ON THE BLUE OF THE SCALE PARAMETER OF THE GENERALIZED
PARETO DISTRIBUTION

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Abstract. In this article, we will study the linear estimation of the scale parameter of the
generalized Pareto distribution (GPD) which has the probability density function (p.d.f.)

\[ f(x) = \begin{cases} 
  \sigma^{-1}(1 - rx/\sigma)^{1/r - 1}, & r \neq 0 \\
  \sigma^{-1}\exp(-x/\sigma), & r = 0.
\end{cases} \]

We first derive the expected value, variances and covariances of the order statistics from the
GPD. Then proceed to find the best linear unbiased estimates of the scale parameter \( \sigma \) based
on a few order statistics selected from a complete sample or a type-II censored sample. Results
of some chosen cases were tabulated.

Introduction

We consider the random variable \( X \) having the generalized Pareto distribution (GPD)
with a probability density function (p.d.f.)

\[ f(x) = \begin{cases} 
  \sigma^{-1}(1 - rx/\sigma)^{1/r - 1}, & r \neq 0 \\
  \sigma^{-1}\exp(-x/\sigma), & r = 0.
\end{cases} \]

and the cumulative distribution function (c.d.f)

\[ F(x) = \begin{cases} 
  1 - (1 - rx/\sigma)^{1/r}, & r \neq 0 \\
  1 - \exp(-x/\sigma), & r = 0.
\end{cases} \]

where \( \sigma \) and \( r \) are, respectively, the scale and shape parameter. The range of \( x \) is
\( 0 \leq x < \infty \) for \( r \leq 0 \) and \( 0 \leq x \leq \sigma/r \) for \( r > 0 \).

Clearly, the standardized random variable \( U = X/\sigma \) has the p.d.f.

\[ f(u) = \begin{cases} 
  (1 - ru)^{1/r - 1}, & r \neq 0 \\
  \exp(-u), & r = 0.
\end{cases} \]

and the c.d.f.

\[ F(u) = \begin{cases} 
  1 - (1 - ru)^{1/r}, & r \neq 0 \\
  1 - \exp(-u), & r = 0.
\end{cases} \]
The range of \( u \) is \( 0 \leq u \leq \infty \) for \( r \leq 0 \) and \( 0 \leq u \leq 1/r \) for \( r > 0 \).

The Pareto distribution was first proposed by Pareto (1897). The generalized Pareto distribution was introduced by Pickands (1975). Maximum likelihood estimation of generalized Pareto was discussed by Davison (1984) and Smith (1984, 1985). Hosking and Wallis (1987) derived estimators of parameters and quantiles by the method of moments and the method of probability weighted moments and restrict attention to the case \(-1/2 < r < 1/2\), for both practical and theoretical reasons. Also, they point out a close connection between generalized Pareto and generalized extreme-value distributions (GEV) with equal value for their shape parameters, and, as Hosking, Wallis, and Wood (1985) remarked, applications of the GEV distributions, particularly in hydrology usually involve the case \(-1/2 < r < 1/2\).

For \( r < 0 \) the distribution has a heavy Pareto-type upper tail. The case \( r = 0 \) is the exponential distribution for which many statistical techniques are available. When \( r > 0 \) the distribution has an upper endpoint at \( \sigma/r \). For \( r = 0 \), \( r = 0.5 \) and \( r = 1 \) the distribution is triangular and uniform respectively.

The applications of the GPD include the use in the analysis of extreme events, in the modeling of large insurance claims, as a failure-time distribution in reliability studies.

Consider a sample of \( n \) order statistics

\[
X_{(1)} < X_{(2)} < \cdots < X_{(n)}
\]

(1.1)

from a continuous distribution with the p.d.f. \( f[(x-\mu)/\sigma]/\sigma \) and the c.d.f. \( F[(x-\mu)/\sigma] \), where \( \mu \) and \( \sigma \) are the location and scale parameters respectively.

Lloyd (1952) obtained the generalized least-square estimators of the location and scale parameters using order statistics. We would like to find the best linear unbiased estimators (BLUE’s) of the parameters based on \( k(\leq n) \) order statistics selected from (1.1).

Let \( E' \) and \( V \) denote the mean vector and the covariance matrix of the standardized order statistics \( (X_{(n_i)} - \mu)/\sigma \) in \( X^* \), where

\[
X^* = (X_{(n_1)} < X_{(n_2)} < \cdots < X_{(n_k)}), \quad 1 \leq n_1 \leq \cdots \leq n_k \leq n
\]

(1.2)

and \( 1' \) denotes the unit vector with \( k \) components.

The BLUE \( \sigma^+ \) of \( \sigma \) while \( \mu \) is known is

\[
\sigma^+ = \frac{E'V^{-1}X^*}{E'V^{-1}E} - \frac{E'V^{-1}1}{E'V^{-1}E} \cdot \mu = \sum_{i=1}^{k} b_i X_{(n_i)} - b_0 \mu
\]

\[
\text{Var} (\sigma^+) = \frac{\sigma^2}{E'V^{-1}E},
\]

\[
\text{RE}(\sigma^+) = \frac{\text{VAR}[\text{BLUE based on (1.1)]}}{\text{VAR}(\sigma^+)}.
\]

Chan and Cheng (1982) obtained an algorithm to compute the coefficients, variance, and relative efficiency of the best linear unbiased estimators of the location, \( \mu \), and/or
scale, $\sigma$, parameters based on $k$ optimally chosen order statistics from a sample which could be a censored sample.

In estimating a parameter by a linear combination of $k(< n)$ chosen order statistics from a sample, the set of $k$ order statistics which gives the minimum variance among all possible choices of $k$ order statistics is the preferred one.

**The BLUE**

In order to find the BLUE, we have to obtain the expected values, the variances and the covariances of the order statistics. Hence we obtain the p.d.f. of order statistics and the joint p.d.f. of any two order statistics and then derive the moments of order statistics.

Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution of continuous type having a p.d.f. $f(x)$ that is positive, provided $a < x < b$. The form of (1.1) denotes the order statistics from this sample. For convenience, we let $X_1 = Y_1, X_2 = Y_2, \ldots, X_n = Y_n$.

The p.d.f of $Y_i$, the $i$th order statistics is given by

$$g_i(y_i) = C \cdot [F(y_i)]^{i-1}[1-F(y_i)]^{-1}f(y_i), \quad a < y_i < b,$$

where $C = \frac{n!}{(i-1)!(n-i)!}$.

The joint p.d.f. of any two order statistics $Y_i$ and $Y_j$, ($Y_i < Y_j$) is given by

$$g_{ij}(y_i, y_j) = D[F(y_i)]^{i-1}[F(y_j)-F(y_i)]^{j-i-1}[1-F(y_j)]^{n-j}f(y_i)f(y_j), \quad a < y_i < y_j < b,$$

where $D = \frac{n!}{(i-1)!(j-1)!(n-j)!}$.

The expected value of the cross product of any two order statistics $Y_i$ and $Y_j$ is

$$E(Y_i Y_j) = \int_{a}^{b} \int_{y_i}^{b} y_i y_j g_{ij}(y_i, y_j) dy_j dy_i,$$

for $a < y_i < y_j < b$, and $1 \leq i \leq j \leq n$.

The expected value of the cross product of any two order statistics $Y_i$ and $Y_j$ is

$$E(Y_i Y_j) = \int_{a}^{b} \int_{y_i}^{b} y_i y_j g_{ij}(y_i, y_j) dy_j dy_i,$$

for $a < y_i < y_j < b$, and $1 \leq i \leq j \leq n$.

When the shape parameter $r > 0$, the range of $x$ is not the same as that of the shape parameter $r < 0$. First, we consider the case $r > 0$. The range of $x$ is $0 \leq x \leq \sigma/r$. Thus, the p.d.f. of $Y_i$, from the GPD, is given by

$$g_i(y) = C \cdot [1-(1-ry/\sigma)^{1/r}]^{i-1}[1-ry/\sigma]^{-i/r} \cdot (1/\sigma)[1-ry/\sigma]^{(1/r)-1},$$

Therefore,

$$E(Y_i^a) = \int_{0}^{\sigma/r} y^a g_i(y) dy$$

$$= \frac{C}{\sigma} \cdot \int_{0}^{\sigma/r} y^a [1-(1-ry/\sigma)^{1/r}]^{i-1}(1-ry/\sigma)^{(n-i-r+1)/r} dy.$$
Using transformation and binomial expansion to express the form \((1 - t^r)^a\), we then have

\[
E(Y_i^a) = \sum_{k=0}^{\alpha} (-1)^k (a_k) \cdot C \cdot (\sigma/r)^a \cdot B[n - i + rk + 1, i]
\]

where \(B[a, b]\) stands for a beta function.

Hence, the first moment and the second moment of \(Y_i\) are

\[
E(Y_i) = \frac{n!}{(i-1)!(n-i)!} \cdot \frac{\sigma}{r} \cdot (B[n-i+1, i] - B[n-i+r+1, i]), \tag{2.5}
\]

and

\[
E(Y_i^2) = \frac{n!}{(i-1)!(n-i)!} \cdot \frac{\sigma^2}{r^2} \cdot (B[n-i+1, i] - 2B[n-i+r+1, i] + B[n-i+2r+1, i]). \tag{2.6}
\]

The joint p.d.f. of \(Y_i, Y_j\) is

\[
g_{ij}(x, y) = D[1 - (1 - rx/\sigma)^{1/r}]^{i-1}[(1 - rx/\sigma)^{1/r} - (1 - ry/\sigma)^{1/r}]^{j-1} \cdot (1 - ry/\sigma)^{n-j} \cdot (1/\sigma)(1 - rx/\sigma)^{(1-r)/r},
\]

then

\[
E(Y_i^a Y_j^b) = \int_0^{\sigma/r} D \cdot x^a \cdot y^b \cdot [1 - (1 - rx/\sigma)^{1/r}]^{i-1} \cdot (1/\sigma^2)(1 - rx/\sigma)^{(1-r)/r} \cdot Q dx,
\]

where

\[
Q = \int_x^{\sigma/r} y^i [(1 - rx/\sigma)^{1/r} - (1 - ry/\sigma)^{1/r}]^{j-1} \cdot (1 - ry/\sigma)^{(n-j-r+1)/r} dy.
\]

After the substitutions, then we can obtain

\[
E(Y_i^a Y_j^b) = D \cdot (\sigma/r)^{i+j} \sum_{a=0}^{i} \cdot \sum_{b=0}^{j} (-1)^{a+b} (s_a)(t_b) \cdot B[n-j+br+1, j-i] \cdot B[n-i+(a+b)r+1, i]. \tag{2.8}
\]

When \(s = t = 1\), the

\[
E(Y_i Y_j) = \frac{D \cdot \sigma^2}{r^2} \cdot \left[ B[n - j + 1, j - i] \cdot B[n - i + 1, i] - B[n - i + r + 1, i] - B[n - j + r + 1, j - i] \cdot B[n - i + r + 1, i] - B[n - i + 2r + 1, i] \right]. \tag{2.9}
\]

Secondly, we consider the case \(r < 0\). Then \(0 \leq x < \infty\). Thus,

\[
E(Y_i^a) = \int_0^{\infty} y^a g_i(y) dy
= \frac{C}{\sigma} \cdot \int_0^{\infty} y^a [1 - (1 - ry/\sigma)^{1/r}]^{i-1} (1 - ry/\sigma)^{(n-i-r+1)/r} dy,
\]
After the substitutions, we can rewrite
\[ E(Y_i^a) = C \cdot (\sigma/r)^a \sum_{k=0}^{a} (-1)^k (a_k) \cdot B[n - i + kr + 1, i]. \]

When \( a = 1 \), then
\[ E(Y_i) = C \cdot (\sigma/r) \cdot \left\{ B[n - i + 1, i] - B[n - i + r + 1, i] \right\}. \quad (2.10) \]

When \( a = 2 \), then
\[ E(Y_i^2) = C \cdot (\sigma/r)^2 \cdot \left\{ B[n - i + 1, i] - 2 \cdot B[n - i + r + 1, i] + B[n - i + 2r + 1, i] \right\}. \quad (2.11) \]

And then,
\[ E(Y_i Y_j^t) = \int_{x}^{\infty} D \cdot x^r \left[ (1 - rx/\sigma)^1/r - (1 - ry/\sigma)^1/r \right] \cdot F \cdot dx \quad (2.12) \]

where
\[ F = \int_{x}^\infty (\sigma/r)^{t+1}/(1 - rx/\sigma)^{1/r - 1} \cdot (1 - ry/\sigma)^{(n-j-r+1)/r} dy \]

After the substitutions, then we can also obtain
\[ E(Y_i^2 Y_j^t) = D \cdot (\sigma/r)^{t+s} \sum_{a=0}^{s} \sum_{b=0}^{t} (-1)^{a+b} (s_a)(b_b) \cdot B[n - j + br + 1, j-i] B[n-i+(a+b)r+1, i]. \quad (2.13) \]

When \( s = t = 1 \), then \( E(Y_i Y_j) \) is equal to the form of (2.9) for the shape parameter \( r < 0 \).

We standardize the random variable of order statistics and express the beta function in terms of gamma functions, then (2.5), (2.6), and (2.9) can be rewritten as follows:
\[ E(Y_i/\sigma) = \frac{1}{r} \cdot \left\{ 1 - \frac{\Gamma(n+1)\Gamma(n-i+r+1)}{\Gamma(n-i+1)\Gamma(n+r+1)} \right\}. \quad (2.14) \]
\[ E(Y_i^2/\sigma^2) = \frac{1}{r^2} \cdot \left\{ \frac{\Gamma(n+1)\Gamma(n-i+2r+1)}{\Gamma(n-i+1)\Gamma(n+2r+1)} \right\}, \quad (2.15) \]
\[ E(Y_i Y_j/\sigma^2) = \frac{1}{r^2} \cdot \left\{ \frac{\Gamma(n+1)\Gamma(n-j+r+1)\Gamma(n-i+2r+1)}{\Gamma(n-i+1)\Gamma(n-i+r+1)\Gamma(n+2r+1)} \right\}, \quad (2.16) \]

where \( \Gamma(a) \) stands for the gamma function. And the covariance of \( Y_i \) and \( Y_j \) is given by
\[ \text{COV}(Y_i/\sigma, Y_j/\sigma) = \frac{1}{r^2} \cdot \left\{ \frac{\Gamma(n+1)\Gamma(n-j+r+1)\Gamma(n-i+2r+1)}{\Gamma(n-j+1)\Gamma(n-i+r+1)\Gamma(n+2r+1)} \right\} - \frac{\Gamma^2(n+1)\Gamma(n-j+r+1)\Gamma(n-i+r+1)}{\Gamma^2(n+r+1)\Gamma(n-i+1)\Gamma(n-r+1)}. \quad (2.17) \]
And (2.14) is the $i$th element component of the mean vector $E'$ and (2.17) is the $(i, j)$th element of the covariance matrix $V$.

Let

$$E' = (E_i), \quad i = 1, \ldots, n, \quad V = \frac{1}{r^2}(V_{ij}), \quad i, j = 1, \ldots, n,$$

denote the means and the variances of the order statistics of the standardized r.v., say, $U(1) < U(2) < \cdots U(n)$.

We rewrite the mean vector and the covariance matrix as

$$E[U(i)] = (1/r) \cdot [1 - A(i)], \quad 1, \ldots, n,$$

and

$$\text{COV}[U(i), U(j)] = (V_{ij}) \cdot (1/r^2) = A_j D_i \cdot (1/r^2), \quad i \leq j; \quad i, j = 1, \ldots, n,$$

where

$$D_i = 1/[A_i B_i] - A_i, \quad A_i = \Gamma(n + 1)\Gamma(n - i + r + 1), \quad B_i = \Gamma(n + 2r + 1)\Gamma(n - i + 2r + 1).$$

We apply the theorem in Graybill (1983) [p.198-199], then the inverse of the matrix $V$, denoted by $\Omega = (\Omega_{ij})$, is a diagonal matrix of the second type having elements:

$$\Omega_{i,1} = \frac{-\left(D_{i-1}A_{i+1} - D_i A_{i-1}\right)}{r^2 (D_{i-1}A_i - D_i A_{i-1})(D_i A_{i+1} - D_{i+1} A_i)},$$

$$\Omega_{i,1} = \Omega_{1,i} = \frac{1}{r^2} \left(D_{i-1}A_i - D_i A_{i-1}\right),$$

$$\Omega_{i,j} = \frac{(n - i + 1) \cdot (n - i + r + 1) \cdot B_i}{-r^2}, \quad i = 1, \ldots, n,$$

where $A_0 = D_{n+1} = 1$, $A_{n+1} = D_0 = 0$.

Applying the generalized least-square method and the results in the above Section, we can obtain the BLUE of $\sigma$ based on $k$ order statistics. When $\mu$ is known, the BLUE $\sigma^+$ of $\sigma$ is given by

$$\sigma^+ = \frac{E' V^{-1} X^*}{E' V^{-1} E},$$

and,

$$\text{VAR}(\sigma^+) = \frac{\sigma^2}{E' V^{-1} E},$$

and, $\text{RE}(\sigma^+) = \frac{\text{VAR(BLUE based on (1.1))}}{\text{VAR}(\sigma^+)}$.

Using the algorithm of Chan and Cheng (1982), we obtain the estimate for the finite sample cases.
The BLUE is the linear unbiased estimate which has the minimum variance of the estimator among the set of all \( nC_k \) estimators based on the \( k \) selected order statistics in (1.1). When \( n \) and/or \( k \) is large, this algorithm for finding the minimum variance of the estimator is time-consuming even with the use of a computer.

Here, the results for the complete sample andensored sample are summarized in tables. The sample size \( n = 12 \); \( k = 2 \); number of left censored observations = \( IL \); number of right censored observations = \( IR \); the optimum ranks \( n_i \); the coefficient, the variance, and the relative efficiencies for the BLUE are all given in Table 1 for \( r = 0.2 \).

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We will now give an example below:

**Example:** Let \( n = 12, \ k = 2, \ IL = 1, \ IR = 3 \). The optimum ranks and the coefficients can easily be found from Table 1. The BULE of \( \sigma \) is

\[
\sigma^+ = 0.379265 \cdot x(6) + 0.696398 \cdot x(9).
\]

And the \( \text{RE}(\sigma^+) \) is 98.1897%.

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Since too many censored cases are there, we cannot list all of them here. A few selected cases for $n = 11, 12$ and $k = 3$ are listed in Table 2. However the program for other different cases can be obtained from the author.

**Conclusion**

In this paper we can see from the tables that most of the relative efficiencies are quite high for the BLUE of $\sigma$. From the results it is easy to obtain the linear estimate of the scale parameter $\sigma$ of the GPD for the various shape parameters. We must notice one point that the moments of order statistics may not exist if the argument of the gamma function is negative integer.

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**References**


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