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A NON-COMMUTATIVE SOBOLEV ESTIMATE AND ITS APPLICATION TO SPECTRAL SYNTHESIS

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Abstract. In [M. K. Vemuri, *Realizations of the canonical representation*, Proc. Indian Acad. Sci. Math. Sci., 118 (2008), 115–131], it was shown that the spectral synthesis problem for the Alpha transform is closely related to the problem of classifying realizations of the canonical representation (of the Heisenberg group). In this paper, it is shown that discrete sets are sets of spectral synthesis for the Alpha transform.

1. Introduction

For $p \in [1,\infty]$, let S^p denote the Schatten class of p^{th} power traceable operators on $L^2(\mathbb{R})$. For $x, y \in \mathbb{R}$, let T_x and M_y denote the translation and modulation operators on $L^2(\mathbb{R})$, i.e. for $s \in L^2(\mathbb{R})$

$$(T_x s)(t) = s(t+x) \qquad \text{and}$$
$$(M_y s)(t) = e^{2\pi i y t} s(t).$$

If $(x_1, y_1) \in \mathbb{R}^2$ and $X \in S^p$, set

$$(x_1, y_1)\cdot X = T_{y_1} M_{-x_1} X M_{x_1} T_{-y_1}.$$

If $q \in L^1(\mathbb{R}^2)$ and $X \in S^p$, set

$$q\cdot X = \iint q(x_1, y_1)(x_1, y_1)\cdot X\, dx_1\, dy_1.$$

Then S^p becomes an $L^1(\mathbb{R}^2)$-module. Note that the previous integral exists by Lemma 2.4.

Main Theorem. *If $X \in S^1$, $\operatorname{tr}(X) = 0$ and $\varepsilon > 0$, then there exists $\rho \in L^1(\mathbb{R}^2)$ with $\hat{\rho} = 1$ on a neighborhood of $(0,0)$ such that*

$$\|\rho\cdot X\|_{S^1} < \varepsilon.$$

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Definition 1.1. If $X \in S^1$, the Alpha transform of X is the function on $\mathbb{R}^2$ defined by

$$\alpha(X)(x,y) = \operatorname{tr}(T_x M_y X).$$

The Alpha transform is related to the module structure on S^1 in the same way that the classical Fourier transform is related to convolution. In fact, we have the following lemma.

Lemma 1.2. *If $q \in L^1(\mathbb{R}^2)$ and $X \in S^1$, then $\alpha(q \cdot X) = \hat{q}\alpha(X)$.*

Proof. Note first that

$$\begin{aligned}
\alpha((x_1,y_1)\cdot X)(x,y) &= \operatorname{tr}(T_x M_y T_{y_1} M_{-x_1} X M_{x_1} T_{-y_1}) \\
&= e^{-2\pi i y y_1} \operatorname{tr}(T_{y_1} T_x M_{-x_1} M_y X M_{x_1} T_{-y_1}) \\
&= e^{-2\pi i (y y_1 + x x_1)} \operatorname{tr}(T_{y_1} M_{-x_1} T_x M_y X M_{x_1} T_{-y_1}) \\
&= e^{-2\pi i (y y_1 + x x_1)} \operatorname{tr}(T_x M_y X) \\
&= e^{-2\pi i (y y_1 + x x_1)} \alpha(X)(x,y).
\end{aligned}$$

The result now follows by integration.

By standard methods, the main theorem leads to the following corollary, which may be viewed as saying that discrete sets are sets of spectral synthesis (more precisely, C-sets) for the Alpha transform.

Main Corollary. *Let $D \subseteq \mathbb{R}^2$ be a discrete set. Let $\varepsilon > 0$. If $X \in S^1$ and $\alpha(X)$ vanishes on D, then there exists $Y \in S^1$ such that $\alpha(Y)$ vanishes on a neighborhood of D and $\|X - Y\|_{S^1} < \varepsilon$.*

It was shown in [6] (see also [5]) that the spectral synthesis problem for the Alpha transform is closely related to the problem of classifying realizations of the canonical representation (of the Heisenberg group).

2. Some Lemmas on Integration

In this section, we prove some slight extensions of [4, Theorem 3.27].

Definition 2.1. Let f be a continuous function defined on $\mathbb{R}^2$. We say that f is *rapidly decreasing* and write $f \in \mathscr{R}(\mathbb{R}^2)$ if for all positive integers n, the function $(x,y) \mapsto (1 + x^2 + y^2)^n f(x,y)$ is bounded. The best constants in the bounds give a countable family of norms which turn $\mathscr{R}(\mathbb{R}^2)$ into a Frechet space.

Let $\mathscr{T}$ be a Frechet space defined by a countable family of *norms.*

Definition 2.2. A function $\Phi : \mathbb{R}^2 \to \mathscr{T}$ is *bounded* if for each norm σ, the function $\sigma \circ \Phi$ is bounded.

Definition 2.3. A function $\Phi : \mathbb{R}^2 \to \mathcal{T}$ is *polynomially bounded* if for each norm σ, there exists a polynomial p such that

$$\sigma \circ \Phi(x, y) \le p(x, y)$$

for all $(x, y) \in \mathbb{R}^2$.

Lemma 2.4. *Let $\Phi : \mathbb{R}^2 \to \mathcal{T}$ be a bounded continuous function. Let $f \in L^1(\mathbb{R}^2)$. Then*

$$\iint f(x, y)\Phi(x, y)\,dx\,dy$$

exists.

Proof. There is a sequence $\{f_k\}$ of compactly supported continuous functions such that $f_k \to f$ in $L^1(\mathbb{R}^2)$. For each k, the integral

$$I_k = \iint f_k(x, y)\Phi(x, y)\,dx\,dy$$

exists by [4, Theorem 3.27]. For each norm σ,

$$\begin{aligned}
\sigma(I_k - I_j) &\le \iint |f_k(x, y) - f_j(x, y)|\sigma(\Phi(x, y))\,dx\,dy \\
&\le \sup \sigma(\Phi(x, y)) \left\| f_k - f_j \right\|_1 \\
&\to 0
\end{aligned}$$

as $k, j \to \infty$. So $\{I_k\}$ is a Cauchy sequence. Since $\mathcal{T}$ is Frechet, there exists $I \in \mathcal{T}$ such that

$$\lim_{k\to\infty} I_k = I.$$

Let Λ be a continuous linear functional on $\mathcal{T}$. Then

$$\begin{aligned}
\Lambda(I) &= \lim_{k\to\infty} \Lambda(I_k) \\
&= \lim_{k\to\infty} \Lambda\left(\iint f_k(x, y)\Phi(x, y)\,dx\,dy\right) \\
&= \lim_{k\to\infty} \iint f_k(x, y)\Lambda(\Phi(x, y))\,dx\,dy \\
&= \iint f(x, y)\Lambda(\Phi(x, y))\,dx\,dy \qquad \text{(by the dominated convergence theorem)} \\
&= \iint \Lambda(f(x, y)\Phi(x, y))\,dx\,dy.
\end{aligned}$$

Therefore, $\iint f(x, y)\Phi(x, y)\,dx\,dy = I$.

Lemma 2.5 *Let $\Phi : \mathbb{R}^2 \to \mathcal{T}$ be a polynomially bounded continuous function. Let $f \in \mathcal{R}(\mathbb{R}^2)$. Then*

$$\iint f(x, y)\Phi(x, y)\,dx\,dy$$

exists.

We omit the proof because it is very similar to the proof of Lemma 2.4.

3. An Interpolation Theorem

Theorem 3.1. *Let* $1 \le p_0, p_1, p_0', p_1' \le \infty$ *and suppose that* $T : L^{p_0}(\mathbb{R}^2) \cap L^{p_1}(\mathbb{R}^2) \to S^{p_0'} \cap S^{p_1'}$ *is a linear transformation which satisfies*

$$\begin{aligned} \|Tf\|_{S^{p_0'}} &\le M_0 \|f\|_{p_0} \qquad \text{and} \\ \|Tf\|_{S^{p_1'}} &\le M_1 \|f\|_{p_1}. \end{aligned}$$

Then for each $f \in L^{p_0}(\mathbb{R}^2) \cap L^{p_1}(\mathbb{R}^2)$ *and each* $t \in (0,1)$, $Tf \in S^{p_t'}$ *and*

$$\|Tf\|_{S^{p_t'}} \le M_t \|f\|_{p_t},$$

where

$$\begin{aligned} \frac{1}{p_t} &= \frac{t}{p_1} + \frac{1-t}{p_0}, \\ \frac{1}{p_t'} &= \frac{t}{p_1'} + \frac{1-t}{p_0'} \qquad \text{and} \\ M_t &= M_0^{1-t} M_1^t. \end{aligned}$$

Proof. This follows immediately from the abstract (Calderon-Lions) interpolation theorem once we know that $\{L^p(\mathbb{R}^2) | p \in [1,\infty]\}$ and $\{S^p | p \in [1,\infty]\}$ form complex interpolation scales. For this, see [2, p38, Example 1 and p44, Proposition 8]. Note that we must take S^∞ to be the space of compact operators with the operator norm for all this to work.

4. Non-commutative Hölder Inequality

Theorem 4.1. *Let* $1 \le p \le \infty$ *and* $p^{-1} + p'^{-1} = 1$. *If* $A \in S^p$ *and* $B \in S^{p'}$, *then* $AB \in S^1$ *and*

$$\|AB\|_{S^1} \le \|A\|_{S^p} \|B\|_{S^{p'}}.$$

Proof. This follows from Theorem 3.1. For details, see [2].

5. The Inverse Alpha Transform

Definition 5.1. Let $f \in L^1(\mathbb{R}^2)$. Then the inverse Alpha transform of f is the bounded operator defined by

$$\Theta(f) = \iint f(x,y) M_{-y} T_{-x} \, dx\, dy$$

Note that this integral exists by Lemma 2.4.

Lemma 5.2. *If $f \in L^1(\mathbb{R}^2)$, then for all $g \in L^2(\mathbb{R})$,*

$$[\Theta(f)g](v) = \int K(v,w)g(w)\,dw \quad a.e.$$

where

$$K(v,w) = \int f(v-w,y)e^{-2\pi i y v}\,dy.$$

Proof. For any $h \in L^2(\mathbb{R})$,

$$\begin{aligned}
\int [\Theta(f)g](v)\overline{h(v)}\,dv &= \int \left(\iint f(x,y)M_{-y}T_{-x}\,dx\,dy\,g\right)(v)\overline{h(v)}\,dv \\
&= \iint f(x,y)\int (M_{-y}T_{-x}g)(v)\overline{h(v)}\,dv\,dx\,dy \\
&\qquad \text{(by definition of integral)} \\
&= \iint f(x,y)\int e^{-2\pi i y v}g(v-x)\overline{h(v)}\,dv\,dx\,dy \\
&= \iiint f(x,y)e^{-2\pi i y v}\,dy\,g(v-x)\,dx\,\overline{h(v)}\,dv \\
&\qquad \text{(by Fubini's theorem)} \\
&= \iiint f(v-w,y)e^{-2\pi i y v}\,dy\,g(w)\,dw\,\overline{h(v)}\,dv \\
&= \iint K(v,w)g(w)\,dw\,\overline{h(v)}\,dv.
\end{aligned}$$

The application of Fubini's theorem is justified by the fact that $F \in L^1(\mathbb{R}^3)$ if

$$F(x,y,v) = f(x,y)e^{-2\pi i y v}g(v-x)\overline{h(v)}.$$

6. A Minimalist Alpha Inversion Formula

Lemma 6.1. *If $X \in S^1$, then*

$$\|\alpha(X)\|_\infty \le \|X\|_{S^1}.$$

Proof. For any $Y \in S^1$, we have $|\mathrm{tr}(Y)| \le \mathrm{tr}(|Y|)$ by the spectral theorem. So for any $X \in S^1$,

$$\begin{aligned}
\|\alpha(X)\|_\infty &= \sup_{(x,y)\in\mathbb{R}^2} |\mathrm{tr}(T_x M_y X)| \\
&\le \sup_{(x,y)\in\mathbb{R}^2} \mathrm{tr}(|T_x M_y X|) \\
&= \mathrm{tr}(|X|) \qquad \text{(since } T_x M_y \text{ is unitary)} \\
&= \|X\|_{S^1}.
\end{aligned}$$

Remark 6.2. Note that if X is given by an integral kernel K of Schwartz class, then

$$\alpha(X)(x,y)=\int e^{2\pi i y v}K(v,v-x)\,dv.$$

Theorem 6.3. *If f is a Schwartz class function on $\mathbb{R}^2$, then*

$$\alpha(\Theta(f))=f.$$

Proof. By the Schwartz-Plancherel theorem for the classical Fourier transform, the kernel K of $\Theta(f)$ is of Schwartz class. So

$$\begin{aligned}\alpha(\Theta(f))(x,y) &= \int e^{2\pi i y v}K(v,v-x)\,dv\\ &= \int e^{2\pi i y v}\int f(x,y')e^{-2\pi i y' v}\,dy'\,dv\\ &= f(x,y) \qquad \text{(by the classical Fourier inversion formula.)}\end{aligned}$$

Theorem 6.4. *If $X\in S^1$ is given by an integral kernel K of Schwartz class, then*

$$X=\Theta(\alpha(X)).$$

Proof. By Lemma 5.2, $\Theta(\alpha(X))$ is given by the kernel $\tilde{K}$, where

$$\tilde{K}(v',w)=\int \alpha(X)(v'-w,y)e^{-2\pi i y v'}\,dy.$$

However, by Remark 6.2 and the classical Fourier inversion formula,

$$\begin{aligned}\tilde{K}(v',w) &= \int\int e^{2\pi i y v}K(v,v-(v'-w))\,dv\,e^{-2\pi i y v'}\,dy\\ &= K(v',w).\end{aligned}$$

7. Non-commutative Riemann-Lebesgue Lemma

Theorem 7.1. *If $f\in L^1(\mathbb{R}^2)$, then $\|\Theta(f)\|_\infty\le\|f\|_1$ and $\Theta(f)\in S^\infty$.*

Proof. Firstly,

$$\begin{aligned}\|\Theta(f)\|_\infty &= \left\|\iint f(x,y)M_{-y}T_{-x}\,dx\,dy\right\|_\infty\\ &\le \iint |f(x,y)|\,\|M_{-y}T_{-x}\|_\infty\,dx\,dy\\ &= \iint |f(x,y)|\,dx\,dy\end{aligned}$$

$$= \|f\|_1.$$

Now, there is a sequence $\{f_k\}$ of compactly supported smooth functions such that $f_k \to f$ in $L^1(\mathbb{R}^2)$. Moreover, for each k, the operator $\Theta(f_k) \in S^\infty$. But, by the previous calculation, $\Theta(f_k) \to \Theta(f)$ in operator norm. Since S^∞ is closed in the operator norm, $\Theta(f) \in S^\infty$.

8. Non-commutative Plancherel Theorem

Theorem 8.1. Θ *extends to an isometry*

$$\Theta : L^2(\mathbb{R}^2) \to S^2.$$

Proof. Assume first that $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. Then $\Theta(f)$ is given by the kernel

$$K(v,w) = \int f(v-w,y)e^{-2\pi i y v}\,dy.$$

So

$$\begin{aligned}
\|\Theta(f)\|_{S^2}^2 &= \iint |K(v,w)|^2 \,dv\,dw \\
&= \iint \left| \int f(v-w,y)e^{-2\pi i y v}\,dy \right|^2 dv\,dw \\
&= \iint |f(u,v)|^2 \,dv\,du \qquad \text{(by the classical Plancherel theorem)} \\
&= \|f\|_2.
\end{aligned}$$

The rest is clear.

9. Non-commutative Hausdorff-Young Theorem

Theomre 9.1. *Let* $1 \le p \le 2$ *and* $p^{-1} + p'^{-1} = 1$. *Then* Θ *extends to a bounded operator*

$$\Theta : L^p(\mathbb{R}^2) \to S^{p'}.$$

Proof. The endpoint estimates are given by Theorem 7.1 and Theorem 8.1. The result now follows from Theorem 3.1.

10. An Approximation Lemma

Recall that if $\varphi, \psi \in L^2(\mathbb{R})$, the operator $\varphi \otimes \overline{\psi} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined by

$$(\varphi \otimes \overline{\psi})(f) = \langle f, \psi \rangle \varphi,$$

and is of rank one. Thus a general finite rank operator is of the form

$$X=\sum_{k=1}^{n}\varphi_k\otimes\overline{\psi_k}\qquad\text{with } \varphi_k,\psi_k\in L^2(\mathbb{R}).$$

Definition 10.1. The *non-commutative Schwartz space* is the space $S^{\mathscr{S}}$ of finite rank operators $X: L^2(\mathbb{R})\to L^2(\mathbb{R})$ such that

$$X=\sum_{k=1}^{n}\varphi_k\otimes\overline{\psi_k}\qquad\text{with } \varphi_k\in\mathscr{S},$$

where $\mathscr{S}$ is the space of Schwartz class functions on $\mathbb{R}$.

Lemma 10.2. *Let* $X\in S^1$, $\operatorname{tr}(X)=0$ *and* $\varepsilon>0$. *Then there exists* $Z\in S^{\mathscr{S}}$ *such that* $\operatorname{tr}(Z)=0$ *and*

$$\|X-Z\|_{S^1}<\varepsilon.$$

Proof. It is well known (see e.g. [1]) that there exists a finite rank operator

$$X_1=\sum_{k=1}^{n}\varphi_k\otimes\overline{\psi_k}$$

such that $\left\|\psi_k\right\|_2=1$ and $\|X-X_1\|_{S^1}<\frac{\varepsilon}{4}$. It is also well known that there exist $\varphi'_k\in\mathscr{S}$ such that $\left\|\varphi_k-\varphi'_k\right\|_2<\frac{\varepsilon}{4n}$. Set

$$X_2=\sum_{k=1}^{n}\varphi'_k\otimes\overline{\psi_k}.$$

Then

$$\begin{aligned}\|X_1-X_2\| &= \Big\|\sum_{k=1}^{n}(\varphi_k-\varphi'_k)\otimes\overline{\psi_k}\Big\|_{S^1}\\ &\le \sum_{k=1}^{n}\left\|\varphi_k-\varphi'_k\right\|_2\left\|\psi_k\right\|_2\\ &< \sum_{k=1}^{n}\frac{\varepsilon}{4n}\\ &= \frac{\varepsilon}{4}.\end{aligned}$$

Fix $W\in S^{\mathscr{S}}$ such that $\|W\|=\operatorname{tr}(W)=1$ and define $Z=X_2-\operatorname{tr}(X_2)W$. Then $Z\in S^{\mathscr{S}}$, $\operatorname{tr}(Z)=0$ and

$$\begin{aligned}\|X_2-Z\|_{S^1} &\le |\operatorname{tr}(X_2)|\\ &= \Big|\sum_{k=1}^{n}\langle\varphi'_k,\psi_k\rangle\Big|\end{aligned}$$

$$\begin{aligned}
&= \Big| \sum_{k=1}^{n} \left(\langle \varphi_k, \psi_k \rangle - \langle \varphi_k - \varphi'_k, \psi_k \rangle \right) \Big| \\
&\le \Big| \sum_{k=1}^{n} \langle \varphi_k, \psi_k \rangle \Big| + \sum_{k=1}^{n} |\langle \varphi_k - \varphi'_k, \psi_k \rangle| \\
&\le |\operatorname{tr}(X_1)| + \sum_{k=1}^{n} \|\varphi_k - \varphi'_k\|_2 \, \|\psi_k\|_2 \\
&< |\operatorname{tr}(X_1)| + \frac{\varepsilon}{4}.
\end{aligned}$$

But

$$\begin{aligned}
|\operatorname{tr}(X_1)| &= |\operatorname{tr}(X) - \operatorname{tr}(X - X_1)| \\
&\le |\operatorname{tr}(X)| + |\operatorname{tr}(X - X_1)| \\
&= |\operatorname{tr}(X - X_1)| \\
&\le \operatorname{tr}(|X - X_1|) \\
&= \|X - X_1\|_{S^1} \\
&< \frac{\varepsilon}{4}.
\end{aligned}$$

Therefore, $\|X_2 - Z\|_{S^1} < \frac{\varepsilon}{2}$. Therefore, $\|X - Z\| < \varepsilon$.

11. The Action of Differential Operators

Definition 11.1. For $\varphi \in \mathscr{S}$, define

$$\begin{aligned}
(P\varphi)(t) &= \frac{d\varphi}{dt} \\
(Q\varphi)(t) &= 2\pi i t \varphi(t).
\end{aligned}$$

Lemma 11.2. *If $X \in S^{\mathscr{S}}$, then $PX, QX \in S^{\mathscr{S}}$. Moreover,*

$$\begin{aligned}
\alpha(PX) &= \left(\frac{\partial}{\partial x} - 2\pi i y \right) \alpha(X) \\
\alpha(QX) &= \frac{\partial}{\partial y} \alpha(X).
\end{aligned}$$

Proof. It follows immediately from the definition that $PX, QX \in S^{\mathscr{S}}$. Since for any $\varphi \in \mathscr{S}$, $\lim_{h\to 0} \frac{T_h - I}{h} \varphi = P\varphi$ in the $L^2(\mathbb{R})$ sense, $\lim_{h\to 0} \frac{T_h - I}{h} X = PX$ in S^1-norm. So

$$\begin{aligned}
\frac{\partial}{\partial x} \alpha(X)(x, y) &= \frac{\partial}{\partial x} \operatorname{tr}(T_x M_y X) \\
&= \lim_{h\to 0} \frac{\operatorname{tr}(T_{x+h} M_y X) - \operatorname{tr}(T_x M_y X)}{h}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h\to 0} \operatorname{tr}\left(\frac{e^{2\pi i y h} T_x M_y T_h X - T_x M_y X}{h}\right)\\
&= \lim_{h\to 0} \operatorname{tr}\left(\frac{e^{2\pi i y h} T_x M_y T_h X - e^{2\pi i y h} T_x M_y X}{h}\right.\\
&\qquad \left. + \frac{e^{2\pi i y h} T_x M_y X - T_x M_y X}{h}\right)\\
&= \lim_{h\to 0} \operatorname{tr}\left(T_x M_y \frac{T_h - I}{h} X\right) + \frac{e^{2\pi i y h} - 1}{h} \operatorname{tr}(T_x M_y X)\\
&= \operatorname{tr}(T_x M_y P X) + 2\pi i y \operatorname{tr}(T_x M_y X)\\
&= (\alpha(PX) + 2\pi i y \alpha(X))(x, y).
\end{aligned}$$

So

$$\alpha(PX) = \left(\frac{\partial}{\partial x} - 2\pi i y\right)\alpha(X).$$

By essentially the same argument, we get

$$\alpha(QX) = \frac{\partial}{\partial y}\alpha(X).$$

Lemma 11.3. *If $X \in S^{\mathscr{S}}$ and $q \in \mathscr{R}(\mathbb{R}^2)$, then* $\operatorname{Im}(q\cdot X) \subseteq \mathscr{S}$.

Proof. Without loss of generality, we may assume $X = \varphi \otimes \overline{\psi}$ with $\varphi \in \mathscr{S}$. Let $g \in L^2(\mathbb{R})$. Since the map $(x, y) \mapsto \langle g, T_y M_{-x}\psi\rangle$ is bounded and continuous, $q(x_1, y_1)\langle g, T_{y_1} M_{-x_1}\psi\rangle \in \mathscr{R}(\mathbb{R}^2)$. Since $(x_1, y_1) \mapsto T_{y_1} M_{-x_1}\varphi$ is a polynomially bounded continuous map $\mathbb{R}^2 \to \mathscr{S}$,

$$I = \iint q(x_1, y_1)\langle g, T_{y_1} M_{-x_1}\psi\rangle T_{y_1} M_{-x_1}\varphi\, dx_1\, dy_1$$

exists in $\mathscr{S}$ by Lemma 2.5. Now, for any $h \in L^2(\mathbb{R})$,

$$\begin{aligned}
\int I(v)\overline{h(v)}\,dv &= \iint q(x_1, y_1)\langle((x_1, y_1)\cdot X)g, h\rangle\, dx_1\, dy_1 \qquad \text{(by definition of integral)}\\
&= \langle (q\cdot X)g, h\rangle.
\end{aligned}$$

It follows that

$$I = (q\cdot X)g \quad a.e.$$

In particular $(q\cdot X)g \in \mathscr{S}$.

Lemma 11.4. *Let $X \in S^{\mathscr{S}}$ and $q \in \mathscr{R}(\mathbb{R}^2)$. Then*

$$P(q\cdot X) = (-2\pi i x_1 q)\cdot X + q\cdot(PX).$$

In particular, $P(q\cdot X) \in S^1$.

Proof. Since for any $\varphi \in \mathscr{S}$,

$$\lim_{h\to 0}\left(\frac{T_h - I}{h}\right)\varphi = P\varphi$$

in the $L^2(\mathbb{R})$-sense and $\operatorname{Im}(q\cdot X) \subseteq \mathscr{S}$, we have

$$P(q\cdot X) = \lim_{h\to 0}\frac{T_h - I}{h}(q\cdot X)$$

in the strong operator topology. For the same reason,

$$PX = \lim_{h\to 0}\frac{T_h - I}{h}X$$

in S^1-norm. So

$$\begin{aligned}
\lim_{h\to 0}\frac{T_h - I}{h}(q\cdot X) &= \lim_{h\to 0}\frac{T_h - I}{h}\iint q(x_1,y_1)(x_1,y_1)\cdot X\,dx_1\,dy_1\\
&= \lim_{h\to 0}\iint q(x_1,y_1)\frac{T_h - I}{h}(x_1,y_1)\cdot X\,dx_1\,dy_1\\
&= \lim_{h\to 0}\iint q(x_1,y_1)\frac{1}{h}\Big(e^{-2\pi i x_1 h}T_{y_1}M_{-x_1}T_hXM_{x_1}T_{-y_1}\\
&\quad -T_{y_1}M_{-x_1}XM_{x_1}T_{-y_1}\Big)\,dx_1\,dy_1\\
&= \lim_{h\to 0}\iint q(x_1,y_1)\left(\frac{e^{-2\pi i x_1 h}-1}{h}T_{y_1}M_{-x_1}T_hXM_{x_1}T_{-y_1}\right.\\
&\quad \left.+T_{y_1}M_{-x_1}\frac{T_h - I}{h}XM_{x_1}T_{-y_1}\right)dx_1\,dy_1\\
&= \iint q(x_1,y_1)(-2\pi i x_1(x_1,y_1)\cdot X + (x_1,y_1)\cdot(PX))\,dx_1\,dy_1\\
&= (-2\pi i x_1 q)\cdot X + q\cdot(PX)
\end{aligned}$$

in S^1-norm. This proves the claim.

Lemma 11.5. *Let $X \in S^{\mathscr{S}}$ and $q \in \mathscr{R}(\mathbb{R}^2)$. Then*

$$Q(q\cdot X) = (-2\pi i y_1 q)\cdot X + q\cdot(QX).$$

In particular, $Q(q\cdot X) \in S^1$.

Proof. This is proved by the same sort of reasoning as Lemma 11.4.

Lemma 11.6. *If $X \in S^{\mathscr{S}}$ and $q \in \mathscr{R}(\mathbb{R}^2)$, then*

$$\alpha(P(q\cdot X)) = \left(\frac{\partial}{\partial x} - 2\pi i y\right)\alpha(q\cdot X).$$

Proof. By Lemma 11.4, we have

$$P(q\cdot X) = (-2\pi i x_1 q)\cdot X + q\cdot(PX).$$

It follows that

$$\begin{aligned}\alpha(P(q\cdot X)) &= \frac{\partial\hat{q}}{\partial x}\alpha(X)+\hat{q}\alpha(PX)\\ &= \frac{\partial\hat{q}}{\partial x}\alpha(X)+\hat{q}\left(\frac{\partial}{\partial x}-2\pi i y\right)\alpha(X)\\ &= \frac{\partial}{\partial x}(\hat{q}\alpha(X))-2\pi i y\hat{q}\alpha(X)\\ &= \left(\frac{\partial}{\partial x}-2\pi i y\right)(\hat{q}\alpha(X))\\ &= \left(\frac{\partial}{\partial x}-2\pi i y\right)\alpha(q\cdot X).\end{aligned}$$

Lemma 11.7. *If $X\in S^{\mathscr{S}}$ and $q\in\mathscr{R}(\mathbb{R}^2)$, then*

$$\alpha(Q(q\cdot X))=\frac{\partial}{\partial y}\alpha(q\cdot X).$$

Proof. This is proved in the same way as Lemma 11.6.

12. The Harmonic Oscillator

Definition 12.1. The *harmonic oscillator* is the differential operator

$$H=P^2+Q^2.$$

Lemma 12.2. *If $X\in S^{\mathscr{S}}$ and $q\in\mathscr{R}(\mathbb{R}^2)$, then $H(q\cdot X)\in S^1$ and*

$$\alpha(H(q\cdot X))=\mathscr{D}\alpha(q\cdot X),$$

where

$$\mathscr{D}=\left(\frac{\partial}{\partial x}-2\pi i y\right)^2+\left(\frac{\partial}{\partial y}\right)^2.$$

Proof. By Lemma 11.4 and Lemma 11.5, we have

$$\begin{aligned}P^2(q\cdot X) &= P((-2\pi i x_1 q)\cdot X)+P(q\cdot(PX)) \qquad \text{and}\\ Q^2(q\cdot X) &= Q((-2\pi i y_1 q)\cdot X)+Q(q\cdot(QX)).\end{aligned}$$

So $P^2(q\cdot X), Q^2(q\cdot X)\in S^1$ and hence $H(q\cdot X)\in S^1$.

By Lemma 11.6 and Lemma 11.7, we have

$$\alpha(P^2(q\cdot X)) = \alpha(P((-2\pi i x_1 q)\cdot X))+\alpha(P(q\cdot(PX)))$$

$$
\begin{aligned}
&= \left(\frac{\partial}{\partial x} - 2\pi i y\right)\alpha((-2\pi i x_1 q)\cdot X) + \left(\frac{\partial}{\partial x} - 2\pi i y\right)\alpha(q\cdot(PX)) \\
&= \left(\frac{\partial}{\partial x} - 2\pi i y\right)(\alpha((-2\pi i x_1 q)\cdot X) + \alpha(q\cdot(PX))) \\
&= \left(\frac{\partial}{\partial x} - 2\pi i y\right)\left(\frac{\partial \hat{q}}{\partial x}\alpha(X) + \hat{q}\left(\frac{\partial}{\partial x} - 2\pi i y\right)\alpha(X)\right) \\
&= \left(\frac{\partial}{\partial x} - 2\pi i y\right)\left(\frac{\partial}{\partial x}(\hat{q}\alpha(X)) - 2\pi i y \alpha(X)\right) \\
&= \left(\frac{\partial}{\partial x} - 2\pi i y\right)^2 (\hat{q}\alpha(X)) \\
&= \left(\frac{\partial}{\partial x} - 2\pi i y\right)^2 \alpha(q\cdot X).
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha(Q^2(q\cdot X)) &= \frac{\partial}{\partial y}\alpha((-2\pi i y_1 q)\cdot X) + \frac{\partial}{\partial y}\alpha(q\cdot(QX)) \\
&= \frac{\partial}{\partial y}(\alpha((-2\pi i y_1 q)\cdot X) + \alpha(q\cdot(QX))) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial \hat{q}}{\partial y}\alpha(X) + \hat{q}\frac{\partial}{\partial y}\alpha(X)\right) \\
&= \frac{\partial^2}{\partial y^2}(\hat{q}\alpha(X)) \\
&= \frac{\partial^2}{\partial y^2}\alpha(q\cdot X).
\end{aligned}
$$

The result now follows.

Theorem 12.3. *There is a complete orthonormal set* $\{\varphi_k\}$ *in* $L^2(\mathbb{R})$ *such that*

$$H\varphi_k = -2\pi(2k+1)\varphi_k.$$

Proof. See [3, Lemma 10.34].

Corollary 12.4. *For* $p > 1$,

$$H^{-1} \in S^p.$$

Proof. This follows from the p-series test.

13. A Versal Constant

For $\delta > 0$, let $B_\delta = \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} < \delta\}$. Fix a radial smooth function τ with support in B_1 and identically 1 in a neighborhood of 0. Set

$$V = \|\check{\tau}\|_1.$$

Set

$$\tau_\delta(x,y) = \tau(x/\delta, y/\delta).$$

Then

$$\begin{aligned}
\check{\tau}_\delta(x_1,y_1) &= \iint e^{2\pi i(xx_1+yy_1)}\tau_\delta(x,y)\,dx\,dy \\
&= \iint e^{2\pi i(xx_1+yy_1)}\tau(x/\delta, y/\delta)\,dx\,dy \\
&= \iint e^{2\pi i\delta(xx_1+yy_1)}\tau(x,y)\delta^2\,dx\,dy \\
&= \delta^2\check{\tau}(\delta x_1, \delta y_1).
\end{aligned}$$

So for any $\delta > 0$,

$$\begin{aligned}
\|\check{\tau}_\delta\|_1 &= \delta^2 \iint |\check{\tau}(\delta x_1, \delta y_1)|\,dx_1\,dy_1 \\
&= \delta^2 \iint |\check{\tau}(x_1,y_1)|\delta^{-2}\,dx_1\,dy_1 \\
&= \|\check{\tau}\|_1 \\
&= V.
\end{aligned}$$

14. The Heart of the Matter

14.1. *If* $X \in S^{\mathscr{S}}$, $\operatorname{tr}(X) = 0$ *and* $\varepsilon > 0$, *then there exists* $\rho \in L^1(\mathbb{R}^2)$ *with* $\hat{\rho} = 1$ *on a neighborhood of* $(0,0)$ *such that*

$$\left\|\rho \cdot X\right\|_{S_1} < \varepsilon.$$

Proof. Let $r = \sqrt{x^2 + y^2}$. Let

$$C_1 = \sup_{B_1} |\nabla \alpha(X)|.$$

Since $\alpha(X)$ is smooth and $\alpha(X)(0,0) = 0$, it follows from the mean value theorem that

$$|\alpha(X)(x,y)| \le C_1 r \qquad \text{on } B_1.$$

Let

$$\begin{aligned} C_2 &= \sup_{B_1} |\Delta\alpha(X)| \\ D_1 &= \sup_{B_1} |\tau| \\ D_2 &= \sup_{B_1} |\nabla\tau| \\ D_3 &= \sup_{B_1} |\Delta\tau|. \end{aligned}$$

Then

$$\begin{aligned} \mathscr{D}(\tau_\delta \alpha(X)) &= \left(\Delta - 4\pi i y \frac{\partial}{\partial x} - 4\pi^2 y^2\right)(\tau_\delta \alpha(X)) \\ &= (\Delta\tau_\delta)\alpha(X) + 2\nabla\tau_\delta \cdot \nabla\alpha(X) + \tau_\delta \Delta\alpha(X) \\ &\quad -4\pi i y \left(\frac{\partial \tau_\delta}{\partial x}\alpha(X) + \tau_\delta \frac{\partial}{\partial x}\alpha(X)\right) \\ &\quad -4\pi^2 y^2 \tau_\delta \alpha(X). \end{aligned}$$

Therefore, on B_δ, we have by the Cauchy–Schwarz inequality that

$$\begin{aligned} |\mathscr{D}(\tau_\delta \alpha(X))| &\le [|\Delta\tau_\delta||\alpha(X)| + 2|\nabla\tau_\delta||\nabla\alpha(X)| + |\tau_\delta||\Delta\alpha(X)| \\ &\quad +4\pi|y|\left(\left|\frac{\partial\tau_\delta}{\partial x}\right||\alpha(X)| + |\tau_\delta|\left|\frac{\partial}{\partial x}\alpha(X)\right|\right) \\ &\quad +4\pi^2|y|^2|\tau_\delta||\alpha(X)|] \\ &\le (|\Delta\tau_\delta||\alpha(X)| + 2|\nabla\tau_\delta||\nabla\alpha(X)| + |\tau_\delta||\Delta\alpha(X)| \\ &\quad +4\pi r(|\nabla\tau_\delta||\alpha(X)| + |\tau_\delta||\nabla\alpha(X)|) + 4\pi^2 r^2 |\tau_\delta||\alpha(X)|) \\ &\le (\delta^{-2} D_3 C_1 r + 2\delta^{-1} D_2 C_1 + D_1 C_2 \\ &\quad +4\pi r(\delta^{-1} D_2 C_1 r + D_1 C_1) + 4\pi^2 r^2 D_1 C_1 r) \\ &\le (\delta^{-2} D_3 C_1 r + \delta^{-1}(2D_2 C_1 + 4\pi D_2 C_1) + (D_1 C_2 + 4\pi D_1 C_1 + 4\pi^2 D_1 C_1)) \end{aligned}$$

$$\leq (A_1\delta^{-2}r + A_2\delta^{-1} + A_3).$$

Moreover, $\mathscr{D}(\tau_\delta\alpha(X))$ is supported in B_δ. Fix $p \in (1,2)$. Then

$$\begin{aligned}\|\mathscr{D}(\tau_\delta\alpha(X))\|_p &\leq A_1\delta^{-2}\left(\iint_{B_\delta} r^p\,dx\,dy\right)^{1/p} + A_2\delta^{-1}\left(\iint_{B_\delta} dx\,dy\right)^{1/p}\\ &\quad + A_3\left(\iint_{B_\delta} dx\,dy\right)^{1/p}\\ &\leq (2\pi)^{1/p}\left(A_1\delta^{-2}\left(\frac{\delta^{p+2}}{p+2}\right)^{1/p} + A_2\delta^{-1}\left(\frac{\delta^2}{2}\right)^{1/p} + A_3\left(\frac{\delta^2}{2}\right)^{1/p}\right)\\ &= (2\pi)^{1/p}\left(A_1(p+2)^{-1/p} + A_2 2^{-1/p}\right)\delta^{2/p-1} + A_3 2^{-1/p}\delta^{2/p}.\end{aligned}$$

Therefore,

$$\lim_{\delta\to 0}\mathscr{D}(\tau_\delta\alpha(X)) = 0 \qquad \text{in } L^p(\mathbb{R}^2).$$

If $p^{-1} + p'^{-1} = 1$, then we have

$$\begin{aligned}\lim_{\delta\to 0} H(\check{\tau}_\delta\cdot X) &= \lim_{\delta\to 0}\Theta(\mathscr{D}(\tau_\delta\alpha(X))) \qquad \text{(by Theorem 6.4 and Lemma 12.2)}\\ &= \Theta\left(\lim_{\delta\to 0}\mathscr{D}(\tau_\delta\alpha(X))\right) \qquad \text{(by Theorem 9.1)}\\ &= 0 \qquad \text{in } S^{p'}.\end{aligned}$$

By Corollary 12.4, $H^{-1} \in S^p$. So by Theorem 4.1, we have

$$\begin{aligned}\lim_{\delta\to 0}\check{\tau}_\delta\cdot X &= H^{-1}(\lim_{\delta\to 0} H(\check{\tau}_\delta\cdot X))\\ &= 0 \qquad \text{in } S^1.\end{aligned}$$

Therefore, there exists $\delta_0 > 0$ such that $\left\|\check{\tau}_{\delta_0}\cdot X\right\|_{S^1} < \varepsilon$. Take $\rho = \check{\tau}_{\delta_0}$. Then $\hat{\rho} = 1$ in a neighborhood of $(0,0)$ and $\left\|\rho\cdot X\right\|_{S^1} < \varepsilon$.

Proof. of the Main Theorem. Now, if $X \in S^1$ and $\operatorname{tr}(X) = 0$, by Theorem 10.2 we can find $X' \in S^{\mathscr{S}}$ such that $\left\|X - X'\right\|_{S^1} < \frac{\varepsilon}{2V}$ and $\operatorname{tr}(X') = 0$. Then by Lemma 14.1 we can find $\rho \in L^1(\mathbb{R}^2)$ such that $\left\|\rho\cdot X'\right\|_{S^1} < \frac{\varepsilon}{2}$. Thus

$$\begin{aligned}\left\|\rho\cdot X\right\|_{S^1} &= \left\|\rho\cdot[X' - (X' - X)]\right\|_{S^1}\\ &\leq \left\|\rho\cdot X'\right\|_{S^1} + \left\|\rho\cdot(X' - X)\right\|_{S^1}\\ &\leq \frac{\varepsilon}{2} + \left\|\rho\right\|_1\left\|X - X'\right\|_{S^1}\\ &< \varepsilon.\end{aligned}$$

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