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THREE-DIMENSIONAL CR-SUBMANIFOLDS IN THE NEARLY KAEHLER SIX-SPHERE SATISFYING B.Y. CHEN'S BASIC EQUALITY

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Abstract. B. Y. Chen introduced in [3] an important Riemannian invariant for a Riemannian manifold and obtained a sharp inequality between his invariant and the squared mean curvature for arbitrary submanifolds in real space forms. In this paper we investigate 3-dimensional CR-submanifolds in the nearly Kaehler 6-sphere which realize the equality case of the inequality.

1. Introduction

Let M^n be an n-dimensional submanifold M^n in an m-dimensional real space form $\tilde{M}^m(c)$ of constant sectional curvature c. In [3] B. Y. Chen introduces a Riemannian invariant δ_M on M^n defined by $\delta_M(p) = \tau(p) - \inf K(p)$, where $\inf K(p)$ is the infimum of the sectional curvature $K(\pi)$ at p, π runs over all planes in the tangent space T_pM^n and τ is the scalar curvature given by $\tau = \sum_{i<j} K(e_i \wedge e_j)$ for an orthonormal basis $e_1, \ldots, e_n$ of T_pM^n. Chen's invariant vanishes trivially when $n = 2$.

In [3] he proves the following basic inequality for arbitrary submanifolds M^n in a real space form $\tilde{M}^m(c)$:

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n+1)(n-2)c, \tag{1.1}$$

where $||H||^2$ is the squared mean curvature. It is a natural and very interesting problem to investigate and to understand submanifolds of dimension ≥ 3 satisfying the equality case of the inequality, i.e.

$$\delta_M = \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n+1)(n-2)c, \tag{1.2}$$

which is known as Chen's basic equality (see, for instance [10]). For such submanifolds there is a canonical distribution defined by

$$\mathcal{D}(p) = \{X \in T_pM^n \,|\, (n-1)h(X,Y) = n\langle X, Y\rangle H, \forall Y \in T_pM^n\},$$

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where h is the second fundamental form of M^n in $\tilde{M}^m(c)$. If the dimension of $\mathcal{D}(p)$ is constant, it is shown in [3] that the distribution $\mathcal{D}$ is completely integrable.

When M is a submanifold of an almost Hermitian manifold $\tilde{M}$, a subspace V of T_pM is called *totally real* if JV is contained in the normal space $T_p^{\perp}M$ of M at p. The submanifold M is called *totally real* if each tangent space of M is totally real; and M is called a CR-submanifold if there exists a differential holomorphic distribution $\mathcal{H}$ on M such that the orthogonal complement $\mathcal{H}^{\perp}$ of $\mathcal{H}$ in TM is a totally real distribution [1]. A CR-submanifold is called *proper* if it is neither totally real (i.e., $\mathcal{H}^{\perp} = TM$) nor holomorphic (i.e., $\mathcal{H} = TM$).

Submanifolds satisfying equality (1.2) have been studied recently by many geometers (see, for instance [2-6, 8-11]). In particular, 3-dimensional totally real submanifolds satisfying Chen's basic equality in the nearly Kaehler 6-sphere $S^6(1)$ have been completely classified in [11] by F. Dillen and L. Vrancken.

In [5] Chen also established the basic inequality for arbitrary submanifolds in complex space forms of constant holomorphic sectional curvature $4c$ as follows:

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n^2+2n-2)c \quad (for \quad c > 0),$$

$$\delta_M \leq \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n+1)(n-2)c \quad (for \quad c < 0).$$

He proved that a submanifold of complex projective space satisfies Chen's basic equality if and only if the submanifold is a totally geodesic complex submanifold. Moreover in [7] he and L. Vrancken completely classified proper CR-submanifolds of complex hyperbolic spaces which satisfy the basic equality.

In complex hyperbolic space, the canonical distribution $\mathcal{D}$ of submanifolds satisfying Chen's basic equality is totally real.

In this paper, we investigate 3-dimensional CR-submanifolds in the nearly Kaehler 6-sphere $S^6(1)$ satisfying Chen's basic equality (1.2) under the condition that $\mathcal{D}$ is totally real. Our main result is the following.

Main Theorem. *There exist no 3-dimensional proper CR-submanifold in $S^6(1)$ satisfying Chen's basic equality under the condition that $\mathcal{D}$ is totally real.*

2. The Nearly Kaehler Structure of $S^6(1)$

We give a brief explanation how the standard nearly Kaehler structure on $S^6(1)$ arises in a natural manner from Cayley multiplication. For elementary facts about Cayley numbers and their automorphism group G_2, we refer the reader to [14] and [17].

The multiplication on Cayley numbers ϑ may be used to define a vector cross product on purely imaginary Cayley numbers $\mathbb{R}^7$ by using the formula

$$u \times v = \frac{1}{2}(uv - vu), \tag{2.1}$$

while the standard inner product on $\mathbb{R}^7$ is given by

$$\langle u, v\rangle = -\frac{1}{2}(uv + vu). \tag{2.2}$$

It is elementary [12] to show that the triple scalar product $\langle u \times v, w\rangle$ is skew symmetric in u, v, w.

Let J be the automorphism of the tangent bundle TS^6 of $S^6(1)$ defined by

$$Ju = x \times u, \quad u \in T_xS^6(1), \quad x \in S^6(1).$$

It is clear that J is an almost complex structure on $S^6(1)$ and J is in fact a nearly Kaehler structure on $S^6(1)$ in the sense that $(\tilde{\nabla}_u J)u = 0$, for any vector u tangent to $S^6(1)$, where $\tilde{\nabla}$ denotes the metric connection of $S^6(1)$. We define the corresponding skew-symmetric (2,1)-tensor field G by

$$G(X, Y) = (\tilde{\nabla}_X J)(Y).$$

We know from [16] that this tensor field satisfies the following property:

$$G(X, JY) + JG(X, Y) = 0. \tag{2.3}$$

For further information on the properties of G, we refer to [6] and [16].

3. Proof of the Main Theorem

We need the following lemmas.

Lemma 3.1. *Let M be a CR-submanifold of a nearly Kaehler manifold $\tilde{M}$. Denote by $T^\perp M = J\mathcal{H}^\perp \oplus \nu$ the orthogonal decomposition of the normal bundle, where $\mathcal{H}^\perp$ is the totally real distribution and ν a complex subbundle of $T^\perp M$. Then the shape operator A satisfies*

$$\langle A_\xi JX, X\rangle = -\langle A_{J\xi} X, X\rangle, \tag{3.1}$$

for any vector field X in the holomorphic distribution $\mathcal{H}$ and ξ in ν.

Proof. For any vector field X in the holomorphic distribution $\mathcal{H}$, we have

$$0 = (\tilde{\nabla}_X J)(X) = \nabla_X JX + h(X, JX) - J(\nabla_X X) - Jh(X, X),$$

where ∇ is the metric connection of M with respect to the induced metric. Hence we obtain $\langle h(X, JX), \xi\rangle = \langle Jh(X, X), \xi\rangle$ for any vector field $\xi \in \nu$.

Lemma 3.2. *Let M be a 3-dimensional proper minimal CR-submanifold in $S^6(1)$. If M satisfies equality* (1.2) *and $\mathcal{D}$ is totally real, then $\mathcal{D}^\perp$ is integrable.*

Proof. Since $\mathcal{D}^\perp$ is a complex plane ($i.e, \mathcal{D}^\perp = \mathcal{H}$), we can choose an orthonormal frame field $\{E_1, E_2\}$ on $\mathcal{D}^\perp$ such that $JE_1 = E_2$. By the minimality of M and lemma

3.2 of [3], the second fundamental h satisfies $h(E_1, JE_2) = h(JE_1, E_2)$. Moreover from (2.3) we have $G(E_1, E_2) = 0$. Hence, by virtue of [1, page 26], $\mathcal{D}^\perp$ is integrable.

A submanifold is said to be *linearlyfull* in $S^m(1)$ if it does not lie in any totally geodesic submanifold of $S^m(1)$.

Lemma 3.3. *Let M be a 3-dimensional proper minimal CR-submanifold in $S^6(1)$. If M satisfies equality* (1.2) *and $\mathcal{D}$ is totally real, then M is linearly full in a totally geodesic $S^5(1)$.*

Proof. Let $\{E_1, E_2, E_3\}$ be an orthonormal frame field on M such that $\{E_1, E_2\} \in \mathcal{H}$, $E_3 \in \mathcal{H}^\perp$ which diagonalize the shape operator A_{JE_3}. We may assume $JE_1 = E_2$. For any $X \in TM$, we have

$$-A_{JE_3}X + \nabla^\perp_X JE_3 = \tilde{\nabla}_X JE_3 = (\tilde{\nabla}_X J)(E_3) + J\nabla_X E_3 + Jh(X, E_3). \tag{3.2}$$

From (3.2) and lemma 3.2 of [3], we have

$$\nabla^\perp_{E_3} JE_3 = J(\nabla_{E_3} E_3) = 0. \tag{3.3}$$

Further from (3.2) and lemma 3.2 of [3], we have

$$-A_{JE_3}E_1 + \nabla^\perp_{E_1} JE_3 = (\tilde{\nabla}_{E_1} J)(E_3) + J(\nabla_{E_1} E_3). \tag{3.4}$$

Since G is skew-symmetric, we find

$$\begin{aligned}(\tilde{\nabla}_{E_1} J)(E_3) &= -(\tilde{\nabla}_{E_3} J)(E_1) \\ &= -\nabla_{E_3} E_2 + J(\nabla_{E_3} E_1)\end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we obtain $\nabla^\perp_{E_1} JE_3 \in J\mathcal{H}^\perp$ which implies $\nabla^\perp_{E_1} JE_3 = 0$ because $J\mathcal{H}^\perp$ is of rank one and JE_3 is of unit length. Similarly, we also have $\nabla^\perp_{E_2} JE_3 = 0$ Hence, JE_3 is a parallel normal vector field. Thus, from the equation of Ricci, we have $[A_{JE_3}, A_\xi] = 0$.

If $A_{JE_3} \equiv 0$, the first normal space ν is parallel i.e. $\nabla^\perp \xi \in \nu$, $\xi \in \nu$ by the parallelism of JE_3. Therefore, by Erbacher's theorem [12], M must be contained in a totally geodesic $S^5(1)$.

If $A_{JE_3} \not\equiv 0$, we put $V = \{p \in M : \det A_{JE_3} \neq 0\}$. In this case, the shape operators take the following form on V:

$$A_{JE_3} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{\xi_i} = \begin{pmatrix} h^i_{11} & 0 & 0 \\ 0 & -h^i_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.6}$$

$$\xi_i \in \nu, \quad i = 1, 2.$$

Combining (3.1) with (3.6) yields $A_{\xi_1} = A_{\xi_2} = 0$ which imply that V is contained in a totally geodesic $S^4(1)$, since JE_3 is a parallel normal vector field. Therefore, $TS^4(1)|_V$ is spanned by $\{E_1, E_2, E_3, JE_3\}$. A result of Gray in [13] shows that this is impossible.

For a given orthonormal frame field $\{E_1, E_2, E_3\}$, define local functions γ_{ij}^k by $\gamma_{ij}^k = \langle \nabla_{E_i} E_j, E_k \rangle$.

Lemma 3.4. *Let M be a 3-dimensional proper minimal CR-submanifold in $S^6(1)$. If M satisfies equality (1.2) and $\mathcal{D}$ is totally real, then, with respect to some suitable orthonormal frame field $\{E_1, E_2, E_3\}$ on M, we have*

$$\gamma_{33}^1 = \gamma_{33}^2 = 0 \tag{3.7}$$

$$\gamma_{11}^3 = \gamma_{22}^3 \tag{3.8}$$

$$\gamma_{12}^3 = -\gamma_{21}^3 \tag{3.9}$$

Proof. Let $\{E_1, E_2, E_3\}$ be an orthonormal frame field mentioned in lemma 3.3. From $A_{JE_3} = 0$ and (3.1), we know that the second fundamental form satisfies

$$h(E_1, E_1) = \phi\xi, \quad h(E_2, E_2) = -\phi\xi, \quad h(E_3, E_3) = 0,$$
$$h(E_1, E_2) = \phi J\xi, \quad h(E_1, E_3) = 0, \quad h(E_2, E_3) = 0,$$

where ϕ is a function and $\xi \in \nu$. Thus, from $(\nabla h)(E_1, E_3, E_3) = (\nabla h)(E_3, E_1, E_3)$, we obtain $\nabla_{E_3} E_3 = 0$. Also from $(\nabla h)(E_3, E_1, E_1) = (\nabla h)(E_1, E_3, E_1)$, we obtain (3.8). Finally from $(\nabla h)(E_1, E_2, E_3) = (\nabla h)(E_2, E_1, E_3)$, we obtain (3.9).

From lemma 3.2, we know that the distribution $\mathcal{D}^\perp$ locally spanned by E_1 and E_2 is an integrable distribution. Hence we get $\gamma_{12}^3 = \gamma_{21}^3 = 0$.

Similarly as [6], we have the following lemmas using lemma 3.4.

Lemma 3.5. *The local function γ_{11}^3 satisfies the following system of differential equations:*

$$E_1(\gamma_{11}^3) = 0, \quad E_2(\gamma_{11}^3) = 0.$$

Lemma 3.6. *Under the hypothesis of Lemma 3.4. Then, on a neighborhood of a given point $p \in M$, M is the warped product of an open interval $(-\epsilon, \epsilon)$ and N^2, N^2 the leaf of the distribution $\mathcal{D}^\perp$ through p.*

From lemma 3.6 we may prove the following lemma.

Lemma 3.7. *Let M be a 3-dimensional minimal CR-submanifold in $S^6(1)$. If M satisfies equality (1.2) and $\mathcal{D}$ is totally real, then M is a totally real submanifold.*

Proof. If M is proper, we obtain from Lemma 3.6 that M is locally a warped product and that the distribution on M determined by the product structure coincide with $\mathcal{D}$ and $\mathcal{D}^\perp$. Moreover, since $h(\mathcal{D}, \mathcal{D}^\perp) = 0$, locally M is immersed as a warped product; furthermore, the first factor is totally geodesic, and therefore we can assume that the first factor of the corresponding warped product decomposition is 1-dimensional. Since the decomposition into a warped product with one-dimensional first factor is unique up to isometries, M is thus immersed as follows [6]:

Let $S^1_+(1) = \{(x_1, x_2) \in \mathbb{R}^2 | \ ||x|| = 1 \text{ and } x_1 > 0\}$ be a half circle, $S^5(1)$ the unit hypersphere of $\mathbb{R}^6$, and N the unit vector orthogonal to the hyperplane containing $S^5(1)$. We parametrize the half circle $S^1_+(1)$ by $(-\frac{\pi}{2}, \frac{\pi}{2}) \to S^1_+(1), \quad t \mapsto (\cos(t), \sin(t))$. Let $f : M^2 \to S^5(1)$ be a minimal immersion of a surface into $S^5(1)$. Then the associated warped product immersion is given by

$$x : (-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos(t)} M^2 \to S^6(1), \quad (t, p) \mapsto \sin(t)N + \cos(t)f(p).$$

Let X be a vector field tangent to M^2. Then

$$x_*(\frac{\partial}{\partial t}) = \cos(t)N - \sin(t)f(p), \quad x_*(X) = \cos(t)f_*(X),$$

and

$$Jx_*(\frac{\partial}{\partial t}) = N \times f(p), \quad Jx_*(X) = \cos(t)\sin(t)N \times f_*(X) + \cos^2(t)Jf_*(X).$$

Since $\mathcal{D} = \text{span}\{\frac{\partial}{\partial t}\}$ is totally real, we have

$$\langle N \times f(p), f_*(X)\rangle = 0. \tag{3.10}$$

Differentiating (3.10) yields

$$\langle N \times f_*(Y), f_*(X)\rangle + \langle N \times p, h(X, Y)\rangle = 0, \tag{3.11}$$

for any vector fields X and Y tangent to M^2. Because the first term in (3.11) is skew symmetric and the second is symmetric, both terms must vanish. Hence we obtain

$$\langle N \times f_*(X), f_*(Y)\rangle = 0$$

which implies $\mathcal{D}^\perp$ is not a complex plane. This is a contradiction. Hence, M must be nonproper.

Lemma 3.8. *Let M be a* 3*-dimensional nonminimal CR-submanifold in* $S^6(1)$ *which satisfies equality* (1.2)*. If $\mathcal{D}$ is totally real, then $H \in J\mathcal{D}$.*

Proof Let $\{E_1, E_2, E_3\}$ be an orthonormal frame field on M such that $\{E_1, E_2\} \in \mathcal{H}$. Without loss of generality we may assume $JE_1 = E_2$. Thus, (3.1) gives

$$\langle A_\xi E_1, E_1\rangle + \langle A_\xi JE_1, JE_1\rangle = \langle A_{J\xi} JE_1, E_1\rangle - \langle A_{J\xi} E_1, JE_1\rangle = 0$$

for every vector field $\xi \in \nu$. Therefore, the mean curvature vector H lies in $J\mathcal{D}$.

From lemma 3.8 we have the following.

Lemma 3.9. *Let M be a* 3*-dimensional nonminimal proper CR-submanifold in* $S^6(1)$ *which satisfies equality* (1.2)*. Then $\mathcal{D}$ is not totally real.*

Proof. If $\mathcal{D}$ is totally real, we have $H \in J\mathcal{D}$ from lemma 3.8. Without loss of generality we may assume JE_3 is parallel to the mean curvature vector. By (3.2), we obtain that JE_3 is a parallel normal vector field. Further, form the equation of Ricci we have $[A_{JE_3}, A_\xi] = 0$ for any $\xi \in \nu$. By lemma 3.2 of [3], this implies that, with respect to a suitable orthonormal frame field with $E_2 = JE_1$, the shape operators either take the form:

$$A_{JE_3} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2a \end{pmatrix}, \quad A_{\xi_i} = \begin{pmatrix} h^i_{11} & h^i_{12} & 0 \\ h^i_{12} & -h^i_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, 2, \tag{3.12}$$

or take the form:

$$A_{JE_3} = \begin{pmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & b+c \end{pmatrix}, \quad A_{\xi_i} = \begin{pmatrix} h^i_{11} & 0 & 0 \\ 0 & -h^i_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i = 1, 2, \tag{3.13}$$

where a, b, c are functions such that $b \neq c$ and $\xi_i \in \nu$.

If the shape operators take the form (3.13), then by combining (3.1) with (3.13), we obtain $A_{\xi_1} = A_{\xi_2} = 0$. Since JE_3 is a parallel normal vector field, this implies that M is contained in a totally geodesic $S^4(1)$. Since the tangent space of this sphere $S^4(1)$ is spanned by $\{E_1, E_2, E_3, JE_3\}$, $S^4(1)$ is an almost complex submanifold in $S^6(1)$ which is a contradiction. Hence the shape operators must take the form (3.12). Therefore, the second fundamental form satisfies

$$\begin{aligned} &h(E_1, E_1) = aJE_3 + \phi\xi, \quad h(E_2, E_2) = aJE_3 - \phi\xi, \quad h(E_3, E_3) = 2aJE_3 \\ &h(E_1, E_3) = 0, \quad h(E_1, E_2) = \phi J\xi, \quad h(E_2, E_3) = 0, \end{aligned} \tag{3.14}$$

for some function ϕ and some $\xi \in \nu$. From (3.14) we find

$$(\nabla h)(E_1, E_2, E_3) = -\gamma^3_{12} aJE_3 - \gamma^1_{13}\phi J\xi + \gamma^2_{13}\phi\xi, \tag{3.15}$$

$$(\nabla h)(E_2, E_1, E_3) = -\gamma^3_{21} aJE_3 - \gamma^2_{23}\phi J\xi - \gamma^1_{23}\phi\xi. \tag{3.16}$$

If we put $W_1 = \{p \in M : a(p) \neq 0\}$ and $W_2 = \{q \in M : \phi(q) \neq 0\}$, then the equation of Codazzi, (3.15) and (3.16) yield

$$\gamma^3_{12} = \gamma^3_{21} = 0 \quad on \quad W_1 \cap W_2$$

which implies that $\mathcal{D}^\perp$ is integrable on $W_1 \cap W_2$. Hence by virtue of [1, page 26], we conclude that $W_1 \cap W_2$ is minimal, which is a contradiction. Hence, $\mathcal{D}$ cannot be totally real.

The main theorem follows from Lemma 3.7 and Lemma 3.9.

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