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Strongly λ-statistically and strongly Valle-Poussin pre-Cauchy sequences in probabilistic metric spaces

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Abstract. We introduce the notions of strongly λ-statistically pre-Cauchy and strongly Valle-Poussin pre-Cauchy sequences in probabilistic metric spaces endowed with strong topology. And we show that these two new notions are equivalent. Strongly λ-statistically convergent sequences are strongly λ-statistically pre-Cauchy sequences, and we give an example to show that there is a sequence in a probabilistic metric space which is strongly λ-statistically pre-Cauchy but not strongly λ-statistically convergent.

Keywords. Probabilistic metric space, strongly λ-statistical convergence, strongly λ-statistical pre-Cauchy, strongly statistical pre-Cauchy, strongly Valle-Poussin pre-Cauchy

1 Introduction:

Throughout the article, λ will always denote a non-decreasing sequence of positive real numbers tending to ∞ such that $\lambda_1 = 1$, $\lambda_{n+1} \leq \lambda_n + 1$, $n \in \mathbb{N}$. And we write Δ_∞ to denote all such sequences λ.

In 1951, in order to study sequences of real numbers similar in some sense to the convergent sequences of real numbers, Fast [4] and Schoenberg [10] independently introduced the notion of statistical convergence of sequences of real numbers. Afterward, in [1], in order to provide a Cauchy-like criterion for the statistical convergent sequences of real numbers, Connor et al. introduced the notion of pre-Cauchy sequences of real numbers and established a relationship between the notions of statistical convergence and pre-Cauchy sequences of real numbers. In 2000, Mursaleen [8] introduced the notion of λ-statistical convergence of sequences of real numbers and established its relationship with the notion of statistical convergence of sequences of real numbers. For a survey on this direction, see [2, 3, 5] and many others.

On the other hand, the concept of Probabilistic metric spaces was introduced by Menger [7] under the name of statistical metric spaces. In this theory, the distance between two points u, v is a distribution function F_{uv}. And the value of the function F_{uv} at any $t > 0$, that is, $F_{uv}(t)$ can be interpreted as the probability that the distance between u and v is less than t. After Menger, the theory of probabilistic metric spaces was studied and developed by Schwiezer and

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Sklar [12, 13, 14], Tardiff [15], Thorp [16] and many others. For a survey on this direction, see the book written by Schwiezer and Sklar [14].

One can define several topologies on this space, however, strong topology is the only one which is getting more attention over the years. In [11], Şençimen et al. remarked that the strong topology is first countable and Hausdorff, it can be characterized in terms of strong convergence of sequences. Moreover, to provide a more general framework for applications, Şençimen et al. [11] extended the notion of strong convergence of sequences to the notion of strong statistical convergence of sequences. Recently, Malik and Das [6] studied the notion of strong λ-statistical convergence of sequences which is a generalization of the notion of strong statistical convergence of sequences.

In point of view of recent applications of probabilistic metric spaces, in this article, we provide a Cauchy-like criterion of strong statistical and strong λ-statistical convergence of sequences in probabilistic metric spaces. We introduce the notion of strongly λ-statistically pre-Cauchy sequences in probabilistic metric spaces. Strongly λ-statistically convergent sequences are strongly λ-statistically pre-Cauchy sequences, and we give an example to show that there is a sequence in a probabilistic metric space which is strongly λ-statistically pre-Cauchy but not strongly λ-statistically convergent. Therefore, in the realm of PM spaces, our notion resolved the question: can one decide if a sequence is strongly λ-statistically convergent without knowing the strong λ-statistical limit of the sequence in advance? Further, this notion provides a tool for the applications of the strong topology on the PM spaces. Also, we introduce the notion of strongly Valle-Poussin pre-Cauchy sequences to understand the notion of strongly λ-statistically pre-Cauchy sequences in probabilistic metric spaces differently. And these two new notions are equivalent.

2 Preliminaries

First, we familiarize reader with the basic concepts of statistical convergence, statistical pre-Cauchy and λ-statistical convergence of sequences of real numbers.

The notion of asymptotic density of the subsets of the set of all natural numbers $\mathbb{N}$ plays a central role in the concept of statistical convergence of sequences. We recall that a set $A \subset \mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{n\to\infty} \frac{|A(n)|}{n},$$

where for all $n \in \mathbb{N}$, $A(n) = \{k \in A : k \leq n\}$.

Definition 1. [5] A sequence $\{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $l \in \mathbb{R}$ if, for every $\varepsilon > 0$, $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$.

Definition 2. [1] A sequence $\{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be statistically pre-Cauchy if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n^2} |\{(j,k) \in [1,n] \times [1,n] : |x_j - x_k| \geq \varepsilon\}| = 0.$$

From now on rest of the article, for each $n \in \mathbb{N}$, we write $I_n = [n - \lambda_n + 1, n]$, where $\lambda = \{\lambda_n\} \in \Delta_\infty$.

Definition 3. [8] A sequence $\{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be λ-statistically convergent to $l \in \mathbb{R}$ if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - l| \geq \varepsilon\}| = 0.$$

In 2000, Mursaleen [8] defined the notion of the generalized de la Valle-Poussin mean for a sequence $x = \{x_k\}$ by the the formula $\frac{1}{\lambda_n} \sum\limits_{k\in I_n} x_k$. Note that whenever $\lambda_n = n$ we have the notion of Cesro mean for the sequence.

Definition 4. [8] A sequence $\{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be strongly (V, λ)-summable to $l \in \mathbb{R}$ if,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k\in I_n} |x_k - l| = 0.$$

Theorem 2.1. *[8] For bounded sequences of real numbers, the notions of λ-statistical convergence and strong (V, λ)-summability are equivalent.*

Now we recall some basic ideas related to the probabilistic metric spaces (briefly PM spaces) (see [12, 13, 14] and many others).

Definition 5. A non decreasing function $f : [-\infty, \infty] \to [0, 1]$ with $f(-\infty) = 0$ and $f(\infty) = 1$ is called a distribution function.

We denote the set of all left continuous distribution functions over $(-\infty, \infty)$ by $\mathcal{D}$.

We consider a partial relation $\leq$ on $\mathcal{D}$ defined by $g \leq f$ if and only if $g(x) \leq f(x)$ for all $x \in [-\infty, \infty]$.

Definition 6. For any $q \in [-\infty, \infty]$ the unit step at q is denoted by ϵ_q and is defined to be a function in $\mathcal{D}$ given by

$$\begin{aligned} \epsilon_q(t) &= 0, \quad -\infty \leq t \leq q \\ &= 1, \quad q < t \leq \infty. \end{aligned}$$

In particular,

$$\begin{aligned} \epsilon_0(t) &= 0, \quad -\infty \leq t \leq 0 \\ &= 1, \quad 0 < t \leq \infty. \end{aligned}$$

Definition 7. A sequence $\{f_n\}_{n\in\mathbb{N}}$ of distribution functions is said to converge weakly to a distribution function f, if the sequence $\{f_n(t)\}_{n\in\mathbb{N}}$ converges to $f(t)$ at each continuity point t of f. In this case, we write $f_n \xrightarrow{w} f$.

Definition 8. The distance between f and g in $\mathcal{D}$ is denoted by $d_L(f, g)$ and is defined to be the infimum of all numbers $h \in (0, 1]$ such that the inequalities

$$f(t-h) - h \leq g(t) \leq f(t+h) + h$$
$$\text{and } g(t-h) - h \leq f(t) \leq g(t+h) + h$$

hold for all $t \in (-\frac{1}{h}, \frac{1}{h})$.

It is known from the literature that d_L is a metric on $\mathcal{D}$ and for any sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ and $f \in \mathcal{D}$, we have

$$f_n \xrightarrow{w} f \text{ if and only if } d_L(f_n, f) \to 0.$$

Definition 9. A non-decreasing real valued function h defined on $[0,\infty]$ that satisfies $h(0) = 0$ and $h(\infty) = 1$ and is left continuous on $(0,\infty)$ is called a distance distribution function (d.d.f. in short).

The set of all distance distribution functions is denoted by $\mathcal{D}^+$. Moreover, the function d_L is a metric on $\mathcal{D}^+$.

Theorem 2.2. *[14] Let $f \in \mathcal{D}^+$ be given. Then for any $t > 0$, $f(t) > 1 - t$ if and only if $d_L(f, \epsilon_0) < t$.*

Definition 10. A triangle function is a binary operation τ on $\mathcal{D}^+$, which is commutative, non-decreasing, associative in each place and ϵ_0 is the identity.

Definition 11. A probabilistic metric space, briefly PM space, is a triplet (P, F, τ) where P is a nonempty set whose elements are the points of the space; F is a function from $P \times P$ into $\mathcal{D}^+$, τ is a triangle function, and the following conditions are satisfied for all $x, y, z \in P$:

1. $F(x, x) = \epsilon_0$
2. $F(x, y) \neq \epsilon_0$ if $x \neq y$
3. $F(x, y) = F(y, x)$
4. $F(x, z) \geq \tau(F(x, y), F(y, z))$.

From now on we denote $F(x, y)$ by F_{xy} and its value at t by $F_{xy}(t)$.

Example 1. Let $f \in \mathcal{D}^+$ be fixed and distinct from ϵ_0 and ϵ_∞. Then we have the equilateral PM space (P, F, M) where F is defined by

$$F_{pq} = \begin{cases} f & ; \quad \text{if } p \neq q \\ \epsilon_0 & ; \quad \text{if } p = q \end{cases}$$

and M is the maximal triangle function.

Definition 12. Let (P, F, τ) be a PM space. For $x \in P$ and $r > 0$, the strong r-neighborhood of x is denoted by $\mathcal{N}_x(r)$ and is defined by

$$\mathcal{N}_x(r) = \{y \in P : F_{xy}(r) > 1 - r\}.$$

In this case, the collection $\mathfrak{N}_x = \{\mathcal{N}_x(r) : r > 0\}$ is said to be the strong neighborhood system at x and the union $\mathfrak{N} = \bigcup_{x\in P} \mathfrak{N}_x$ is said to be the strong neighborhood system for P.

From the Theorem 2.2, we have $\mathcal{N}_x(r) = \{y \in P : d_L(F_{xy}, \epsilon_0) < r\}$. If τ is continuous, then the strong neighborhood system $\mathfrak{N}$ determines a Hausdorff topology for P. This topology is said to be the strong topology for P.

Definition 13. Let (P, F, τ) be a PM space. Then for any $r > 0$, the subset $\mathfrak{V}(r)$ of $P \times P$ given by

$$\mathfrak{V}(r) = \{(x, y) : F_{xy}(r) > 1 - r\}$$

is said to be the strong r-vicinity.

Theorem 2.3. *[14] Let (P, F, τ) be a PM space and τ be continuous. Then for any $r > 0$, there is an $s > 0$ such that $\mathfrak{V}(s) \circ \mathfrak{V}(s) \subset \mathfrak{V}(r)$, where $\mathfrak{V}(s) \circ \mathfrak{V}(s) = \{(x, z) :$ for some y, (x, y) and $(y, z) \in \mathfrak{V}(s)\}$.*

Note 1. By the virtue of the hypothesis of the Theorem 2.3, we can say, for any $r > 0$, there is an $s > 0$ such that $F_{ab}(r) > 1 - r$ whenever $F_{ac}(s) > 1 - s$ and $F_{cb}(s) > 1 - s$. Equivalently, we can write for any $r > 0$, there is an $s > 0$ such that $d_L(F_{ab}, \epsilon_0) < r$ whenever $d_L(F_{ac}, \epsilon_0) < s$ and $d_L(F_{cb}, \epsilon_0) < s$.

From now on throughout this work, P will always denote the PM space (P, F, τ) endowed with the strong topology.

Definition 14. [11] Let (P, F, τ) be a PM space. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in P is said to be strongly convergent to $l \in P$ if, for every $t > 0$, $\exists$ a natural number k_0 such that

$$x_k \in \mathcal{N}_l(t) \qquad \text{whenever } k \geq k_0.$$

In this case, we write $F\text{-}\lim_{k\to\infty} x_k = l$ or $x_k \xrightarrow{F} l$.

Definition 15. [11] Let (P, F, τ) be a PM space. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in P is said to be strongly statistically convergent to $l \in P$ if, for any $t > 0$

$$d(\{k \in \mathbb{N} : F_{x_k l}(t) \leq 1 - t\}) = 0 \qquad \text{or} \qquad d(\{k \in \mathbb{N} : x_k \notin \mathcal{N}_l(t)\}) = 0.$$

In this case, we write $st^F\text{-}\lim_{k\to\infty} x_k = l$.

Definition 16. [6] A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in a PM space (P, F, τ) is said to be strongly λ-statistically convergent to $l \in P$ if, for every $t > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : F_{x_k l}(t) \leq 1 - t\}| = 0.$$

or

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : x_k \notin \mathcal{N}_l(t)\}| = 0.$$

In this case, we write $st_\lambda^F\text{-}\lim_{k\to\infty} x_k = l$ or simply as $x_k \xrightarrow{st_\lambda^F} l$.

3 Strongly λ-statistical pre-Cauchy

In this section, we introduce the notion of strongly λ-statistically pre-Cauchy and establish a Cauchy like criterion for a strong λ-statistically convergent sequence in probabilistic metric spaces.

Definition 17. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in a PM space (P, F, τ) is said to be strongly λ-statistically pre-Cauchy if for every $t > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \left|\{(j, k) \in I_n \times I_n : F_{x_j x_k}(t) \leq 1 - t\}\right| = 0.$$

Note 2. In the above Definition 17, if we replace λ_n by n for each $n \in \mathbb{N}$, then we have the notion of strong statistically pre-Cauchy.

Theorem 3.1. *If* $\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0$*, then the notion of strongly statistically pre-Cauchy implies the notion of strongly* λ*-statistically pre-Cauchy.*

Proof. Let $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in a PM space (P, F, τ) and x is strongly statistically pre-Cauchy. Let $t > 0$ be given. Then for $t > 0$, we have

$$\begin{aligned} & \frac{1}{n^2}|\{(j,k) \in [1,n] \times [1,n] : F_{x_j x_k}(t) \leq 1 - t\}| \\ \geq \ & \frac{1}{n^2}|\{(j,k) \in I_n \times I_n : F_{x_j x_k}(t) \leq 1 - t\}| \\ = \ & \frac{\lambda_n^2}{n^2}\frac{1}{\lambda_n^2}|\{(j,k) \in I_n \times I_n : F_{x_j x_k}(t) \leq 1 - t\}|. \end{aligned}$$

Now $\{\lambda_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of positive numbers such that $\lambda_1 = 1$, $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_n \leq n$ for all n and hence the quotient $\frac{\lambda_n}{n}$ is nonnegative and bounded above by 1. So we have $0 < \liminf_{n\to\infty} \frac{\lambda_n^2}{n^2} \leq 1$ and since $\lim_{n\to\infty} \frac{1}{n^2}|\{(j,k) \in [1,n] \times [1,n] : F_{x_j x_k}(t) \leq 1 - t\}| = 0$, so

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2}|\{(j,k) \in I_n \times I_n : F_{x_j x_k}(t) \leq 1 - t\}| = 0.$$

Hence x is strongly λ-statistically pre-Cauchy. □

Theorem 3.2. *Let* (P, F, τ) *be a PM space. If a sequence* $x = \{x_k\}_{k\in\mathbb{N}}$ *is strongly* λ*-statistically convergent, then it is strongly* λ*-statistically pre-Cauchy in* P*.*

Proof. Let the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ be strongly λ-statistically convergent to $l \in P$. Let $t > 0$ be given. Then there exists $s > 0$ such that $d_L(F_{uw}, \epsilon_0) < t$ whenever $d_L(F_{uv}, \epsilon_0) < s$, $d_L(F_{vw}, \epsilon_0) < s$ and $u, v, w \in P$. Let

$$E_n = \{k \in I_n : F_{x_k l}(s) \leq 1 - s\}.$$

Then $\lim_{n\to\infty} \frac{|E_n|}{\lambda_n} = 0$. Now let $F_n = I_n \setminus E_n$. Then $\lim_{n\to\infty} \frac{|F_n|}{\lambda_n} = 1$. Then for $j, k \in F_n$, we have $F_{x_j l}(s) > 1 - s$ and $F_{x_k l}(s) > 1 - s$, that is, $d_L(F_{x_j l}, \epsilon_0) < s$ and $d_L(F_{x_k l}, \epsilon_0) < s$. Thus for $j, k \in F_n$, we have $d_L(F_{x_j x_k}, \epsilon_0) < t$, that is, $F_{x_j x_k}(t) > 1 - t$. Hence

$$F_n \times F_n \subset \{(j,k) \in I_n \times I_n : F_{x_j x_k}(t) > 1 - t\}.$$

This implies

$$\left[\frac{|F_n|}{\lambda_n}\right]^2 \leq \frac{1}{\lambda_n^2}\left|\{(j,k) \in I_n \times I_n : F_{x_j x_k}(t) > 1 - t\}\right|.$$

Since $\lim_{n\to\infty} \frac{|F_n|}{\lambda_n} = 1$, so

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2}\left|\{(j,k) \in I_n \times I_n : F_{x_j x_k}(t) > 1 - t\}\right| = 1.$$

Thus

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2}\left|\{(j,k) \in I_n \times I_n : F_{x_j x_k}(t) \leq 1 - t\}\right| = 0.$$

Hence the sequence x is strongly λ-statistically pre-Cauchy in P. □

Now we state a necessary and sufficient condition for a sequence to be strongly λ-statistically pre-Cauchy in a probabilistic metric space.

Theorem 3.3. *Let (P, F, τ) be a PM space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in P. Then the sequence x is strongly λ-statistically pre-Cauchy if and only if*

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \sum_{j,k\in I_n} d_L(F_{x_jx_k}, \epsilon_0) = 0.$$

Proof. At first, we assume that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \sum_{j,k\in I_n} d_L(F_{x_jx_k}, \epsilon_0) = 0.$$

Let $t > 0$ be given. Since $\lambda_n > 0$ for all $n \in \mathbb{N}$ and $t > 0$, so we have

$$\begin{aligned} &\frac{1}{\lambda_n^2} \sum_{j,k\in I_n} d_L(F_{x_jx_k}, \epsilon_0) \\ \geq \quad & t(\frac{1}{\lambda_n^2} \left|\left\{(j,k) \in I_n \times I_n : d_L(F_{x_jx_k}, \epsilon_0) \geq t\right\}\right|) \geq 0. \end{aligned}$$

Thus from the squeeze lemma for limits, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \left|\left\{(j,k) \in I_n \times I_n : d_L(F_{x_jx_k}, \epsilon_0) \geq t\right\}\right| = 0.$$

In other words,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \left|\left\{(j,k) \in I_n \times I_n : F_{x_jx_k}(t) \leq 1 - t\right\}\right| = 0.$$

Hence the sequence x is strongly λ-statistically pre-Cauchy in P.

Conversely, we assume that x is strong λ-statistically pre-Cauchy sequence in P. Let $t > 0$ be given. Choose $s > 0$ and $t_0 > 0$ so that $\frac{s}{2} + t_0 < t$. Since $d_L(F_{pq}, \epsilon_0) \leq 1$ for every $p, q \in P$ and $\lambda_n > 0$ for $n \in \mathbb{N}$, so we have

$$\begin{aligned} &\frac{1}{\lambda_n^2} \sum_{j,k\in I_n} d_L(F_{x_jx_k}, \epsilon_0) \\ = \quad & \frac{1}{\lambda_n^2} \sum_{\substack{d_L(F_{x_jx_k}, \epsilon_0) < \frac{s}{2} \\ j,k\in I_n}} d_L(F_{x_jx_k}, \epsilon_0) + \frac{1}{\lambda_n^2} \sum_{\substack{d_L(F_{x_jx_k}, \epsilon_0) \geq \frac{s}{2} \\ j,k\in I_n}} d_L(F_{x_jx_k}, \epsilon_0) \\ \leq \quad & \frac{s}{2} + \frac{1}{\lambda_n^2} \left|\left\{(j,k) \in I_n \times I_n : d_L(F_{x_jx_k}, \epsilon_0) \geq \frac{s}{2}\right\}\right|. \end{aligned}$$

Now, since x is strongly λ-statistically pre-Cauchy, so there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\frac{1}{\lambda_n^2} \left|\left\{(j,k) \in I_n \times I_n : d_L(F_{x_jx_k}, \epsilon_0) \geq \frac{s}{2}\right\}\right| < t_0.$$

Thus for all $n \geq n_0$, we have

$$\frac{1}{\lambda_n^2} \sum_{j,k\in I_n} d_L(F_{x_jx_k}, \epsilon_0) < \frac{s}{2} + t_0 < t.$$

Hence
$$\lim_{n\to\infty}\frac{1}{\lambda_n^2}\sum_{j,k\in I_n} d_L(F_{x_jx_k},\epsilon_0)=0.$$

□

4 Strongly Valle-Poussin pre-Cauchy

Now we treat the condition of the Theorem 3.3 as a new definition of pre-Cauchy and named it strongly Valle-Poussin pre-Cauchy. In particular, when $\lambda_n = n$ we call it strongly Cesro pre-Cauchy.

Definition 18. A sequence $x=\{x_k\}_{k\in\mathbb{N}}$ in a PM space (P,F,τ) is said to be strongly Valle-Poussin pre-Cauchy if, for every $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$,
$$\frac{1}{\lambda_n^2}\sum_{j,k\in I_n} d_L(F_{x_jx_k},\epsilon_0)<\varepsilon.$$

Theorem 4.1. *Let (P,F,τ) be a PM space and $x=\{x_k\}_{k\in\mathbb{N}}$ be a sequence in P. Then the sequence x is strongly λ-statistically pre-Cauchy if and only if x is strongly Valle-Poussin pre-Cauchy.*

Proof. Directly follows from the Theorem 3.3. □

Theorem 4.2. *If $\lim\limits_{n\to\infty}\frac{\lambda_n}{n}=1$, then the notion of strong Valle-Poussin pre-Cauchy implies the notion of strong Cesro pre-Cauchy.*

Proof. Let $x=\{x_k\}_{k\in\mathbb{N}}$ be a sequence in a PM space (P,F,τ) and x be strongly Valle-Poussin pre-Cauchy. Let $\varepsilon>0$ be given. Then there exists $n_0\in\mathbb{N}$ such that for all $n\geq n_0$, we have
$$\frac{1}{\lambda_n^2}\sum_{j,k\in I_n} d_L(F_{x_jx_k},\epsilon_0)<\frac{\varepsilon}{2}$$

and
$$|\frac{\lambda_n}{n}-1|<\sqrt{\frac{\varepsilon}{2}}.$$

Also, we have $d_L(F_{pq},\epsilon_0)\leq 1$ for every $p,q\in P$ and $\lambda_n\leq n$ for all $n\in\mathbb{N}$. Thus for all $n\geq n_0$, we have

$$\begin{aligned}\frac{1}{n^2}\sum_{j,k\in[1,n]} d_L(F_{x_jx_k},\epsilon_0) &= \frac{1}{n^2}\sum_{j,k\in[1,n-\lambda_n]} d_L(F_{x_jx_k},\epsilon_0)+\frac{1}{n^2}\sum_{j,k\in I_n} d_L(F_{x_jx_k},\epsilon_0)\\ &\leq \frac{(n-\lambda_n)^2}{n^2}+\frac{\lambda_n^2}{n^2}\frac{1}{\lambda_n^2}\sum_{j,k\in I_n} d_L(F_{x_jx_k},\epsilon_0)\\ &< \frac{\varepsilon}{2}+\frac{1}{\lambda_n^2}\sum_{j,k\in I_n} d_L(F_{x_jx_k},\epsilon_0)\end{aligned}$$

$$\begin{aligned} &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Hence x is strongly Cesro pre-Cauchy. □

Corollary 4.3. *If $\lim_{n\to\infty} \frac{\lambda_n}{n} = 1$, then the notion of strong λ-statistically pre-Cauchy implies the notion of strong statistically pre-Cauchy.*

Proof. Directly follows from the Theorem 3.3 and the Theorem 4.2. □

Lemma 4.1. *[6] Let (P, F, τ) be a PM space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in P. If the sequence x is strongly λ-statistical convergent in P, then it has a subsequence $\{x_{q_n}\}_{n\in\mathbb{N}}$ which is strongly convergent to the same limit.*

In the very next theorem, we state a sufficient condition for a strongly λ-statistically pre-Cauchy sequence to be strongly λ-statistically convergent in a PM space (P, F, τ).

Theorem 4.4. *Let (P, F, τ) be a PM space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a strongly λ-statistically pre-Cauchy sequence in P. If the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ has a subsequence $\{x_{q_k}\}_{k\in\mathbb{N}}$ which is strongly convergent to $l \in P$ and the following condition holds*

$$0 < \liminf_{n\to\infty} \frac{1}{\lambda_n} |\{q_k \in I_n : k \in \mathbb{N}\}| < \infty,$$

then x is strongly λ-statistically convergent to l.

Proof. Let $t > 0$ be given. Then there exists $s > 0$ such that for all $u, v, w \in P$, we have

$$d_L(F_{uw}, \epsilon_0) < t \quad \textit{whenever} \quad d_L(F_{uv}, \epsilon_0) < s \quad \textit{and} \quad d_L(F_{vw}, \epsilon_0) < s. \tag{4.1}$$

Since the subsequence $\{x_{q_k}\}_{k\in\mathbb{N}}$ is strongly convergent to l, so there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, we have $d_L(F_{x_{q_k} l}, \epsilon_0) < s$. Let $Q = \{q_k : k \geq k_0; k \in \mathbb{N}\}$ and $P(t) = \{k : d_L(F_{x_k l}, \epsilon_0) \geq t\}$. Then from (1) we have,

$$\begin{aligned} &\frac{1}{\lambda_n^2} \left|\{(j,k) \in I_n \times I_n : d_L(F_{x_j x_k}, \epsilon_0) \geq s\}\right| \\ \geq\ &\frac{1}{\lambda_n^2} \sum_{j,k\in I_n} \chi_{Q\times P(t)}(j,k) \\ =\ &\frac{1}{\lambda_n} |\{q_j \in Q : q_j \in I_n\}| \times \frac{1}{\lambda_n} |\{k \in P(t) : k \in I_n\}|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\lambda_n^2} \left|\{(j,k) \in I_n \times I_n : d_L(F_{x_j x_k}, \epsilon_0) \geq s\}\right| \\ \geq\ &\lim_{n\to\infty} \frac{1}{\lambda_n} |\{q_j \in Q : q_j \in I_n\}| \times \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in P(t) : k \in I_n\}|. \end{aligned}$$

Again since x is strongly λ-statistically pre-Cauchy, so we have for $s > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^2} \left|\{(j,k) \in I_n \times I_n : d_L(F_{x_j x_k}, \epsilon_0) \geq s\}\right| = 0.$$

Thus

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{q_j\in Q: q_j\in I_n\}\right|\times\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{k\in P(t): k\in I_n\}\right|=0.$$

Now by the given hypothesis, we have $0<\liminf\limits_{n\to\infty}\frac{1}{\lambda_n}\left|\{k\in I_n: k\in\mathbb{N}\}\right|<\infty$.

Therefore,

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{k\in B(t): k\in I_n\}\right|=0.$$

Thus

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{k\in I_n: d_L(F_{x_k l},\epsilon_0)\geq t\}\right|=0.$$

Hence x is strongly λ-statistically convergent to l. □

Note 3. The above result suggests us that there must exist a sequence in a probabilistic metric space which is strongly λ-statistically pre-Cauchy, however, the sequence is not strongly λ-statistically convergent.

Example 2. Let $P=\mathbb{R}$ with the usual metric d and $K(x)=1-e^{-x}$. Then $K\in\mathcal{D}^+$. Define a function $F: P\times P\to\mathcal{D}^+$ by

$$F(p,q)(t)=F_{pq}(t)=K(\tfrac{t}{d(p,q)})=1-e^{\frac{-t}{|p-q|}}\text{ for all } p,q\in P \text{ and } t>0.$$

Also, we make the convention that $K(x/0)=K(\infty)=1$ for $x>0$ and $K(0/0)=K(0)=0$. Then (P,F,τ) becomes a PM space, where τ is the continuous triangle function. Let $\lambda_n=n-3$ for $n>3$ and $\lambda_n=1$ for $1\leq n\leq 3$. Now define a sequence x in P in the following way. For $n,k\in\mathbb{N}$ such that $(n-1)!<k\leq n!$ we define

$$x_k=\sum_{v=1}^{n}\tfrac{1}{v},\text{ and let } x=\{x_k\}_{k\in\mathbb{N}}.$$

Clearly, x has no strongly convergent subsequence by the construction of the sequence. Consequently, by the Lemma 4.1 x is not strongly λ-statistically convergent. However, we show that x is strongly λ-statistically pre-Cauchy. Let $t_1>0$ be given. At first, we define $K_n(t)=1-e^{\frac{-tn}{2}}$ for $t>0$. Clearly, $K_n(t)$ is a distance distribution function and for $t>0$, $K_n(t)$ weakly converges to ϵ_0. Then for that $t_1>0$ there exists an positive integer n_0 such that for all $n\geq n_0$ we have $d_L(K_n(t),\epsilon_0)<t_1$. Choose $n>n_0$ and $n\geq 4$. Then if $n!<n_1\leq(n+1)!$ and $(n-1)!<j,k\leq n_1$ then we have, $|x_j-x_k|<\frac{2}{n}$. It follows that for $t_1>0$ and $n!<n_1\leq(n+1)!$, we have

$$\begin{aligned}&\frac{1}{n_1^2}\left|\left\{(j,k)\in[4,n_1]\times[4,n_1]: d_L(F_{x_jx_k},\epsilon_0)<t_1; j,k\leq n_1\right\}\right|\\ \geq\ &\frac{1}{n_1^2}[n_1-(n-1)!]^2\\ \geq\ &[1-\frac{1}{n}]^2.\end{aligned}$$

Since $\lim\limits_{n\to\infty}[1-\frac{1}{n}]^2=1$, it follows that x is strongly λ-statistically pre-Cauchy.

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