definition 3. $(\bar{x}, \bar{y}) \in E$ is a weak efficient solution of MOFBLPP if there does not exist any feasible solution $(x, y) \in E$ such that

$$\frac{F_k(x, y)}{F'_k(x, y)} < \frac{F_k(\bar{x}, \bar{y})}{F'_k(\bar{x}, \bar{y})}, \quad k = 1, 2, \dots, p.$$

Definition 4. $(\bar{x}, \bar{y}) \in E$ is a local weak efficient solution of MOFBLPP if there exist neighbourhoods U_0 of \bar{x} and V_0 of \bar{y} such that for any feasible solution $(x, y) \in (U_0 \times V_0) \cap E$, the following does not hold

$$\frac{F_k(x, y)}{F'_k(x, y)} < \frac{F_k(\bar{x}, \bar{y})}{F'_k(\bar{x}, \bar{y})}, \quad k = 1, 2, \dots, p.$$

4 ∂^* -Asymptotic generalized convexity

Let $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$ be a function. Let F admits a $(CF)\partial^*$ $F(\bar{x}, \bar{y})$ at (\bar{x}, \bar{y}) .

We now define the notions of ∂^* -asymptotic generalized convex functions in terms of convexifactors on the lines of Luu [28]:

Definition 5. The function F is said to be ∂^* -asymptotic pseudoconvex at (\bar{x}, \bar{y}) iff for all $(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$

$$F(x, y) < F(\bar{x}, \bar{y}) \Rightarrow \langle \xi^*, (x - \bar{x}, y - \bar{y}) \rangle < 0 \quad \text{for all } \xi^* \in \text{clconv } \partial^* F(\bar{x}, \bar{y}).$$

Remark 4. (i) If F is a differentiable function and $\partial^* F(\bar{x}, \bar{y})$ is an upper regular convexifactor of F at (\bar{x}, \bar{y}) , then, $\partial^* F(\bar{x}, \bar{y}) = \{\nabla F(\bar{x}, \bar{y})\}$ and the above definition reduces to the definition of pseudoconvex function.

(ii) If F is a locally Lipschitz function and $\text{clconv } \partial^* F(\bar{x}, \bar{y}) = \partial^c F(\bar{x}, \bar{y})$, where $\partial^c F(\bar{x}, \bar{y})$ is the Clarke subdifferential, then, the above definition reduces to the definition of ∂^c -pseudoconvex function given by Bector et al. [4].

Definition 6. The function F is said to be ∂^* -asymptotic quasiconvex at (\bar{x}, \bar{y}) iff for all $(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$

$$F(x, y) \leq F(\bar{x}, \bar{y}) \Rightarrow \langle \xi^*, (x - \bar{x}, y - \bar{y}) \rangle \leq 0 \quad \text{for all } \xi^* \in \text{clconv } \partial^* F(\bar{x}, \bar{y}).$$

Remark 5. (i) If F is a differentiable function and $\partial^* F(\bar{x}, \bar{y})$ is an upper regular convexifactor of F at (\bar{x}, \bar{y}) , then, $\partial^* F(\bar{x}, \bar{y}) = \{\nabla F(\bar{x}, \bar{y})\}$ and the above definition reduces to the definition of quasiconvex function.

- (ii) If F is a locally Lipschitz function and $\text{clconv } \partial^* F(\bar{x}, \bar{y}) = \partial^c F(\bar{x}, \bar{y})$, then, the above definition reduces to the definition of ∂^c -quasiconvex function given by Bector et al. [4].

Now we give an example of ∂^* -asymptotic quasiconvex function.

Example 1. Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$F(x, y) = \begin{cases} \sqrt{x^2 y}, & x \geq 0, y > 0, \\ x, & x \geq 0, y = 0, \\ \sqrt{-xy}, & x \geq 0, y < 0, \\ -\sqrt{-xy^2}, & x < 0, y \leq 0, \\ \sqrt{x^2 y}, & x < 0, y > 0. \end{cases}$$

Convexifactor of F at $(0, 0)$ is given by

$$\partial^* F(0, 0) = \left\{ \left(x^* + \frac{1}{n}, \frac{1}{n} \right) : x^* \geq 0, n \in N \right\} \cup \{(x^*, 0) : x^* \geq 0\}.$$

F is ∂^* -asymptotic quasiconvex at $(\bar{x}, \bar{y}) = (0, 0)$.

Following is the example of ∂^* -asymptotic pseudoconvex function.

Example 2. Let $F : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by

$$F(x, y) = \begin{cases} -\sqrt{-x} - \sqrt{-y}, & x \leq 0, y \leq 0 \\ 0, & x > 0, y < 0 \\ \sqrt{-x}, & x < 0, y > 0 \\ \sqrt{x} + \sqrt{y}, & x \geq 0, y \geq 0 \sim \{x = 0, y = 0\} \end{cases}$$

Convexifactor of F at $(0, 0)$ is given by

$$\partial^* F(0, 0) = \left\{ \left(x^* + \frac{1}{n}, y^* + \frac{1}{n} \right) : x^* > 0, y^* > 0, n \in N \right\}.$$

F is ∂^* -asymptotic pseudoconvex at $(\bar{x}, \bar{y}) = (0, 0)$ as

$$\begin{aligned} F(x, y) < F(0, 0) &\Rightarrow \text{(i) } x < 0, y < 0 \\ &\text{or} \\ &\text{(ii) } x < 0, y = 0 \\ &\text{or} \\ &\text{(iii) } x = 0, y < 0 \end{aligned}$$

To show

$$\langle \xi, (x - 0, y - 0) \rangle < 0, \text{ for all } \xi \in \text{clconv } \partial^* F(0, 0)$$

where

$$\xi = \lim_{n \rightarrow \infty} \xi_n, \text{ for some } \xi_n \in \text{conv } \partial^* F(0, 0)$$

$$\xi_n = (\xi_{1n}, \xi_{2n}) = \lambda \left(x^* + \frac{1}{n}, y^* + \frac{1}{n} \right) + (1 - \lambda) \left(x_1^* + \frac{1}{n}, y_1^* + \frac{1}{n} \right), \quad 0 \leq \lambda \leq 1$$

Taking limit, we get

$$\xi = \lim_{n \rightarrow \infty} \xi_n = (\lambda x^* + (1 - \lambda)x_1^*, \lambda y^* + (1 - \lambda)y_1^*)$$

Now

$$\begin{aligned} \langle \xi, (x, y) \rangle &= (\lambda x^* + (1 - \lambda)x_1^*, \lambda y^* + (1 - \lambda)y_1^*) \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \lambda x^* x + (1 - \lambda)x_1^* x + \lambda y^* y + (1 - \lambda)y_1^* y \\ &< 0 \quad \text{for all cases (i), (ii) and (iii)} \end{aligned}$$

Hence

$$F(x, y) < F(0, 0) \Rightarrow \langle \xi, (x, y) \rangle < 0 \quad \text{for all } \xi \in \text{cl conv } \partial^* F(0, 0).$$

We now introduce a nonsmooth CQ on the lines of CQ3 Jourani [21] in terms of convexifactors for RMOFBLPP. Let us define the following sets.

$$I(\bar{x}, \bar{y}) = \{i \in I : g_i(\bar{x}, \bar{y}) = 0\}, \quad J(\bar{x}, \bar{y}) = \{j \in J : G_j(\bar{x}, \bar{y}) = 0\}.$$

Let $(\bar{x}, \bar{y}) \in E$ be feasible for RMOFBLPP. We assume that the functions $f, g_i, i \in I(\bar{x}, \bar{y}), G_j, j \in J(\bar{x}, \bar{y})$ admit convexifactors $\partial^* f(\bar{x}, \bar{y}), \partial^* g_i(\bar{x}, \bar{y}), i \in I(\bar{x}, \bar{y})$ and $\partial^* G_j(\bar{x}, \bar{y}), j \in J(\bar{x}, \bar{y})$ respectively at (\bar{x}, \bar{y}) .

The problem RMOFBLPP is said to satisfy ∂^* -MOFBLPP CQ at (\bar{x}, \bar{y}) if for every $\gamma \geq 0, \tau_j \geq 0, j \in J(\bar{x}, \bar{y}), \mu_i \geq 0, i \in I(\bar{x}, \bar{y})$ (not all zero)

$$(0, 0) \notin \text{cl} \left[\begin{array}{l} \gamma \text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}) + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \text{conv } \partial^* G_j(\bar{x}, \bar{y}) \\ + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \text{conv } \partial^* g_i(\bar{x}, \bar{y}) \end{array} \right].$$

5 Necessary optimality conditions

In this section, we derive KKT type necessary optimality conditions for MOFBLPP in the form of the following theorem.

Here, we define $\phi_k(\bar{x}, \bar{y}) = \frac{F_k(\bar{x}, \bar{y})}{F'_k(\bar{x}, \bar{y})}, \varphi_k(x, y) = F_k(x, y) - \phi_k(\bar{x}, \bar{y})F'_k(x, y), k = 1, 2, \dots, p.$

Theorem 5.1. *Let (\bar{x}, \bar{y}) be a local weak efficient solution of MOFBLPP. Suppose that $F_k, k = 1, 2, \dots, p$ admit bounded convexifactors $\partial^* F_k(\bar{x}, \bar{y}), k = 1, 2, \dots, p$ at (\bar{x}, \bar{y}) which are upper regular and $F'_k, k = 1, 2, \dots, p$ admit bounded convexifactors $\partial^* F'_k(\bar{x}, \bar{y}), k = 1, 2, \dots, p$ at (\bar{x}, \bar{y}) which are upper regular if $\phi_k(\bar{x}, \bar{y}) > 0$ and lower regular if $\phi_k(\bar{x}, \bar{y}) < 0$. Let f admits a bounded convexifactor $\partial^* f(\bar{x}, \bar{y})$ at (\bar{x}, \bar{y}) which is upper regular and $G_j, j \in J, g_i, i \in I$ admit bounded convexifactors $\partial^* G_j(\bar{x}, \bar{y}), j \in J$ and $\partial^* g_i(\bar{x}, \bar{y}), i \in I$ respectively at (\bar{x}, \bar{y}) . Suppose that $\partial^* \varphi_k, k = 1, 2, \dots, p, \partial^* f, \partial^* G_j, j \in J, \partial^* g_i, i \in I$ are upper semicontinuous at (\bar{x}, \bar{y}) . Also, assume that the functions $F_k, F'_k, k = 1, 2, \dots, p, G_j, j \in J$ are continuous and convex. Let ∂^* -MOFBLPP CQ be satisfied at (\bar{x}, \bar{y}) . Assume that for some $k, k = 1, 2, \dots, p, \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y})\partial^* F'_k(\bar{x}, \bar{y})$ is an upper regular convexifactor of $F_k - \phi_k F'_k$ at (\bar{x}, \bar{y}) and for some $k_0 \neq k, \partial^* F_{k_0}(\bar{x}, \bar{y}) - \phi_{k_0}(\bar{x}, \bar{y})\partial^* F'_{k_0}(\bar{x}, \bar{y})$ is a lower regular convexifactor of $F_{k_0} - \phi_{k_0} F'_{k_0}$ at (\bar{x}, \bar{y}) . Suppose that*

$$\partial^* \left(\sum_{k=1}^p w_k (F_k - \phi_k F'_k) \right) (\bar{x}, \bar{y}) = \sum_{k=1}^p \partial^* (w_k (F_k - \phi_k F'_k)) (\bar{x}, \bar{y})$$

$$= \sum_{k=1}^p w_k (\partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y}))$$

where $w_k > 0$, $k = 1, 2, \dots, p$ and $\sum_{k=1}^p w_k = 1$. Also, assume that $\partial^*(f(\bar{x}, \bar{y}) - V(\bar{x})) = \partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}$. Then, there exist scalars $w'_k \geq 0$, (not all zero), $k = 1, 2, \dots, p$, $\gamma \geq 0$, $\tau_j \geq 0$, $j \in J$, $\mu_i \geq 0$, $i \in I$ with $\sum_{k=1}^p w'_k + \gamma + \sum_{j \in J} \tau_j + \sum_{i \in I} \mu_i = 1$ such that

$$(0, 0) \in \text{cl} \left[\begin{array}{l} \sum_{k=1}^p w'_k \{ \text{conv } \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \text{conv } \partial^* F'_k(\bar{x}, \bar{y}) \} \\ + \gamma \text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}) \\ + \sum_{j \in J} \tau_j \text{conv } \partial^* G_j(\bar{x}, \bar{y}) + \sum_{i \in I} \mu_i \text{conv } \partial^* g_i(\bar{x}, \bar{y}) \end{array} \right].$$

Proof. Since (\bar{x}, \bar{y}) is a local weak efficient solution of MOFBLPP, hence by Remark 3, (\bar{x}, \bar{y}) is a local weak efficient solution of RMOFBLPP.

Our proof consists of the following steps.

At first we will prove that (\bar{x}, \bar{y}) is a local weak efficient solution of (P₁)

$$\begin{aligned} (\text{P}_1) \quad & \text{Minimize}_{x,y} (F_1(x, y) - \phi_1(\bar{x}, \bar{y})F'_1(x, y), \dots, F_p(x, y) - \phi_p(\bar{x}, \bar{y})F'_p(x, y)) \\ & \text{subject to } (x, y) \in E. \end{aligned}$$

On contrary suppose that there exists $(x_1, y_1) \in (U_0 \times V_0) \cap E$ such that

$$F_k(x_1, y_1) - \phi_k(\bar{x}, \bar{y})F'_k(x_1, y_1) < F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y})F'_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p.$$

Now

$$F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y})F'_k(\bar{x}, \bar{y}) = 0, \quad k = 1, 2, \dots, p.$$

Therefore, we have

$$F_k(x_1, y_1) - \phi_k(\bar{x}, \bar{y})F'_k(x_1, y_1) < 0, \quad k = 1, 2, \dots, p.$$

That is,

$$\frac{F_k(x_1, y_1)}{F'_k(x_1, y_1)} < \frac{F_k(\bar{x}, \bar{y})}{F'_k(\bar{x}, \bar{y})}, \quad k = 1, 2, \dots, p,$$

which is contradiction to the fact that (\bar{x}, \bar{y}) is a local weak efficient solution of RMOFBLPP.

Hence (\bar{x}, \bar{y}) is a local weak efficient solution of (P₁).

Next, we prove that (\bar{x}, \bar{y}) is a local optimal solution of the scalar problem (P₂).

By weighted sum approach, we have

$$\begin{aligned} (\text{P}_2) \quad & \text{Minimize}_{x,y} \left(\sum_{k=1}^p w_k (F_k(x, y) - \phi_k(\bar{x}, \bar{y})F'_k(x, y)) \right) \\ & w_k > 0, \quad k = 1, 2, \dots, p, \quad \sum_{k=1}^p w_k = 1 \end{aligned}$$

subject to $(x, y) \in E$.

Since (\bar{x}, \bar{y}) is a local weak efficient solution of (P_1) , there does not exist any feasible solution $(x, y) \in (U_0 \times V_0) \cap E$ such that the following hold

$$F_k(x, y) - \phi_k(\bar{x}, \bar{y})F'_k(x, y) < F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y})F'_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p,$$

where $F'_k(x, y) > 0$, for all $k = 1, 2, \dots, p$ and $(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\Rightarrow w_k(F_k(x, y) - \phi_k(\bar{x}, \bar{y})F'_k(x, y)) < w_k(F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y})F'_k(\bar{x}, \bar{y})),$$

as $w_k > 0$, $k = 1, 2, \dots, p$.

Thus, for any feasible solution $(x, y) \in (U_0 \times V_0) \cap E$, following does not hold

$$\sum_{k=1}^p w_k(F_k(x, y) - \phi_k(\bar{x}, \bar{y})F'_k(x, y)) < \sum_{k=1}^p w_k(F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y})F'_k(\bar{x}, \bar{y})).$$

Therefore (\bar{x}, \bar{y}) is a local optimal solution of (P_2) .

Since (\bar{x}, \bar{y}) solves (P_2) , it solves the following unconstrained problem

$$\text{Minimize } H(x, y) = \max_{x, y} \left\{ \begin{array}{l} \sum_{k=1}^p w_k \varphi_k(x, y) - \sum_{k=1}^p w_k \varphi_k(\bar{x}, \bar{y}), f(x, y) - V(x), \\ G_1(x, y), \dots, G_{m_2}(x, y), \\ g_1(x, y), \dots, g_{m_1}(x, y) \end{array} \right\}.$$

Now,

$$\sum_{k=1}^p w_k \varphi_k(\bar{x}, \bar{y}) = 0.$$

By Lemma 2.2, we have

$$(0, 0) \in \text{clconv } \partial^* H(\bar{x}, \bar{y})$$

where using Theorem 3.2 [17],

$$\partial^* H(\bar{x}, \bar{y}) = \text{conv} \left\{ \begin{array}{l} \partial^* \left(\sum_{k=1}^p w_k \varphi_k(\bar{x}, \bar{y}) \right) \cup \partial^*(f(\bar{x}, \bar{y}) - V(\bar{x})) \cup \bigcup_{j \in J(\bar{x}, \bar{y})} \partial^* G_j(\bar{x}, \bar{y}) \\ \cup \bigcup_{i \in I(\bar{x}, \bar{y})} \partial^* g_i(\bar{x}, \bar{y}) \end{array} \right\}.$$

Thus,

$$(0, 0) \in \text{clconv} \left(\text{conv} \left\{ \begin{array}{l} \partial^* \left(\sum_{k=1}^p w_k \varphi_k(\bar{x}, \bar{y}) \right) \cup \partial^*(f(\bar{x}, \bar{y}) - V(\bar{x})) \cup \bigcup_{j \in J(\bar{x}, \bar{y})} \partial^* G_j(\bar{x}, \bar{y}) \\ \cup \bigcup_{i \in I(\bar{x}, \bar{y})} \partial^* g_i(\bar{x}, \bar{y}) \end{array} \right\} \right),$$

which implies that there exists a sequence

$$(x_n, y_n) \in \text{conv} \left(\text{conv} \left\{ \begin{array}{l} \partial^* \left(\sum_{k=1}^p w_k \varphi_k(\bar{x}, \bar{y}) \right) \cup \partial^*(f(\bar{x}, \bar{y}) - V(\bar{x})) \cup \bigcup_{j \in J(\bar{x}, \bar{y})} \partial^* G_j(\bar{x}, \bar{y}) \\ \cup \bigcup_{i \in I(\bar{x}, \bar{y})} \partial^* g_i(\bar{x}, \bar{y}) \end{array} \right\} \right),$$

such that $(x_n, y_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$.

Since convexfactors are in general nonconvex sets, and there exist sequences of scalars

$$\{\lambda'^n\}, \lambda'^n \geq 0, \{\gamma^n\}, \gamma^n \geq 0, \{\tau_j^n\}, \tau_j^n \geq 0, j \in J(\bar{x}, \bar{y}), \{\mu_i^n\}, \mu_i^n \geq 0, i \in I(\bar{x}, \bar{y}), n \in N$$

with $\lim_{n \rightarrow \infty} \left[\lambda'^n + \gamma^n + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j^n + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i^n \right] = 1$ such that

$$\begin{aligned} (x_n, y_n) \in & \lambda'^n \text{conv } \partial^* \left(\sum_{k=1}^p w_k \varphi_k(\bar{x}, \bar{y}) \right) + \gamma^n (\text{conv } \partial^* (f(\bar{x}, \bar{y}) - V(\bar{x}))) \\ & + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j^n \text{conv } \partial^* G_j(\bar{x}, \bar{y}) + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i^n \text{conv } \partial^* g_i(\bar{x}, \bar{y}). \end{aligned}$$

That is,

$$\begin{aligned} (x_n, y_n) \in & \lambda'^n \text{conv} \left(\sum_{k=1}^p w_k (\partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y})) \right) \\ & + \gamma^n (\text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})) \\ & + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j^n \text{conv } \partial^* G_j(\bar{x}, \bar{y}) + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i^n \text{conv } \partial^* g_i(\bar{x}, \bar{y}). \end{aligned}$$

Using convex hull property of subsets A_1 and A_2 of \mathbb{R}^{n_1} , $\text{conv}(A_1 + A_2) = \text{conv}(A_1) + \text{conv}(A_2)$ and of a subset A of \mathbb{R}^{n_1} , $\text{conv}(\lambda A) = \lambda \text{conv}(A)$, $\lambda \in \mathbb{R}$, we get

$$\begin{aligned} (x_n, y_n) \in & \sum_{k=1}^p w'_k \text{conv} \{ \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y}) \} \\ & + \gamma^n (\text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})) \\ & + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j^n \text{conv } \partial^* G_j(\bar{x}, \bar{y}) + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i^n \text{conv } \partial^* g_i(\bar{x}, \bar{y}) \end{aligned}$$

where

$$w'_k = \lambda'^n w_k, k = 1, 2, \dots, p, \lim_{n \rightarrow \infty} \left[\sum_{k=1}^p w'_k + \gamma^n + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j^n + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i^n \right] = 1.$$

Since $\{w'_k\}$, $\{\gamma^n\}$, $\{\tau_j^n\}$, $j \in J(\bar{x}, \bar{y})$, $\{\mu_i^n\}$, $i \in I(\bar{x}, \bar{y})$ are bounded sequences, we may assume that $w'_k \rightarrow w'_k$, $\gamma^n \rightarrow \gamma$, $\tau_j^n \rightarrow \tau_j$, $\mu_i^n \rightarrow \mu_i$ as $n \rightarrow \infty$.

Thus, we have

$$(0, 0) \in \text{cl} \left[\begin{aligned} & \sum_{k=1}^p w'_k \text{conv} \{ \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y}) \} \\ & + \gamma (\text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})) \\ & + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \text{conv } \partial^* G_j(\bar{x}, \bar{y}) + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \text{conv } \partial^* g_i(\bar{x}, \bar{y}) \end{aligned} \right],$$

with $\sum_{k=1}^p w'_k + \gamma + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i = 1$.

Again, using the convex hull property of subsets A_1 and A_2 of \mathbb{R}^{n_1} , $\text{conv}(A_1 + A_2) = \text{conv}(A_1) + \text{conv}(A_2)$ and of a subset A of \mathbb{R}^{n_1} , $\text{conv}(\lambda A) = \lambda \text{conv}(A)$, $\lambda \in \mathbb{R}$, we get

$$(0, 0) \in \text{cl} \left[\begin{array}{l} \sum_{k=1}^p w'_k \{ \text{conv } \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \text{conv } \partial^* F'_k(\bar{x}, \bar{y}) \} \\ + \gamma (\text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})) \\ + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \text{conv } \partial^* G_j(\bar{x}, \bar{y}) + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \text{conv } \partial^* g_i(\bar{x}, \bar{y}) \end{array} \right],$$

with $\sum_{k=1}^p w'_k + \gamma + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i = 1$.

Thus, there exist

$$\begin{aligned} \xi_k^n &\in \text{conv } \partial^* F_k(\bar{x}, \bar{y}), \quad \xi'_k{}^n \in \text{conv } \partial^* F'_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p, \\ \zeta^n &\in \text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}), \\ \eta_i^n &\in \text{conv } \partial^* g_i(\bar{x}, \bar{y}), \quad i \in I(\bar{x}, \bar{y}), \quad \gamma_j^n \in \text{conv } \partial^* G_j(\bar{x}, \bar{y}), \quad j \in J(\bar{x}, \bar{y}) \end{aligned}$$

such that

$$\lim_{n \rightarrow \infty} \left[\sum_{k=1}^p w'_k (\xi_k^n - \phi_k(\bar{x}, \bar{y}) \xi'_k{}^n) + \gamma \zeta^n + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \eta_i^n + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \gamma_j^n \right] = (0, 0),$$

with

$$\sum_{k=1}^p w'_k + \gamma + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i = 1.$$

For any arbitrary $(v_1, v_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, we have

$$\lim_{n \rightarrow \infty} \left[\begin{array}{l} \sum_{k=1}^p w'_k \langle (\xi_k^n - \phi_k(\bar{x}, \bar{y}) \xi'_k{}^n), (v_1, v_2) \rangle + \gamma \langle \zeta^n, (v_1, v_2) \rangle \\ + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \langle \eta_i^n, (v_1, v_2) \rangle + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \langle \gamma_j^n, (v_1, v_2) \rangle \end{array} \right] = 0, \quad (5.1)$$

with $\sum_{k=1}^p w'_k + \gamma + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i = 1$.

Suppose that $w' = (w'_1, \dots, w'_p) = 0$.

Then, we have, from (5.1),

$$\lim_{n \rightarrow \infty} \left[\gamma \langle \zeta^n, (v_1, v_2) \rangle + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \langle \eta_i^n, (v_1, v_2) \rangle + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \langle \gamma_j^n, (v_1, v_2) \rangle \right] = 0, \quad (5.2)$$

with $\gamma + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i = 1$.

From ∂^* -MOFBLPP CQ, for every $\gamma \geq 0$, $\tau_j \geq 0$, $j \in J(\bar{x}, \bar{y})$, $\mu_i \geq 0$, $i \in I(\bar{x}, \bar{y})$ (not all zero), for all, in particular for $\zeta^n \in \text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})$, $\eta_i^n \in \text{conv } \partial^* g_i(\bar{x}, \bar{y})$, $i \in I(\bar{x}, \bar{y})$, $\gamma_j^n \in \text{conv } \partial^* G_j(\bar{x}, \bar{y})$, $j \in J(\bar{x}, \bar{y})$, we have

$$\lim_{n \rightarrow \infty} \left[\gamma \zeta^n + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \gamma_j^n + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \eta_i^n \right] \neq (0, 0).$$

Thus, there exists $(v_1, v_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ such that

$$\lim_{n \rightarrow \infty} \left[\gamma \langle \zeta^n, (v_1, v_2) \rangle + \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \langle \eta_i^n, (v_1, v_2) \rangle + \sum_{j \in J(\bar{x}, \bar{y})} \tau_j \langle \gamma_j^n, (v_1, v_2) \rangle \right] \neq 0,$$

which contradicts (5.2). Hence $w' \neq 0$. Putting $\mu_i = 0$ for $i \notin I(\bar{x}, \bar{y})$ and $\tau_j = 0$ for $j \notin J(\bar{x}, \bar{y})$ we establish the result. \square

Remark 6. If $\partial^* F_k(\bar{x}, \bar{y})$ are convexifactors of F_k , $k = 1, 2, \dots, p$ at (\bar{x}, \bar{y}) which are upper regular and $\partial^* F'_k(\bar{x}, \bar{y})$ are convexifactors of F'_k , $k = 1, 2, \dots, p$ at (\bar{x}, \bar{y}) which are upper regular if $\phi_k(\bar{x}, \bar{y}) > 0$ and lower regular if $\phi_k(\bar{x}, \bar{y}) < 0$, $k = 1, \dots, p$, then, using Remark 1 and Remark 2, we have that $\partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y})$, $k = 1, 2, \dots, p$ is a convexifactor of $F_k - \phi_k F'_k$, $k = 1, 2, \dots, p$ at (\bar{x}, \bar{y}) . Also, if we assume that for some k , $k = 1, 2, \dots, p$, $\partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y})$ is an upper regular convexifactor of $F_k - \phi_k F'_k$ at (\bar{x}, \bar{y}) and for some $k_0 \neq k$, $\partial^* F_{k_0}(\bar{x}, \bar{y}) - \phi_{k_0}(\bar{x}, \bar{y}) \partial^* F'_{k_0}(\bar{x}, \bar{y})$ is a lower regular convexifactor of $F_{k_0} - \phi_{k_0} F'_{k_0}$ at (\bar{x}, \bar{y}) , then, using Remark 1, Lemma 2.1(ii) and above mentioned fact, we have that

$$\begin{aligned} \partial^* \left(\sum_{k=1}^p w_k (F_k - \phi_k F'_k) \right) (\bar{x}, \bar{y}) &= \sum_{k=1}^p \partial^* (w_k (F_k - \phi_k F'_k)) (\bar{x}, \bar{y}) \\ &= \sum_{k=1}^p w_k (\partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y})) \end{aligned}$$

is a convexifactor of $\sum_{k=1}^p w_k (F_k - \phi_k F'_k)$ at (\bar{x}, \bar{y}) .

Remark 7. If f admits a convexifactor $\partial^* f(\bar{x}, \bar{y})$ which is upper regular, then V will also have an upper regular convexifactor $\partial^* V(\bar{x})$, and using Remark 2, $-\partial^* V(\bar{x})$ will be a lower regular convexifactor of $-V$ at \bar{x} . Then, using Remark 1 and Lemma 2.1(ii), $\partial^* (f(\bar{x}, \bar{y}) - V(\bar{x})) = \partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}$ is a convexifactor of $f - V$ at (\bar{x}, \bar{y}) .

Now we give an example to illustrate above theorem.

Example 3. Consider the problem

$$\begin{aligned} &\underset{x, y}{\text{Minimize}} \left[\frac{F_1(x, y)}{F'_1(x, y)}, \frac{F_2(x, y)}{F'_2(x, y)} \right] \\ &\text{subject to } G(x, y) \leq 0, \quad y \in \psi(x), \end{aligned}$$

where for each $x \in \mathbb{R}$, $\psi(x)$ is the set of optimal solutions to the following optimization problem

$$\begin{aligned} &\underset{y}{\text{Minimize}} f(x, y) \\ &\text{subject to } g(x, y) \leq 0, \end{aligned}$$

where $F_1, F_2, F'_1, F'_2, G, f, g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are defined by

$$F_1(x, y) := \begin{cases} x + y, & x \geq 0, y \geq 0, \\ x + y^2, & x \geq 0, y < 0, \\ |x| + y^2, & x < 0, y \leq 0, \\ y, & x < 0, y > 0, \end{cases}$$

$$\begin{aligned}
F_2(x, y) &:= \begin{cases} x + |y|, & x \geq 0, y \in \mathbb{R}, \\ x^2 + |y|, & x < 0, y \in \mathbb{R}, \end{cases} \\
F'_1(x, y) &:= \begin{cases} 1 + x^2, & x \in \mathbb{R}, y \geq 0 \sim \{x < 0, y > 0 \ \& \ x = 0, y > 0\} \\ 1 + 2y + 3y^2, & x \in \mathbb{R}, y < 0, \\ 1 + 2y^2, & x \leq 0, y > 0, \end{cases} \\
F'_2(x, y) &:= \begin{cases} 1 + x + y^2, & x \geq 0, y \geq 0; x > 0, y < 0, \\ 1 + 2x + 3x^2, & x < 0, y \in \mathbb{R}; x = 0, y < 0 \end{cases} \\
G(x, y) &:= y, \\
f(x, y) &:= \begin{cases} e^x + y + y^2, & x \in \mathbb{R}, y \leq 0 \\ x + y, & x \in \mathbb{R}, y > 0 \end{cases} \\
g(x, y) &:= y^2 + y
\end{aligned}$$

The corresponding value function for lower-level problem is given by $V(x) := e^x$, for all $x \in \mathbb{R}$ and the set $\psi(x)$ of optimal solutions to the lower-level problem is given by

$$\psi(x) := \{0\}, \quad \text{for all } x \in \mathbb{R}.$$

There exists no feasible solution $(x, y) \in E$ such that the following hold

$$\frac{F_1(x, y)}{F'_1(x, y)} < \frac{F_1(\bar{x}, \bar{y})}{F'_1(\bar{x}, \bar{y})} \quad \text{and} \quad \frac{F_2(x, y)}{F'_2(x, y)} < \frac{F_2(\bar{x}, \bar{y})}{F'_2(\bar{x}, \bar{y})},$$

where $E := \left\{ (x, y) : x \in \mathbb{R}, -\frac{1}{2} \leq y \leq 0 \right\}$.

Thus, $(\bar{x}, \bar{y}) = (0, 0)$ is a weak efficient solution of the problem.

It can be seen that F_1, F_2 admit CFs $\partial^* F_1(0, 0) = \{(1, 1), (0, 1), (1, 0), (-1, 0)\}$, $\partial^* F_2(0, 0) = \{(1, 1), (0, -1), (1, -1), (0, 1)\}$ respectively at $(0, 0)$ which are upper regular and F'_1, F'_2 admit CFs $\partial^* F'_1(0, 0) = \{(0, 0), (0, 1), (0, 2)\}$, $\partial^* F'_2(0, 0) = \{(1, 0), (2, 0)\}$ respectively at $(0, 0)$ which are lower regular. G and g admit CFs $\partial^* G(0, 0) = \{(0, 1)\}$, $\partial^* g(0, 0) = \{(0, 1)\}$ respectively at $(0, 0)$. f admits a CF $\partial^* f(0, 0) = \{(1, 1)\}$ at $(0, 0)$ which is upper regular. $\partial^* V(0) \times \{0\} = \{(1, 0)\}$, $\text{conv}(\partial^* f(0, 0) - \partial^* V(0) \times \{0\}) = \{(0, 1)\}$.

$$\phi_1(0, 0) = 0, \quad \phi_2(0, 0) = 0.$$

∂^* -MOFBLPP CQ is satisfied at $(\bar{x}, \bar{y}) = (0, 0)$.

Then, there exist scalars $w'_1 = \frac{1}{3}$, $w'_2 = \frac{1}{3}$, $\gamma = \frac{1}{3}$, $\mu = 0$, $\tau = 0$ with $w'_1 + w'_2 + \gamma + \mu + \tau = 1$ such that

$$(0, 0) \in \text{cl} \left[\begin{array}{l} \sum_{k=1}^2 w'_k \{ \text{conv } \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \text{conv } \partial^* F'_k(\bar{x}, \bar{y}) \} \\ + \gamma (\text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})) \\ + \tau \text{conv } \partial^* G(\bar{x}, \bar{y}) + \mu \text{conv } \partial^* g(\bar{x}, \bar{y}) \end{array} \right].$$

6 Sufficient optimality conditions

Now we prove sufficient optimality conditions for MOFBLPP. We shall now onwards assume that MOFBLPP has at least one optimal solution.

Theorem 6.1. *Let $(\bar{x}, \bar{y}) \in E$. Assume that $\partial^* F_k(\bar{x}, \bar{y})$ are convexifactors of F_k , $k = 1, 2, \dots, p$ at (\bar{x}, \bar{y}) which are upper regular and $\partial^* F'_k(\bar{x}, \bar{y})$ are convexifactors of F'_k , $k = 1, 2, \dots, p$ at (\bar{x}, \bar{y}) which are upper regular if $\phi_k(\bar{x}, \bar{y}) > 0$ and lower regular if $\phi_k(\bar{x}, \bar{y}) < 0$, $k = 1, 2, \dots, p$. Suppose that $\partial^* \varphi_k(x, y) = \partial^*(F_k - \phi_k F'_k)(x, y) = \partial^* F_k(x, y) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(x, y)$, $k = 1, 2, \dots, p$. Let φ_k , $k = 1, 2, \dots, p$ be ∂^* -asymptotic pseudoconvex, $f - V$, g_i , $i \in I(\bar{x}, \bar{y})$ and G_j , $j \in J(\bar{x}, \bar{y})$ be ∂^* -asymptotic quasiconvex at (\bar{x}, \bar{y}) . Suppose that there exist scalars $w'_k \geq 0$ (not all zero), $k = 1, 2, \dots, p$, $\gamma \geq 0$, $\tau_j \geq 0$, $j \in J$, $\mu_i \geq 0$, $i \in I$ with $\sum_{k=1}^p w'_k + \gamma + \sum_{j \in J} \tau_j + \sum_{i \in I} \mu_i = 1$ such that the following condition is satisfied*

$$(0, 0) \in \text{cl} \left[\begin{array}{l} \sum_{k=1}^p w'_k \{ \text{conv } \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \text{conv } \partial^* F'_k(\bar{x}, \bar{y}) \} \\ + \gamma (\text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})) \\ + \sum_{j \in J} \tau_j \text{conv } \partial^* G_j(\bar{x}, \bar{y}) + \sum_{i \in I} \mu_i \text{conv } \partial^* g_i(\bar{x}, \bar{y}) \end{array} \right]. \quad (6.1)$$

Then, (\bar{x}, \bar{y}) is a weak efficient solution of MOFBLPP.

Proof. Suppose that (\bar{x}, \bar{y}) is not a weak efficient solution of MOFBLPP, then by Remark 3, (\bar{x}, \bar{y}) is not a weak efficient solution of RMOFBLPP.

Thus, there exists a feasible solution $(x, y) \in E$ such that

$$\frac{F_k(x, y)}{F'_k(x, y)} < \frac{F_k(\bar{x}, \bar{y})}{F'_k(\bar{x}, \bar{y})}, \quad k = 1, 2, \dots, p,$$

which implies that $F_k(x, y) - \phi_k(\bar{x}, \bar{y}) F'_k(x, y) < 0$, $k = 1, 2, \dots, p$.

Since $F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) F'_k(\bar{x}, \bar{y}) = 0$, $k = 1, 2, \dots, p$, we have

$$F_k(x, y) - \phi_k(\bar{x}, \bar{y}) F'_k(x, y) < F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) F'_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p.$$

That is, we have

$$\varphi_k(x, y) < \varphi_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p.$$

Using ∂^* -asymptotic pseudoconvexity of φ_k , $k = 1, 2, \dots, p$, we get

$$\langle \xi_k^0, (x - \bar{x}, y - \bar{y}) \rangle < 0 \quad \text{for all } \xi_k^0 \in \text{clconv } \partial^* \varphi_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p. \quad (6.2)$$

Now,

$$\begin{aligned} \text{conv } \partial^* \varphi_k(\bar{x}, \bar{y}) &= \text{conv } \partial^*(F_k - \phi_k F'_k)(\bar{x}, \bar{y}) \\ &= \text{conv} \{ \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y}) \}, \quad k = 1, 2, \dots, p. \end{aligned} \quad (6.3)$$

Using the convex hull property of subsets A_1 and A_2 of \mathbb{R}^{n_1} , $\text{conv}(A_1 + A_2) = \text{conv}(A_1) + \text{conv}(A_2)$ and of a subset A of \mathbb{R}^{n_1} , $\text{conv}(\lambda A) = \lambda \text{conv}(A)$, $\lambda \in \mathbb{R}$, we get

$$\begin{aligned} &\text{conv} \{ \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \partial^* F'_k(\bar{x}, \bar{y}) \} \\ &= \text{conv } \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \text{conv } \partial^* F'_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p. \end{aligned} \quad (6.4)$$

Thus, combining (6.3) and (6.4), we get

$$\text{conv } \partial^* \varphi_k(\bar{x}, \bar{y}) = \text{conv } \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \text{conv } \partial^* F'_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p. \quad (6.5)$$

Now, by (6.1), there exist

$$\begin{aligned} \xi_k^n &\in \text{conv } \partial^* F_k(\bar{x}, \bar{y}), \quad \xi_k'^n \in \text{conv } \partial^* F_k'(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p, \\ \zeta^n &\in \text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}), \\ \eta_i^n &\in \text{conv } \partial^* g_i(\bar{x}, \bar{y}), \quad i \in I, \quad \gamma_j^n \in \text{conv } \partial^* G_j(\bar{x}, \bar{y}), \quad j \in J \text{ such that} \\ \lim_{n \rightarrow \infty} \left[\sum_{k=1}^p w'_k (\xi_k^n - \phi_k(\bar{x}, \bar{y}) \xi_k'^n) + \gamma \zeta^n + \sum_{i \in I} \mu_i \eta_i^n + \sum_{j \in J} \tau_j \gamma_j^n \right] &= (0, 0). \end{aligned}$$

Then, for all $(x, y) \in E$, we have

$$\lim_{n \rightarrow \infty} \left[\sum_{k=1}^p w'_k \left\langle (\xi_k^n - \phi_k(\bar{x}, \bar{y}) \xi_k'^n), (x - \bar{x}, y - \bar{y}) \right\rangle + \gamma \left\langle \zeta^n, (x - \bar{x}, y - \bar{y}) \right\rangle \right. \\ \left. + \sum_{i \in I} \mu_i \left\langle \eta_i^n, (x - \bar{x}, y - \bar{y}) \right\rangle + \sum_{j \in J} \tau_j \left\langle \gamma_j^n, (x - \bar{x}, y - \bar{y}) \right\rangle \right] = 0. \quad (6.6)$$

Now, from (6.2), since $\xi_k^0 \in \text{clconv } \partial^* \varphi_k(\bar{x}, \bar{y})$, $k = 1, 2, \dots, p$, there exists $\xi_k^{0n} \in \text{conv } \partial^* \varphi_k(\bar{x}, \bar{y})$, $k = 1, 2, \dots, p$ such that

$$\lim_{n \rightarrow \infty} \xi_k^{0n} = \xi_k^0, \quad k = 1, 2, \dots, p. \quad (6.7)$$

Using above in (6.2), we get

$$\lim_{n \rightarrow \infty} \left\langle \xi_k^{0n}, (x - \bar{x}, y - \bar{y}) \right\rangle < 0, \quad \text{for some } \xi_k^{0n} \in \text{conv } \partial^* \varphi_k(\bar{x}, \bar{y}), \quad k = 1, 2, \dots, p.$$

Thus, for every $\xi_k^0 \in \text{clconv } \partial^* \varphi_k(\bar{x}, \bar{y})$, $k = 1, 2, \dots, p$, there exists $\xi_k^{0n} \in \text{conv } \partial^* \varphi_k(\bar{x}, \bar{y})$, $k = 1, 2, \dots, p$ such that (6.7) holds.

In particular, it is true for $\xi_k^0 \in \text{clconv } \partial^* \varphi_k(\bar{x}, \bar{y})$, $k = 1, 2, \dots, p$ for which there exists $\xi_k^{0n} \in \text{conv } \partial^* \varphi_k(\bar{x}, \bar{y})$ such that

$$\xi_k^0 = \lim_{n \rightarrow \infty} \xi_k^{0n} = \lim_{n \rightarrow \infty} (\xi_k^n - \phi_k(\bar{x}, \bar{y}) \xi_k'^n),$$

where $\xi_k^n \in \text{conv } \partial^* F_k(\bar{x}, \bar{y})$, $\xi_k'^n \in \text{conv } \partial^* F_k'(\bar{x}, \bar{y})$, $k = 1, 2, \dots, p$.

Thus, we get

$$\lim_{n \rightarrow \infty} \left[\sum_{k=1}^p w'_k \left\langle (\xi_k^n - \phi_k(\bar{x}, \bar{y}) \xi_k'^n), (x - \bar{x}, y - \bar{y}) \right\rangle \right] < 0 \text{ as } \xi_k^n \in \text{conv } \partial^* F_k(\bar{x}, \bar{y}), \\ \xi_k'^n \in \text{conv } \partial^* F_k'(\bar{x}, \bar{y}), \quad w'_k \geq 0 \text{ (not all zero)}. \quad (6.8)$$

Since (x, y) is feasible for RMOFBLPP, we have

$$f(x, y) - V(x) \leq 0 \leq f(\bar{x}, \bar{y}) - V(\bar{x}).$$

By ∂^* -asymptotic quasiconvexity of $f - V$, we get

$$\left\langle \zeta, (x - \bar{x}, y - \bar{y}) \right\rangle \leq 0, \quad \text{for all } \zeta \in \text{cl}(\text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})).$$

In particular, it is true for ζ such that $\zeta = \lim_{n \rightarrow \infty} \zeta^n$, where $\zeta^n \in \text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})$.

That is,

$$\lim_{n \rightarrow \infty} \langle \zeta^n, (x - \bar{x}, y - \bar{y}) \rangle \leq 0, \text{ as } \zeta^n \in \text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}),$$

which gives

$$\lim_{n \rightarrow \infty} \gamma \langle \zeta^n, (x - \bar{x}, y - \bar{y}) \rangle \leq 0, \text{ as } \zeta^n \in \text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\}) \text{ and } \gamma \geq 0. \quad (6.9)$$

Now, for each $i \in I(\bar{x}, \bar{y})$, $g_i(x, y) \leq 0 = g_i(\bar{x}, \bar{y})$.

Using ∂^* -asymptotic quasiconvexity of g_i , $i \in I(\bar{x}, \bar{y})$, we have

$$\langle \eta_i, (x - \bar{x}, y - \bar{y}) \rangle \leq 0, \text{ for all } \eta_i \in \text{clconv } \partial^* g_i(\bar{x}, \bar{y}), \text{ } i \in I(\bar{x}, \bar{y}).$$

Then, as earlier, we have

$$\lim_{n \rightarrow \infty} \langle \eta_i^n, (x - \bar{x}, y - \bar{y}) \rangle \leq 0, \text{ as } \eta_i^n \in \text{conv } \partial^* g_i(\bar{x}, \bar{y}), \text{ for } i \in I(\bar{x}, \bar{y}).$$

Thus, we have

$$\lim_{n \rightarrow \infty} \left\langle \sum_{i \in I(\bar{x}, \bar{y})} \mu_i \eta_i^n, (x - \bar{x}, y - \bar{y}) \right\rangle \leq 0, \text{ as } \mu_i \geq 0 \text{ and } \eta_i^n \in \text{conv } \partial^* g_i(\bar{x}, \bar{y}).$$

Taking $\mu_i = 0$, $i \notin I(\bar{x}, \bar{y})$, we get

$$\lim_{n \rightarrow \infty} \left\langle \sum_{i \in I} \mu_i \eta_i^n, (x - \bar{x}, y - \bar{y}) \right\rangle \leq 0, \text{ as } \mu_i \geq 0 \text{ and } \eta_i^n \in \text{conv } \partial^* g_i(\bar{x}, \bar{y}). \quad (6.10)$$

Similarly, we have

$$G_j(x, y) \leq 0 = G_j(\bar{x}, \bar{y}), \text{ for each } j \in J(\bar{x}, \bar{y}).$$

Using ∂^* -asymptotic quasiconvexity of G_j , $j \in J(\bar{x}, \bar{y})$, taking $\tau_j = 0$, $j \notin J(\bar{x}, \bar{y})$ and proceeding as above, we get

$$\lim_{n \rightarrow \infty} \left\langle \sum_{j \in J} \tau_j \gamma_j^n, (x - \bar{x}, y - \bar{y}) \right\rangle \leq 0 \text{ as } \tau_j \geq 0, \text{ } j \in J(\bar{x}, \bar{y}) \text{ and } \gamma_j^n \in \text{conv } \partial^* G_j(\bar{x}, \bar{y}). \quad (6.11)$$

Adding (6.8), (6.9), (6.10) and (6.11), we get

$$\lim_{n \rightarrow \infty} \left[\begin{aligned} & \sum_{k=1}^p w'_k \left\langle (\xi_k^n - \phi_k(\bar{x}, \bar{y}) \xi_k'^n), (x - \bar{x}, y - \bar{y}) \right\rangle + \gamma \langle \zeta^n, (x - \bar{x}, y - \bar{y}) \rangle \\ & + \sum_{i \in I} \mu_i \langle \eta_i^n, (x - \bar{x}, y - \bar{y}) \rangle + \sum_{j \in J} \tau_j \langle \gamma_j^n, (x - \bar{x}, y - \bar{y}) \rangle \end{aligned} \right] < 0,$$

which is contradiction to (6.6). Hence (\bar{x}, \bar{y}) is a weak efficient solution of MOFBLPP. \square

Following example illustrates the above theorem.

Example 4. Consider the problem

$$\text{Minimize}_{x, y} \left[\begin{array}{l} F_1(x, y) \quad F_2(x, y) \\ F'_1(x, y) \quad F'_2(x, y) \end{array} \right]$$

subject to $G(x, y) \leq 0$, $y \in \psi(x)$,

where for each $x \in \mathbb{R}$, $\psi(x)$ is the set of optimal solutions to the following optimization problem

$$\begin{aligned} & \underset{y}{\text{Minimize}} \quad f(x, y) \\ & \text{subject to} \quad g(x, y) \leq 0, \end{aligned}$$

where $F_1, F_2, F'_1, F'_2, G, f, g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are defined by

$$\begin{aligned} F_1(x, y) &:= \begin{cases} \frac{-|y|}{2}, & x \geq 0, y \geq 0, \\ \frac{-x-y}{2}, & x < 0, y \in \mathbb{R}; x = 0, y < 0, \\ \frac{|y|}{2}, & x > 0, y < 0 \end{cases} \\ F_2(x, y) &:= \begin{cases} \frac{-|y|}{2}, & x > 0, y > 0; x = 0, y \geq 0, \\ \frac{|y|}{2}, & x \geq 0, y < 0, \\ \frac{x^2+y^2}{2}, & x > 0, y = 0, \\ \frac{-x-y}{2}, & x < 0, y \in \mathbb{R} \end{cases} \\ F'_1(x, y) &:= \begin{cases} 1 - |x| - y, & x \geq 0, y \geq 0, \\ 1 + |x| - y, & x < 0, y \geq 0, \\ 1 + |x|, & x \leq 0, y < 0, \\ 1 - |x|, & x > 0, y < 0 \end{cases} \\ F'_2(x, y) &:= \begin{cases} 1 - x - y, & x \geq 0, y \geq 0, \\ 1 - x - y, & x < 0, y > 0, \\ 1 - x, & x < 0, y \leq 0, \\ 1 - x, & x \geq 0, y < 0 \end{cases} \\ G(x, y) &:= x, \\ f(x, y) &:= \begin{cases} x + y^2, & x \in \mathbb{R}, y \leq 0 \sim \{x < 0, y = 0\} \\ x + y, & x \geq 0, y > 0; x < 0, y \geq 0 \end{cases} \\ g(x, y) &:= y^2 + y. \end{aligned}$$

The corresponding value function for lower-level problem is given by

$$V(x) := x \quad \text{for all } x \in \mathbb{R}$$

and the set $\psi(x)$ of optimal solutions to the lower-level problem is given by

$$\psi(x) := \{0\}, \quad \text{for all } x \in \mathbb{R}.$$

There exists no feasible solution $(x, y) \in E$ such that the following hold

$$\frac{F_1(x, y)}{F'_1(x, y)} < \frac{F_1(\bar{x}, \bar{y})}{F'_1(\bar{x}, \bar{y})} \quad \text{and} \quad \frac{F_2(x, y)}{F'_2(x, y)} < \frac{F_2(\bar{x}, \bar{y})}{F'_2(\bar{x}, \bar{y})},$$

where $E := \{(x, 0) : x \leq 0\}$. Thus, $(\bar{x}, \bar{y}) = (0, 0)$ is a weak efficient solution of the problem.

It can be seen that φ_1, φ_2 are ∂^* -asymptotic pseudoconvex at $(0, 0)$ with respect to

$$\partial^* \varphi_1(0, 0) = \partial^* F_1(0, 0)$$

$$\begin{aligned}
 &= \left\{ \left(\frac{-1}{2} + \frac{1}{n}, \frac{-1}{2} + \frac{1}{n} \right), \left(0 + \frac{1}{n}, \frac{-1}{2} + \frac{1}{n} \right) \right\} \cup \left\{ \left(-\frac{1}{2}, -\frac{1}{2} \right), \left(0, -\frac{1}{2} \right) \right\}, \\
 \partial^* \varphi_2(0, 0) &= \partial^* F_2(0, 0) \\
 &= \left\{ \left(0 + \frac{1}{n}, \frac{-1}{2} + \frac{1}{n} \right), \left(\frac{-1}{2} + \frac{1}{n}, \frac{-1}{2} + \frac{1}{n} \right) \right\} \cup \left\{ \left(0, \frac{-1}{2} \right), \left(\frac{-1}{2}, \frac{-1}{2} \right) \right\}
 \end{aligned}$$

respectively. $\phi_1(0, 0) = 0$, $\phi_2(0, 0) = 0$. G and g are ∂^* -asymptotic quasiconvex at $(0, 0)$ with respect to $\partial^* G(0, 0) = \{(1 + \frac{1}{n}, 0)\} \cup \{(1, 0)\}$ and $\partial^* g(0, 0) = \{(0, 1 + \frac{1}{n})\} \cup \{(0, 1)\}$ respectively. $f - V$ is ∂^* -asymptotic quasiconvex at $(0, 0)$ with respect to $\partial^* f(0, 0) - \partial^* V(0) \times \{0\} = \{(0 + \frac{1}{n}, 1 + \frac{1}{n}), (0 + \frac{1}{n}, 0 + \frac{1}{n})\} \cup \{(0, 1), (0, 0)\}$.

Then, there exist scalars $w'_1 = \frac{1}{4}$, $w'_2 = \frac{1}{4}$, $\gamma = \frac{1}{4}$, $\mu = \frac{1}{4}$, $\tau = 0$ with $w'_1 + w'_2 + \gamma + \mu + \tau = 1$ such that

$$(0, 0) \in \text{cl} \left[\begin{array}{l} \sum_{k=1}^2 w'_k \{ \text{conv } \partial^* F_k(\bar{x}, \bar{y}) - \phi_k(\bar{x}, \bar{y}) \text{ conv } \partial^* F'_k(\bar{x}, \bar{y}) \} \\ + \gamma (\text{conv}(\partial^* f(\bar{x}, \bar{y}) - \partial^* V(\bar{x}) \times \{0\})) \\ + \tau \text{ conv } \partial^* G(\bar{x}, \bar{y}) + \mu \text{ conv } \partial^* g(\bar{x}, \bar{y}) \end{array} \right].$$

7 Conclusions

In this article, necessary and sufficient optimality conditions using the concept of convexifactors are developed for multiobjective fractional bilevel programming problem. For that, a suitable CQ in terms of convexifactors has been introduced. Under the assumptions of asymptotic pseudoconvexity and asymptotic quasiconvexity on the functions, sufficient optimality conditions are established for the problem.

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