TAMKANG JOURNAL OF MATHEMATICS
Volume 47, Number 1, 39-50, March 2016
doi:10.5556/j.tkjm.47.2016.1889

This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

ON THE EXISTENCE OF TWO STATIONARY SOLUTIONS FOR A FREE BOUNDARY PROBLEM DESCRIBING CELL MOTILITY

HARUNORI MONOBE

Abstract. This paper is concerned with the existence of stationary solutions for a free boundary problem related to cell motility. In recent years, the author and Ninomiya [6] showed that there exist at least two stationary solutions with disk-shaped domains in isotropic boundary conditions. In this paper, it will be shown that there exist exactly two stationary solutions for the free boundary problem under the same boundary conditions. The proof is based on the weak maximum principle and the mean-valued theorem.

1. Introduction

Keratocyte is a cell observed in corneal stroma, fish epidermis and so on. The cell usually maintains the shape in a disk and stays in the same place. However, when corneal stroma and fish epidermis are scratched, the cell moves to the scratched wound to heal. In that time, the cell changes shape from a disk to half-moon and locomotes. It is noted that this change of form is mainly affected by the effect of plasma streaming and actin filaments (F-actin). In recent years, some mathematical models focused on the relationship between cell locomotion and actin filaments were proposed. For instance, Mogilner et al. [2, 3, 4] proposed some mathematical models describing crawling nematode sperm cell.

In this paper, we treat another mathematical model describing cell motility, which is proposed by Tamiki Umeda (see [5, 6]). The model is a free boundary problem as follows :

$$\begin{cases} u_t = d\Delta u + k_1 U - u + k_2 & \text{in } Q := \bigcup_{t>0} \Omega(t) \times \{t\}, \\ u = 1 + A\kappa & \text{on } \Gamma := \bigcup_{t>0} \partial\Omega(t) \times \{t\}, \\ V = \gamma U - 1 - A\kappa & \text{on } \Gamma, \\ u = \phi \ge k_2 & \text{in } \Omega(0) \times \{t = 0\}, \end{cases} \tag{1.1}$$

Received March 31, 2015, accepted May 18, 2015.
2010 *Mathematics Subject Classification*. 92C17, 35R35, 34K21.
Key words and phrases. Cell locomotion, free boundary problems, stationary solutions.

where $\Omega(t)$ is an unknown region in $\mathbb{R}^2$ $(t \in [0,\infty))$, $\partial\Omega(t)$ is the boundary of $\Omega(t)$, $u = u(x,y,t) : Q \to \mathbb{R}$ is an unknown function, U is a time-dependent function represented by

$$U = U(t) = C_0 - \int\int_{\Omega(t)} u\,dxdy,$$

$V = V(x,y,t)$ and $\kappa = \kappa(x,y,t)$ stand for the outer normal velocity and the inward curvature of $\partial\Omega(t)$, respectively, and $\gamma = \gamma(\mathbf{n})$ is a positive function defined in $\partial\Omega(t)$, $\mathbf{n} = \mathbf{n}(x,y,t)$ is the outer normal unit vector to $\partial\Omega(t)$ and d, k_1, k_2, A are positive constants.

Biologically, $\Omega(t)$ stands for a thin two-dimensional sheet occupied by F-actin in a single cell. The functions u, U and constant C_0 represent the density of F-actin, the concentration of G-action and the total amount of G-actin, respectively. Moreover γ represents an activity of polymerization rate of G-actin. The activity is anisotropic and determined by various factors, e.g., chemotaxis outside the cell and the chemical reaction inside the cell. For the explanation about each terms in (1.1), see [5] and [6] for details.

The author [5] considered the behavior of solutions $(u,\Omega(t))$ for (1.1) with spherically symmetric initial data, where γ is a positive constant. It was shown that, if A is sufficiently small, then there exist global-in-time solutions and blow-up solutions depending on initial data. More precisely, if the radius $s_0 = s(0)$ of the initial domain $\Omega(0)$ exists in an interval $[\alpha_2, \beta_2]$ and the initial function ϕ satisfies

$$k_2 < \phi < 1 + A/s_0 (= 1 + A\kappa(x,y,0)) \tag{1.2}$$

for any $(x,y) \in \Omega(0)$, then solutions $(u,\Omega(t))$ of (1.1) exist for any time and satisfy that the radius $s(t)$ of $\Omega(t)$ always exists in the interval $[\alpha_2,\beta_2]$. Similarly, if the radius s_0 of $\Omega(0)$ is smaller than $\beta_1 (< \alpha_2)$ and ϕ satisfies (1.2), then $\Omega(t)$ shrinks to a single point in a finite time, i.e., $\max_{(x,y)\in\overline{\Omega(t)}} u(\cdot,t) = 1 + A/s(t)$ goes to infinity in a finite time. For the definition of $\alpha_1, \alpha_2, \beta_1$ and β_2, see Assumption 2 later (or see Condition 2 in [5]). From this result, it is expected that there exist two stationary solutions (u_1,Ω_1) and (u_2,Ω_2) such that the radii of Ω_1 and Ω_2 are contained in intervals $[\beta_1,\alpha_1]$ and $[\alpha_2,\beta_2]$, respectively. It is interesting to investigate whether stationary solutions with disk-shaped domains for (1.1) exist or not. One of the reasons for the interesting is that some cells, e.g., keratocyte stay the same position with disk-shaped domains when the activity of polymerization rate of G-actin is constant.

Recently, the author and Ninomiya [6] gave a partial answer to the expectation. They showed the existence of traveling wave solutions of (1.1) with the velocity $c \geq 0$ under the condition

$$\gamma = \gamma(\mathbf{n}) = \eta + \xi\cos\theta,$$

where $\eta > \xi \geq 0$ and θ is the angle between $\mathbf{n}$ and x-axis. In particular, if $\xi = 0$, then the velocity c of traveling wave solution is zero and the traveling wave solution corresponds to the stationary solution of (1.1) with $\gamma = \eta > 0$. Hence it has been already shown that there exist at least two stationary solutions with disk-shaped domains in (1.1).

In this paper, we will identify the number of stationary solutions for (1.1) under the condition that $\gamma = \eta > 0$. In other words, we examine the number of (u, Ω) satisfying the following problem :

$$\begin{cases} d\Delta u + k_1 U - u + k_2 = 0 & \text{in } \Omega, \\ u = 1 + A\kappa & \text{on } \partial\Omega, \\ 0 = \gamma U - 1 - A\kappa & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where

$$U = C_0 - \int\int_\Omega u\, dx dy. \tag{1.4}$$

Definition 1. We call (u, Ω) a stationary solution of (1.1) if $(u, \Omega) \in C^{2+\alpha}(\overline{\Omega}) \times C^4$ satisfies (1.3), where u, U are positive, Ω is a bounded domain and $\alpha \in (0, 1)$.

Throughout this paper, we impose some assumptions on k_1, k_2, C_0 and γ.

Assumption 1. *k_1, k_2, C_0 and γ satisfy the following conditions :*

(H1) *γ is a positive constant,*

(H2) *$k_1/\gamma - 1 + k_2 < 0$,*

(H3) *$\gamma C_0 - 1 > 0$.*

The following assumption is of help to define the range of A :

Assumption 2. *$\bar{A}$ is a small constant such that there exist at least four points $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \left(0, \sqrt{C_0/\pi k_2}\right)$ such that*

$$\gamma(C_0 - \pi\alpha_i(\bar{A} + \alpha_i)) = 1 + \frac{\bar{A}}{\alpha_i}, \tag{1.5}$$

$$\gamma(C_0 - \pi k_2 \beta_i^2) = 1 + \frac{\bar{A}}{\beta_i} \tag{1.6}$$

for $i = 1, 2$, where $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$.

Notation 1. *We will use the symbols $\alpha_1(A)$, $\alpha_2(A)$, $\beta_1(A)$ and $\beta_2(A)$ to denote four points satisfying (1.5) and (1.6) for a given constant $A \in (0, \bar{A}]$.*

Next we state our main result :

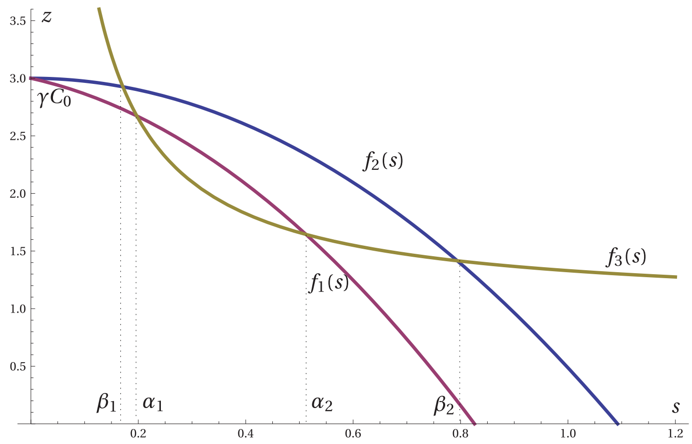

Figure 1: This graph shows three functions $z = f_1(s) = \gamma(C_0 - \pi s(\bar{A} + s))$, $z = f_2(s) = \gamma(C_0 - \pi k_2 s^2)$ and $z = f_3(s) = 1 + \bar{A}/s$, where $\gamma = 1, C_0 = 3, k_2 = 0.8, \bar{A} = 0.33$. The conditions (1.5) and (1.6) in Assumption 2 correspond to conditions $f_1(\alpha_i) = f_3(\alpha_i)$ and $f_2(\beta_i) = f_3(\beta_i)$, respectively.

Theorem 1. *Assume that k_1, k_2, C_0, γ and $\bar{A}$ satisfy Assumption 1 and 2, respectively. Then there is a positive constant $A_0 \le \bar{A}$ such that, for any $A \in (0, A_0]$, there exist exactly two stationary solutions (u_j, Ω_j) $(j = 1, 2)$ of (1.3) with the following properties :*

(P1) *Ω_j is a disk-shaped domain,*

(P2) *u_j is spherically symmetric and $u_j(\mathbf{x_1}) > u_j(\mathbf{x_2})$ if $|\mathbf{x_1}| > |\mathbf{x_2}|$, where $\mathbf{x_1} = (x_1, y_1)$ and $\mathbf{x_2} = (x_2, y_2)$.*

(P3) *For any $(x, y) \in \Omega_j$, it holds that*

$$k_2 < u_j < 1 + \frac{A}{s_j}, \quad U_j = C_0 - \int\int_{\Omega_j} u_j \, dx dy > 0,$$

where s_1 and s_2 represent the radius of Ω_1 and Ω_2, respectively, and they satisfy

$$s_1 \in [\beta_1(A), \alpha_1(A)], \quad s_2 \in [\alpha_2(A), \beta_2(A)].$$

In Theorem 1, (P1) means that the shape of cell is disked-shape. The positivity of u and U in (P3) imply that the density of F-actin and the concentration of G-actin in a cell are positive, respectively. In addition, (P2) means that F-actin is concentrated in the neighborhood of cell membrane. This result is similar to the experimental observation that F-actin are assembled around the leading edge in the the membrane (see [7]). According to the result of [5], it is expected that (u_1, Ω_1) and (u_2, Ω_2) are stable and unstable, respectively. As mentioned above, some cells stay the same position with a disk-shaped domain under the isotropic activity conditions. Thus Ω_2 is coincided with the phenomena. On the other hand, we confirm that if the initial domain $\Omega(0)$ of (1.1) is a disk-shaped domain whose radius is smaller than $\beta_1(A)(< s_1)$, then the domain $\Omega(t)$ of (1.1) shrinks to a single point, which is the center of $\Omega(0)$, at a finite time (see [5]). Such a behavior is not observed in cell locomotion. From this, we expect that the existence of the stationary solution (u_1, Ω_1) is caused by a lack of the effect of thickness of cell membrane, size of actin filaments and so on.

This paper is organized as follows: In Section 2, we confirm that solutions of (1.3) have to be spherically symmetric. This claim is derived from the boundary condition $A\kappa = \gamma U - 1$. In addition, we review the previous results for (1.3) such that, for a sufficiently small A satisfying (1.5) and (1.6), there exist at least two solutions of (1.3). Moreover we introduce auxiliary function and domain $(u(s), \Omega(s))$ satisfying

$$\begin{cases} d\Delta u + \dfrac{k_1}{\gamma}\left(1 + \dfrac{A}{s}\right) - u + k_2 = 0 & \text{in } \Omega(s), \\ u = 1 + \dfrac{A}{s} & \text{on } \partial\Omega(s), \end{cases}$$

where $\Omega(s)$ is a disk-shaped domain with the radius s. In particular, we give an estimate related to

$$\frac{\partial}{\partial s}\left(C_0 - \int_{\Omega(s)} u(s)\, dx dy\right)$$

by using a comparison function. The estimate is useful for investigating the number of solutions for (1.3).

In Section 3, we complete the proof of Theorem 1. To this end, we will verify the number of constants s which satisfy

$$\gamma\left(C_0 - \int_{\Omega(s)} u(s)\, dx dy\right) = 1 + \frac{A}{s}.$$

Actually, it will be shown that, if $(u(s), \Omega(s))$ satisfies this equality for a certain $s > 0$, then the pair $(u(s), \Omega(s))$ corresponds to the solution of (1.3). As a result, we will show that, for a sufficiently small constant A, there exist exactly two solutions (u_1, Ω_1) and (u_2, Ω_2) of (1.3), where the radii s_1 of Ω_1 and s_2 of Ω_2 satisfy $s_1 \in [\beta_1(A), \alpha_1(A)]$ and $s_2 \in [\alpha_2(A), \beta_2(A)]$, respectively.

After the proof of Theorem 1, we briefly discuss the structure of solutions for (1.3). In particular, we remark that it is possible that the saddle-node bifurcation takes place with respect to A.

2. A priori estimate and auxiliary lemmas

In this section, we introduce a priori estimate of (u, Ω) for (1.3) and auxiliary lemmas. At first, we confirm that, if there exists a solution of (1.3), then the solution must be spherically symmetric. Second, we review the previous result in [5] which gives an upper and lower estimates of u. Finally, we will prepare auxiliary lemmas which contributes to examine the number of solutions for (1.3).

2.1. Known results

Note that γ is a positive constant by Assumption 1. From the boundary condition and the interface equation of (1.3), we verify that u and Ω are spherically symmetric.

Lemma 1. *If there exists a classical solution* (u,Ω) *of* (1.3)*, then* Ω *is a disk-shaped domain and* u *is a spherically symmetric function. Moreover, if* $|\mathbf{x_1}|>|\mathbf{x_2}|$*, then it holds that* $u_j(\mathbf{x_1})>u_j(\mathbf{x_2})$*, where* $\mathbf{x_1}=(x_1,y_1)$ *and* $\mathbf{x_2}=(x_2,y_2)$.

Proof. Suppose that there exists a classical solution (u,Ω) of (1.3). Then U is a positive constant. In addition, γ is also a positive constant by $(H1)$ of Assumption 1. Since the interface equation of (1.3) satisfies

$$\gamma U = 1 + A\kappa, \tag{2.1}$$

the curvature κ is a constant. According to the definition of solutions for (1.3), κ is a positive constant. This means that Ω is a disk-shaped domain. By (2.1), we know that the solution u has to satisfy the following elliptic problem :

$$\begin{cases} d\Delta u + \dfrac{k_1}{\gamma}\left(1+\dfrac{A}{s}\right) - u + k_2 = 0 & \text{in } B_s, \\ u = 1 + \dfrac{A}{s} & \text{on } \partial B_s, \end{cases} \tag{2.2}$$

where κ, U are positive constants and B_s is a disk-shaped domain with the radius s. By standard elliptic theory [1], there exists a unique solution $u \in C^{2+\alpha}(\overline{\Omega})$ of (2.2). Fortunately, we can solve (2.2) with the help of a modified Bessel function $v(r)$ satisfying

$$\begin{cases} v_{rr} + \dfrac{1}{r}v_r - v + k_1 U + k_2 = 0 & \text{in } \left(0, \dfrac{s}{d}\right), \\ v_r(0) = 0, \ v\left(\dfrac{s}{d}\right) = 1 + \dfrac{A}{s}. \end{cases} \tag{2.3}$$

We note that the modified Bessel function $v(r)$ is represented by

$$v(r) = 1 + \frac{A}{s} - a\left(\frac{s}{d}\right) + \frac{a(s/d)}{I_0(s/d)} I_0(r),$$

where

$$a\left(\frac{s}{d}\right) := \left(1 - \frac{k_1}{\eta}\right)\left(1 + \frac{A}{s}\right) - k_2,$$

$$I_0(r) := \frac{1}{\pi}\int_0^{\pi} \exp(r\cos\theta)\, d\theta = \sum_{m=0}^{\infty}\left(\frac{r^m}{m!2^m}\right)^2.$$

As a result, it is confirmed that there exists a spherically symmetric solution u of (2.2). In addition, since the derivative of $v(r)$ with respect to r is positive, we immediately confirm that $v(r)$ is a monotone increasing function. Therefore we complete the proof. $\square$

From Lemma 1, we only have to investigate the number of spherically symmetric solutions with disk-shaped domains for (1.3). One of difficulties of our problem is that we have to confirm the condition (1.4), i.e.,

$$U = C_0 - \int\int_{\Omega} u\,dxdy.$$

To overcome this difficulty, we first ignore the restrict (1.4) and investigate the property of solutions of an elliptic problem with a given domain B_s as follows :

$$\begin{cases} d\Delta u + \dfrac{k_1}{\gamma}\left(1+\dfrac{A}{s}\right) - u + k_2 = 0 & \text{in } B_s, \\ u = 1 + \dfrac{A}{s} & \text{on } \partial B_s. \end{cases} \tag{2.4}$$

We note that (2.2) was appeared in the proof Lemma 1. From now on, $u(s)$ stands for the solution u of (2.4) with a disk-shaped domain B_s.

The following lemma is of help to a priori estimate of U :

Lemma 2 (cf.[6])**.** *Let s be a given positive-constant. Then there exists a unique solution $u(s) \in C^{2+\alpha}(\overline{B_s})$ of* (2.4) *with the property*

$$k_2 < u(s) < 1 + \frac{A}{s} \quad in\ B_s.$$

Proof. The existence of solutions for (2.4) is guaranteed by standard elliptic theory [1]. Applying the weak maximum principle for (2.3), we obtain the desire estimate. Here we used $(H2)$ and $(H3)$ of Assumption 1. See [6] for the details. □

In addition, we review the previous result related to the existence of solutions for (1.3).

Lemma 3 (cf. [6])**.** *Assume that k_1, k_2, γ and C_0 satisfy Assumption 1. Then there is a positive constant A_* such that, for any $A \in (0, A_*)$, there exist at least two stationary solutions (u_j, Ω_j) $(j = 1,2)$ of* (1.3) *with the property* (P1), (P2) *and* (P3) *in Theorem 1.*

Here we explain the reason why Assumption 2 is needed. By Lemma 2, we immediately obtain the following estimate :

$$C_0 - \pi s(A+s) < C_0 - \int_{\Omega(s)} u(s)\,dxdy < C_0 - k_2\pi s^2. \tag{2.5}$$

If there exists a positive constant s such that

$$\gamma\left(C_0 - \int_{\Omega(s)} u(s)\,dxdy\right) = 1 + \frac{A}{s}, \tag{2.6}$$

then $(u(s),\Omega(s))$ of (2.3) corresponds to a solution of (1.3) because U satisfies (1.4). Considering (2.5) and (2.6), we know that the necessary condition for solutions of (1.3) to exist is that there exists at least a constant $s_* > 0$ satisfying

$$\gamma\left(C_0 - k_2\pi s_*^2\right) > 1 + \frac{A}{s_*}. \tag{2.7}$$

In this paper, due to technical reasons to find two constant s_1 and s_2 satisfying (2.6), the author imposed more stronger assumption, i.e., Assumption 2 which ensures the existence of s_{**} satisfying

$$\gamma\left(C_0 - \pi s_{**}(A + s_{**})\right) > 1 + \frac{A}{s_{**}} \tag{2.8}$$

provided that $A \le \bar{A}$. Note that it follows from Assumption 1 that s_{**} in (2.8) satisfies the inequality (2.7). The size of A_* in Lemma 3 is also determined by the same reason.

2.2. Auxiliary lemmas

As mentioned in subsection 2.1, we investigate the number of constants s satisfying (2.6) to find solutions (u,Ω) of (1.3). To this end, we will give estimates for the derivative of

$$C_0 - \int_{B_s} u(s)\,dxdy \tag{2.9}$$

with respect to s. Here we transform (2.3) into an elliptic problem defined in a fixed domain which is independent of s. Define $w(\tau;s) = u(s)$, where $\tau = |\mathbf{x}|/s$ and $\mathbf{x} = (x,y)$. Then (2.3) is rewritten by

$$\begin{cases} \dfrac{d}{s^2} w_{\tau\tau} + \dfrac{d}{s^2\tau} w_\tau + \dfrac{k_1}{\gamma}\left(1 + \dfrac{A}{s}\right) - w + k_2 = 0 & \text{in } (0,1), \\ w_r(0) = 0, \quad w(1) = 1 + \dfrac{A}{s}, \end{cases} \tag{2.10}$$

and (2.9) is represented by

$$C_0 - 2\pi s^2 \int_0^1 \tau\, w(\tau;s)\,d\tau. \tag{2.11}$$

From now on, we will investigate a priori estimate for the derivative of (2.11) with respect to s.

The following lemma provides us detailed informations about the derivative of $w(1;s)$ with respect to s:

Lemma 4. *Let w_s be the derivative of w with respect to s. Define $\overline{w}_s$ by*

$$\overline{w}_s = \overline{w}_s(\tau;s) = \frac{\sigma}{d}(1-\tau^2) - \frac{A}{s^2},$$

where $\sigma = k_1(2s + A)/4\gamma + A/2$. Then it holds that

$$w_s \le \overline{w}_s \qquad \text{in} \quad [0,1].$$

Proof. By (2.10), w_s satisfies

$$\begin{cases} d(w_s)_{\tau\tau} + \dfrac{d}{\tau}(w_s)_\tau + \dfrac{k_1}{\gamma}(2s+A) - 2sw - s^2 w_s + 2k_2 s = 0 \quad \text{in } (0,1), \\ (w_s)_\tau(0) = 0, \quad w_s(1) = -\dfrac{A}{s^2}. \end{cases} \tag{2.12}$$

Set $X = \overline{w}_s - w_s$. Then it follows from simple calculations that X satisfies

$$\begin{cases} dX_{\tau\tau} + \dfrac{d}{\tau}X_\tau - \dfrac{k_1}{\gamma}(2s+A) + 2sw - s^2 X - 2k_2 s \\ \qquad\qquad - d(\overline{w}_s)_{\tau\tau} - \dfrac{d}{\tau}(\overline{w}_s)_\tau + s^2\overline{w}_s = 0 \quad \text{in } (0,1), \\ X_\tau(0) = 0, \quad X(1) = 0. \end{cases} \tag{2.13}$$

From now, we show that $X \geq 0$ in $[0,1]$. Suppose that there exists a point $\tau_* \in [0,1)$ such that $X(\tau_*) < 0$ and $X(\tau) \geq X(\tau_*)$ for any $\tau \in [0,1]$. Then $X_{\tau\tau}(\tau_*) \geq 0$, $X_\tau(\tau_*) = 0$ and $X(\tau_*) < 0$. On the other hand, we have the following positivity at $\tau = \tau_*$:

$$\begin{aligned} &-\frac{k_1}{\gamma}(2s+A) + 2sw - 2k_2 s - d(\overline{w}_s)_{\tau\tau} - \frac{d}{\tau}(\overline{w}_s)_\tau + s^2\overline{w}_s \\ &\geq -\frac{k_1}{\gamma}(2s+A) + 2\sigma + 2\sigma - A + \frac{s^2\sigma}{d}(1-\tau^2) \\ &\geq A + \frac{s^2\sigma}{d}(1-\tau^2) > 0. \end{aligned}$$

This estimate implies that the left-hand side of interior condition in (2.13) is positive at $\tau = \tau_*$. This contradicts the fact that the right-hand side of (2.13) is zero. Since $X \geq 0$ in $[0,1)$, we have the desired inequality. ☐

The following two lemmas imply that, if A is sufficiently small, then the derivative of $1 + A/s$ with respect to s approaches infinity in $[\beta_1(A), \alpha_1(A)]$ and zero in $[\alpha_2(A), \beta_2(A)]$, respectively.

Lemma 5. *There exists a constant $A_1 \in (0, \bar{A}]$ such that, for a fixed constant $A \in (0, A_1]$, it holds that*

$$-2\gamma\pi(A+s) > \frac{\partial}{\partial s}\left(1 + \frac{A}{s}\right)$$

for any $s \in [\beta_1(A), \alpha_1(A)]$.

Proof. By the definition of $\alpha_1(\bar{A})$, we obtain

$$\gamma\pi(\bar{A} + \alpha_1(\bar{A})) - \frac{1}{\alpha_1(\bar{A})}(\gamma C_0 - 1) = -\frac{\bar{A}}{\{\alpha_1(\bar{A})\}^2}.$$

Note that $\alpha_1(A)$ is a monotone function with respect to A and converges to zero as A tends to zero. Thus there exists a constant $A_{0,1} \in (0, \bar{A}]$ such that $-1 > -A_{0,1}/s^2$ for any $s \in [0, \alpha_1(A_{0,1})]$. Thus we have

$$\begin{aligned}-2\pi\gamma(A_{0,1}+s) - \frac{\partial}{\partial s}\left(1+\frac{A_{0,1}}{s}\right) &= -2\pi\gamma(A_{0,1}+s) + \frac{A_{0,1}}{s^2}\\ &\geq -2\pi\gamma\left(A_{0,1}+\sqrt{A_{0,1}}\right)+1\end{aligned}$$

for any $s \in [0, \alpha_1(A_{0,1})]$. Let $A_{0,2}$ be a positive constant satisfying $-2\pi\gamma(A_{0,2}+\sqrt{A_{0,2}})+1>0$. Taking $A_1 = \min\{A_{0,1}, A_{0,2}\}$, we obtain the desired estimate for $A \in (0, A_1]$. $\square$

Lemma 6. *There exists a constant $A_2 \in (0, \bar{A}]$ such that, for a fixed constant $A \in (0, A_2]$, it holds that*

$$\gamma(-2\pi k_2 s + A) < \frac{\partial}{\partial s}\left(1+\frac{A}{s}\right)$$

for any $s \in [\alpha_2(A), \beta_2(A)]$.

Proof. Recall that $\alpha_2(A)$ is monotone increasing with respect to A. Thus it satisfies that $\alpha_2(\bar{A}) \leq \alpha_2(A)$ for any $A \in (0, \bar{A})$. Let A be a constant in $(0, \bar{A})$. Then we obtain

$$\begin{aligned}\gamma(-2\pi k_2 s + A) - \frac{\partial}{\partial s}\left(1+\frac{A}{s}\right) &= \gamma(-2\pi k_2 s + A) + \frac{A}{s^2}\\ &\leq -2\pi\gamma k_2 \alpha_2(\bar{A}) + A\gamma + \frac{A}{\{\alpha_2(\bar{A})\}^2}\end{aligned} \tag{2.14}$$

for any $s \in [\alpha_2(A), \infty)$. Letting A_2 be a positive constant in $(0, \bar{A}]$ such that the right-hand side of (2.14) is negative, we obtain the desire estimate. $\square$

3. Proof of Theorem 1

In this section, to complete the proof of Theorem 1, we confirm the number of constants s satisfying

$$\gamma\left(C_0 - 2\pi s^2 \int_0^1 \tau w(\tau; s)\, d\tau\right) = 1 + \frac{A}{s} \tag{3.1}$$

instead of (2.6). Let A be a small constant such that $A \leq A_0 := \min\{A_1, A_2\}$. Differentiating (2.11) with respect to s, we have

$$\begin{aligned}\frac{\partial}{\partial s}\left(C_0 - 2\pi s^2 \int_0^1 \tau w(\tau; s)\, d\tau\right) &= -2\pi\left(\int_0^1 2s\tau w(\tau; s) + s^2 \tau w_s(\tau; s)\, d\tau\right)\\ &= -2\pi\left\{k_2 s + \frac{k_1}{2\gamma}(2s+A) + d(w_s)_\tau(1)\right\}.\end{aligned} \tag{3.2}$$

The last equality is derived from (2.12).

We show that there exist exactly two constants s_1 and s_2 satisfying (3.1), where $s_1 \in [\beta_1(A), \alpha_1(A)]$ and $s_2 \in [\alpha_2(A), \beta_2(A)]$, respectively. As seen in Lemma 4, $(w_s)_\tau(1)$ is non-positive. Hence we have

$$\begin{aligned}\gamma\frac{\partial}{\partial s}\left(C_0 - 2\pi s^2\int_0^1 \tau w(\tau;s)\,d\tau\right) &> -2\pi\gamma\left\{k_2 s + \frac{k_1}{2\gamma}(2s+A)\right\},\\ &> -2\pi\gamma(s+A),\\ &> \frac{\partial}{\partial s}\left(1+\frac{A}{s}\right)\end{aligned}$$

for any $s \in [\beta_1(A), \alpha_1(A)]$. Note that $(H2)$ in Assumption 1. Remember that it follows from Assumption 2 that

$$\gamma\left(C_0 - 2\pi\beta_1(A)^2\int_0^1 \tau w(\tau;\beta_1(A))\,d\tau\right) < 1 + \frac{A}{\beta_1(A)}$$

and

$$\gamma\left(C_0 - 2\pi\alpha_1(A)^2\int_0^1 \tau w(\tau;\alpha_1(A))\,d\tau\right) > 1 + \frac{A}{\alpha_1(A)}.$$

Therefore there exists at most a point $s \in [\beta_1(A), \alpha_1(A)]$ satisfying (3.1).

On the other hand, it follows from Lemma 4 that

$$-d(w_s)_\tau(1) \le -d(\overline{w}_s)_\tau(1) = \frac{k_1}{2\gamma}(2s+A) + A. \tag{3.3}$$

According to (3.2) and (3.3), we obtain

$$\begin{aligned}\gamma\frac{\partial}{\partial s}\left(C_0 - 2\pi s^2\int_0^1 \tau w(\tau;s)\,d\tau\right) &< \gamma(-2\pi k_2 s + A)\\ &< \frac{\partial}{\partial s}\left(1+\frac{A}{s}\right)\end{aligned}$$

for any $s \in [\alpha_2(A), \beta_2(A)]$. From Assumption 2, $\alpha_2(A)$ and $\beta_2(A)$ satisfy

$$\gamma\left(C_0 - 2\pi\alpha_2(A)^2\int_0^1 \tau w(\tau;\alpha_2(A))\,d\tau\right) > 1 + \frac{A}{\alpha_2(A)}$$

and

$$\gamma\left(C_0 - 2\pi\beta_2(A)^2\int_0^1 \tau w(\tau;\beta_2(A))\,d\tau\right) < 1 + \frac{A}{\beta_2(A)}.$$

Therefore there exists at most a point $s \in [\alpha_2(A), \beta_2(A)]$ satisfying (2.6). In conclusion, we complete the proof of Theorem 1.

4. Conclusions

In this paper, we obtained the result that there exist exactly two solutions for (1.3) with a small constant A satisfying $A \le A_0 \le \bar{A}$. Thus we naturally encounter a question of existences of solutions for (1.3) provided $A > \bar{A}$. It is not easy to answer the question when A is slightly larger than $\bar{A}$. However we can confirm that there exist no solutions (u, Ω) of (1.3) provided A is sufficiently larger than $\bar{A}$. Assume that there exists a solution of (1.3). By Lemma 1, Ω is a disk-shaped domain with a radius s_*. In addition, it follows from a similar argument of Lemma 2 that

$$f_1(s_*) < \gamma U < f_2(s_*),$$

where f_1 and f_2 stand for functions appeared in Figure 1. Since we can easily take a large constant A such that $f_2(s) < f_3(s)$ in $[0, \infty)$, it holds that $\gamma U < 1 + A/s_*$. This result implies that there exist no solutions (u, Ω) of (1.3) under the condition that A is sufficiently large and suggests that the saddle-node bifurcation takes place with respect to A. Thus it is also expected that there exists exactly two solutions for (1.3) even if $A \in (A_0, \bar{A})$ and the bifurcation point is larger than $\bar{A}$. Hence Assumptions 1 and 2 are essential. Incidentally, if $(H3)$ is not satisfied, i.e., $\gamma C_0 \leq 1$, then we can also confirm that there are no stationary solutions of (1.3).

Acknowledgements

The author would like to thank Professor Hirokazu Ninomiya for valuable suggestions. I am partially supported by Meiji Institute for Advanced Study of Mathematical Sciences and Grant-in-Aid for Young Scientists (B) (No. 15K17595) from the Japan Society for the Promotion of Science.

References

[1] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, New York, 1998.

[2] A. Mogilner and L. Edelstein-Keshet, *Regulation of actin dynamics in rapidly moving cells*, A quantitative analysis. Biophys. J., **83** (2002), 1237–1258

[3] A. Mogilner, J. Stajic and C. W. Wolgemuth, *Redundant mechanisms for stable cell locomotion revealed by minimal models*, Biophys J., **101** (2011), 545–553.

[4] A. Mogilner and B. Rubinstein et al, *Actin-myosin viscoelastic flow in the keratocyte lamellipod*, Bio. J., **97** (2009), 1853–1863.

[5] H. Monobe, *Behavior of solutions for a free boundary problem describing amoeba motion*, Differential and Integral Equations, **25** (2012), 93–116.

[6] H. Monobe and H. Ninomiya, *Multiple existence of traveling waves of a free boundary problem describing cell motility*, Discrete and Continuous Dynamical Systems Series B, **19** (2014), 789–799.

[7] J. V. Small, M. Herzog and K. Anderson, *Actin filament organization in the fish keratocyte lamellipodium*, J. Cell Biol., **129** (1995), 1275–1286.

Meiji Institute of Mathematical Sciences Meiji University, 4-21-1 Nakano, Nakano-ku, Tokyo, 164-8525, Japan.

E-mail: te12001@meiji.ac.jp