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DISCRETE OU-IANG TYPE INEQUALITIES

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Abstract. Discrete Ou-Iand type inequalities are established which find applications in stability of difference equations with sublinear perturbations.

In [1], Ou-Iang derived the following result: Let $x,\ f : [0,\infty) \to R$ be nonnegative and continuous functions which satisfy

$$x^2(t) \le C^2 + 2\int_0^t f(s)x(s)ds, \quad C > 0,$$

then

$$x(t) \le C + \int_0^t f(s)ds, \quad t \in [0,\infty).$$

Since such a result is useful when dealing with stability of differential equations, several extensions [2, 3, 4] have then been obtained for continuous functions of the form $x : [0,\infty) \to [0,\infty)$ which satisfy

$$x^p(t) \le C + \int_0^t f(s)x^p(s)ds + \int_0^t g(s)x^q(s)ds, \quad p, q > 0,\ t \in [0,\infty),$$

where $f,\ g : [0,\infty) \to R$ are continuous and nonnegative functions.

It is equally important to consider stability of difference equations. For this reason, we will consider in this note nonnegative sequences of the form $\{x_n\}_{n=\alpha}^\infty$ which satisfy

$$x_n^p \le \gamma_n + \sum_{m=\alpha}^{n-1} f_m x_m^p + \sum_{m=\alpha}^{n-1} g_m x_m^q, \quad n \ge \alpha, \tag{1}$$

where α is an integer, $p, q \ge 0$ and $\{f_n\}_{n=\alpha}^\infty$, $\{g_n\}_{n=\alpha}^\infty$, $\{\gamma_n\}_{n=\alpha}^\infty$ are nonnegative sequences.

We remark that when $n = \alpha$, the two sums on the right hand side of the above inequality are taken to be zero. This practice is in line with the general convention that empty sums are taken to be zero, and empty products to be one. We remark further that when $q = 1$, $\gamma_n = \gamma$, $f_m = f$ and $g_m = g$, the functional inequality (1) has been

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considered by Pang and Agarwal [6]. For other discrete Ou-Iang type inequalities, see [7, 8].

In case $p = 1$, $q \geq 0$ and $\{\gamma_n\}_{n=\alpha}^{\infty}$ is the null sequence, (1) implies

$$\Delta x_n \leq f_n x_n + g_n x_n^q, \quad n \geq \alpha.$$

According to [5, Lemma 15.5], we can then assert that

$$x_n \leq \left(\prod_{i=\alpha}^{n-1}(1+f_i)\right)\left\{x_\alpha^{1-q} + (1-q)\sum_{m=\alpha}^{n-1} g_m \left(\prod_{i=\alpha}^{m}\frac{1}{1+f_i}\right)^{1-q}\right\}^{1/(1-q)} \tag{2}$$

for $n = \alpha, \alpha+1, \dots, \beta$, provided that $q \neq 1$, and

$$x_\alpha^{1-q} + (1-q)\sum_{m=\alpha}^{n-1} g_m \left(\prod_{i=\alpha}^{m}\frac{1}{1+f_i}\right)^{1-q} > 0, \quad n = \alpha, \alpha+1, \dots, \beta.$$

The case where $p = q > 0$ can also be dealt with easily by means of a discrete Gronwall type result of Sugiyama [5, Corollary 15.3]: If $\{u_n\}_{n=\alpha}^{\infty}$ satisfies

$$u_n \leq a_n + \sum_{m=\alpha}^{n-1} b_m u_m, \quad n \geq \alpha,$$

for some nonnegative sequence $\{b_n\}_{n=\alpha}^{\infty}$, then

$$u_n \leq a_n + \sum_{m=\alpha}^{n-1} b_m a_m \prod_{i=m+1}^{n-1}(1+b_i), \quad n \geq \alpha.$$

Indeed, suppose $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative sequence which satisfies

$$x_n^p \leq \gamma_n + \sum_{m=\alpha}^{n-1}(f_m + g_m)x_m^p, \quad n \geq \alpha,$$

where $p > 0$, $\{\gamma_n\}_{n=\alpha}^{\infty}$ is a nonnegative and nondecreasing sequence, and $\{f_n\}_{n=\alpha}^{\infty}$, $\{g_n\}_{n=\alpha}^{\infty}$ are nonnegative sequences. Then according to the result of Sugiyama, we have

$$x_n^p \leq \gamma_n + \sum_{m=\alpha}^{n-1}(f_m + g_m)\gamma_m \prod_{i=m+1}^{n-1}(1+f_i+g_i), \quad n \geq \alpha.$$

Next, we consider the case where $p \neq q$ and $q \geq p - 1 \geq 0$.

Theorem 1. *Let p, q be distinct positive real numbers such that $q \geq p - 1 \geq 0$. Furthermore, let $\{f_n\}_{n=\alpha}^{\infty}$ and $\{g_n\}_{n=\alpha}^{\infty}$ be nonnegative sequences, and $\{\gamma_n\}_{n=0}^{\infty}$ be a* *nonnegative and nondecreasing sequence. If* $\{x_n\}_{n=\alpha}^{\infty}$ *is a nonnegative solution of the functional inequality* (1)*, then*

$$x_j \leq \left\{ \prod_{m=\alpha}^{j-1} \frac{p+f_m}{p} \right\} \left\{ \gamma_j^{(p-q)/p} + \frac{p-q}{p} \sum_{m=\alpha}^{j-1} g_m \left(\prod_{i=\alpha}^{m} \frac{p}{p+f_i} \right)^{p-q} \right\}^{1/(p-q)}$$

for each $j \geq \alpha$*, provided that*

$$\gamma_j^{(p-q)/p} + \frac{p-q}{p} \sum_{m=\alpha}^{n-1} g_m \left(\prod_{i=\alpha}^{m} \frac{p}{p+f_n} \right)^{p-q} > 0, \quad n = \alpha, \alpha+1, \dots, j.$$

Proof. Let j be an arbitrary integer greater than or equal to α. Let $\{z_n\}_{n=\alpha}^{j}$ be defined by

$$z_n = \gamma_j + \sum_{m=\alpha}^{n-1} f_m x_m^p + \sum_{m=\alpha}^{n-1} g_m x_m^q, \quad n = \alpha, \alpha+1, \dots, j,$$

then it is easily seen (see e.g. [5, Lemma 15.1]) that $x_\alpha^p = \gamma_\alpha \leq \gamma_j = z_\alpha$ and from (1), we have

$$x_n^p \leq \gamma_j + \sum_{m=\alpha}^{n-1} f_m x_m^p + \sum_{m=\alpha}^{n-1} g_m x_m^q = z_n,$$

or, $x_n \leq z_n^{1/p}$ for $n = \alpha, \alpha+1, \dots, j$. Since

$$0 \leq \Delta z_n = f_n x_n^p + g_n x_n^q \leq f_n z_n + g_n z_n^{q/p}, \quad n = \alpha, \alpha+1, \dots, j-1,$$

by means of the mean value theorem, we see that

$$\begin{aligned} z_{n+1}^{1/p} - z_n^{1/p} &\leq \frac{1}{p} z_n^{1/p-1} \Delta z_n \\ &\leq \frac{1}{p} z_n^{1/p-1} (f_n z_n + g_n z_n^{q/p}) \\ &= \frac{f_n}{p} z_n^{1/p} + \frac{g_n}{p} (z_n^{1/p})^{1-p+q} \end{aligned}$$

for $n = \alpha, \alpha = 1, \dots, j-1$. In view of (2) and the fact that $0 \leq 1-p+q \neq 1$, we then obtain

$$\begin{aligned} z_n^{1/p} &\leq \left(\prod_{i=\alpha}^{n-1} \frac{p+f_i}{p} \right) \left\{ z_\alpha^{(p-q)/p} + (p-q) \sum_{m=\alpha}^{n-1} \frac{g_m}{p} \left(\prod_{i=\alpha}^{m} \frac{p}{p+f_i} \right)^{p-q} \right\}^{1/(p-q)} \\ &= \left(\prod_{i=\alpha}^{n-1} \frac{p+f_i}{p} \right) \left\{ \gamma_j^{(p-q)/p} + \frac{p-q}{p} \sum_{m=\alpha}^{n-1} g_m \left(\prod_{i=\alpha}^{m} \frac{p}{p+f_i} \right)^{p-q} \right\}^{1/(p-q)} \end{aligned}$$

for $n = \alpha, \alpha+1, \ldots, j$, provided that

$$\gamma_j^{(p-q)/p} + \frac{p-q}{p} \sum_{m=\alpha}^{n-1} g_m \left(\prod_{i=\alpha}^{m} \frac{p}{p+f_i} \right)^{p-q} > 0, \quad n = \alpha, \alpha+1, \ldots, j. \tag{3}$$

Thus,

$$x_j \le z_j^{1/p} \le \left(\prod_{i=\alpha}^{j-1} \frac{p+f_i}{p} \right) \left\{ \gamma_j^{(p-q)/p} + \frac{p-q}{p} \sum_{m=\alpha}^{j-1} g_m \left(\prod_{i=\alpha}^{m} \frac{p}{p+f_i} \right)^{p-q} \right\}^{1/(p-q)}$$

for each $j \ge \alpha$, provided that (3) holds. The proof is complete.

As an example, let $p \ge 1$, let $\{\gamma_n\}_{n=\alpha}^{\infty}$ be a nonnegative and nondecreasing sequence, and let $\{f_n\}_{n=\alpha}^{\infty}$, $\{g_n\}_{n=\alpha}^{\infty}$ be nonnegative sequences. If the real sequence $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of the functional inequality

$$x_n^p \le \gamma_n + p \sum_{m=\alpha}^{n-1} f_m x_m^p + p \sum_{m=\alpha}^{n-1} g_m x_m^{p-1}, \quad n \ge \alpha,$$

then in view of Theorem 1, we have

$$x_j \le \left(\prod_{i=\alpha}^{j-1} (1+f_i) \right) \left\{ \gamma_j^{1/p} + \sum_{m=\alpha}^{j-1} g_m \left(\prod_{i=\alpha}^{m} \frac{1}{1+f_i} \right) \right\}, \quad j \ge \alpha,$$

provided that

$$\gamma_j^{1/p} + \sum_{m=\alpha}^{n-1} g_m \left(\prod_{i=\alpha}^{m} \frac{1}{1+f_i} \right) > 0, \quad n = \alpha, \alpha+1, \ldots, j, \tag{4}$$

for $j \ge \alpha$. Note that if $\gamma_\alpha > 0$, then $\gamma_j \ge \gamma_\alpha > 0$ for $j \ge \alpha$, and hence (4) will be satisfied automatically.

As another example, let $0 < q < 1$, let $\gamma > 0$, and let $\{f_n\}_{n=\alpha}^{\infty}$, $\{g_n\}_{n=\alpha}^{\infty}$ be nonnegative sequences. If the real sequence $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of the functional inequality

$$x_n \le \gamma + \sum_{m=\alpha}^{n-1} f_m x_m + \sum_{m=\alpha}^{n-1} g_m x_m^q, \quad n \ge \alpha, \tag{5}$$

then in view of Theorem 1, we have [5, Theorem 15.6]

$$x_j \le \left(\prod_{i=\alpha}^{j-1} (1+f_i) \right) \left\{ \gamma^{1-q} + (1-q) \sum_{m=\alpha}^{j-1} g_m \left(\prod_{i=\alpha}^{m} \frac{1}{1+f_i} \right)^{1-q} \right\}^{1/(1-q)}, \quad j \ge \alpha,$$

since

$$\gamma^{1-q} + (1-q) \sum_{m=\alpha}^{n-1} g_m \left(\prod_{i=\alpha}^{m} \frac{1}{1+f_i} \right)^{1-q} > 0, \quad n \ge \alpha.$$

As a final example, consider the perturbed difference equation

$$\Delta x_n = h_n x_n + G(n, x_n), \quad n = 0, 1, 2, \ldots,$$

where $\{h_n\}_{n=0}^{\infty}$ is a real sequence,

$$|G(n,t)| \le g_n |t|^q, \quad n = 0, 1, 2, \ldots; \; 0 < q < 1,$$

and $\{g_n\}$ is a nonnegative sequence. Let $\{x_n\}_{n=0}^{\infty}$ be the unique solution of this equation which satisfies the initial condition $x_0 = \gamma \ne 0$. Then

$$|x_n| \le |\gamma| + \sum_{m=0}^{n-1} |h_m||x_m| + \sum_{m=0}^{n-1} g_n |x_m|^q, \quad n = 0, 1, 2, \ldots.$$

This functional inequality has the same form as (5), and hence

$$|x_n| \le \left(\prod_{i=0}^{n-1}(1+|h_i|)\right)\left\{|\gamma|^{1-q} + (1-q)\sum_{m=0}^{n-1} g_m \left(\prod_{i=0}^{m}\frac{1}{1+|h_i|}\right)^{1-q}\right\}^{1/(1-q)}$$

for $n \ge 0$.

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