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# A different approach for multi-level distance labellings of path structure networks

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**Abstract**. For a positive integer $k$, a radio $k$-labelling of a simple connected graph $G = (V, E)$ is a mapping $f$ from the vertex set $V(G)$ to a set of non-negative integers such that $|f(u)-f(v)| \geqslant k+1-d(u,v)$ for each pair of distinct vertices $u$ and $v$ of $G$, where $d(u,v)$ is the distance between $u$ and $v$ in $G$. The *span* of a radio $k$-coloring $f$, denoted by $span_f(G)$, is defined as $\max_{v\in V(G)} f(v)$ and the *radio $k$-chromatic number of* $G$, denoted by $rc_k(G)$, is $\min_f\{ span_f(G)\}$ where the minimum is taken over all radio $k$-labellings of $G$. In this article, we present results of radio $k$-chromatic number of path $P_n$ for $k \in \{n-1, n-2, n-3\}$ in different approach but simple way.

***Keywords.*** Frequency assignment problem, radio $k$-coloring, radio $k$-chromatic number, span

## 1 Introduction

In the frequency assignment problem (FAP) the task is to assign radio frequencies to transmitters at different locations without causing interference and also minimizing the span. FAP plays an important role in wireless networking and is a well-studied interesting problem. Due to rapid growth of wireless networks and to the relatively scarce radio spectrum the importance of FAP is growing significantly. In 1980, Hale [5] has modelled FAP as a Graph labelling problem (in particular as a generalized graph coloring problem) which is an active area of research now. In 1988, Roberts [6] proposed an FAP with two levels of interference which Griggs adapted to graphs and extended to a more general graph problem of distance-constrained labelling [4] as follows:

Let $j_1, j_2, \ldots, j_d \in \mathbb{N}$ be any integers, traditionally assumed that $j_1 \geq j_2 \geq \ldots \geq j_t$. An $L(j_1, j_2, \ldots, j_d)$-labelling of a graph $G(V(G), E(G))$ is an assignment $f : V(G) \to \{0, 1, 2, \ldots\}$ such that $|f(u) - f(v)| \geq j_t$ for all pairs of vertices $u$, $v$ whose distance in $G$ is equal to $t$ $(t = 1, 2, \ldots, d)$. The *span* of an $L(j_1, j_2, \ldots, j_d)$-labelling $f$ is the largest label assigned by $f$ to the vertices of $G$. The $\lambda_{j_1,j_2,\ldots,j_d}(G)$ is defined as the smallest possible span taken over all $L(j_1, j_2, \ldots, j_d)$-labellings of $G$, i.e.,

$$\lambda_{j_1,j_2,\ldots,j_d}(G) = \min_f \max_{v\in V(G)} f(v).$$

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Motivated by the problem of channel assignment to FM radio stations of Federal Communications Commission of the United States, Chartrand et al. [7, 8] introduced the following concept of radio $k$-labellings of graphs.

For a positive integer $k$, a radio $k$-labelling of a simple connected graph $G = (V, E)$ is a mapping $f$ from the vertex set $V(G)$ to a set of non-negative integers such that $|f(u) - f(v)| \geqslant k+1-d(u,v)$ for each pair of distinct vertices $u$ and $v$ of $G$, where $d(u,v)$ is the distance between $u$ and $v$ in $G$. The *span* of a radio $k$-coloring $f$, denoted by $span_f(G)$, is defined as $\max\limits_{v\in V(G)} f(v)$ and the *radio $k$-chromatic number of* $G$, denoted by $rc_k(G)$, is $\min\limits_{f}\{span_f(G)\}$ where the minimum is taken over all radio $k$-labellings of $G$. For some specific values of $k$ there are specific names for radio $k$-labellings as well as the radio $k$-chromatic number in the literature. For $k = \text{diam}(G)$, $\text{diam}(G)-1$ $\text{diam}(G)-2$, the $rc_k(G)$ is known as *radio number* ($\mathsf{rn}(G)$), *antipodal number* ($ac(G)$) and *nearly antipodal number* ($ac'(G)$) of $G$, respectively.

The $rc_k(G)$ for $k = n-1, n-2, n-3$ are known and each of them are in different papers [14, 11, 15]. We obtained an improve lower bound for $rc_k(P_n)$ for any $k$ and use this bound to determine $rc_k(P_n)$, for $k = n-1, n-2, n-3$ that unifying the proof given in [14, 11, 15]. The lower bound presented in this paper is used to prove the above conjecture for $k = n-4$. This method can be extended to prove the above conjecture for most of the cases and this will be reported in a subsequent paper.

The importance and complexities to prove the existing results on $P_n$ motivated us to combine and reprove all the results in a simple way. In this article, we determine the exact value of $\mathsf{rn}(P_n)$ in Section 3, then we give a lower bound of $rc_k(P_n)$ in Section 4, and use it in Section 5 to find the exact value $ac(P_n)$, $ac'(P_n)$ in a simple but different approach.

## 2 Preliminaries

**Definition 1.** For a path $P_n = (v_0, v_1, \dots, v_{n-1})$, a middle vertex is called a *centroid* of $P_n$. For even integer $n$, $P_n$ has two centroids, namely, $v_{\frac{n}{2}-1}$ and $v_{\frac{n}{2}+1}$ whereas for even integer $n$, $P_n$ has unique centroid $v_{\frac{n-1}{2}}$. We always fix a centroid $s = v_t$ for paths $P_n$. Then the left and right branch, denoted by $L(P_n)$ and $R(P_n)$, are the $(v_0, v_{t-1})$ section and $(v_{t+1}, v_{n-1})$ section of $P_n$, respectively.

**Definition 2.** Let $s$ be the centroid of $n$-vertex path $P_n$. Define the *level* of $u \in V(P_n)$ (with respect to $s$) by $L(u) = d(s,u)$. A vertex $u$ of $T$ is in level $l$ if $L(u) = l$. For two paths $P : (u,s)$ and $Q : (v,s)$, define $\phi(u,v)$ is the length of $P \cap Q$. The *weight* of $P_n$,denoted by $w(P_n)$, is defined by

$$w(P_n) = \sum_{u\in V(P_n)} L(u).$$

It is known that $w(P_n)$ does not depend on the choice of $s$.

**Lemma 1.** Let $P_n$ be an $n$-vertex path and $s$ be a centroid of $P_n$. Then for distinct vertices $u, v \in V(P_n)$ the following hold.

(a) $w(P_n) = \begin{cases} \frac{n^2}{4}, & \text{if } n \text{ is even;} \\ \frac{n^2-1}{4}, & \text{if } n \text{ is odd.} \end{cases}$

(b) There exists a sequence $u_0, u_1, u_2, \dots, u_{n-1}$ of vertices of $P_n$ such that no two consecutive vertices are in same branch of $P_n - \{s\}$.

(c) If $s \in \{u_0, u_{n-1}\}$ and $\{u_0, u_{n-1}\}/S$ is not in maximal cardinality branch with respect to the centroid $s$, then then there exist no alternating sequence $\gamma : u_0, u_1, u_2, \ldots, u_{n-1}$ of vertices of $P_n$ with respect to the centroid $S$.

## 3 Radio labelling of Path

Let $V(P_n) = \{v_0, v_1, v_2, \ldots, v_{n-1}\}$ be the vertex set of an $n$-vertex path $P_n$. A radio labelling is a one-to-one function. On the other hand, any one-to-one integral function $f$ on $V(P_n)$, with $0 \in f(V)$, induces an ordering of $V(P_n)$, which is a line-up of the vertices with increasing images. We denote this ordering by $U(f)$, where $V(P_n) = U(f) = \{u_0, u_1, u_2, \ldots, u_{n-1}\}$ with

$$0 = f(u_0) < f(u_1) < f(u_2) < \ldots < f(u_{n-1}).$$

Notice, if $f$ is a radio labelling, then the span of $f$ is $f(u_{n-1})$. Now from the radio conditions we have the following for $0 \leq i \leq n-2$

$$f(u_{i+1}) - f(u_i) \geq n - d(u_i, u_{i+1}). \tag{3.1}$$

To make it an equality, we add a positive quantity $J_f(u_i, u_{i+1})$, called *jump* of $f$ from $u_i$ to $u_{i+1}$, in right hand side of the inequality (3.1). Therefore,

$$f(u_{i+1}) - f(u_i) = n - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1}).$$

Summing up these $n-1$ equations, we have

$$\begin{aligned} f(u_{n-1}) &= \sum_{i=0}^{n-2}[f(u_{i+1}) - f(u_i)] + f(u_0) \\ &= \sum_{i=0}^{n-2}[n - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1})] + f(u_0) \\ &\geq n(n-1) - 2\sum_{i=0}^{n-2} L_S(u_i) + L_S(u_0) + L_S(u_{n-1}) \\ &+ \sum_{i=0}^{n-2}[J_f(u_i, u_{i+1}) + 2\phi(u_i, u_{i+1})] + f(u_0) \qquad (3.2) \\ &= n(n-1) - 2w(P_n) + f(u_0) + L_S(u_0) + L_S(u_{n-1}) + \sigma(f) \end{aligned}$$

where $\sigma(f) = \sum_{i=0}^{n-2} \sigma_f(u_i, u_{i+1})$ and $\sigma_f(u_i, u_{i+1}) = J_f(u_i, u_{i+1}) + 2\phi(u_i, u_{i+1})$. Here *total jump* $J(f) = \sum_{i=0}^{n-2} J_f(u_i, u_{i+1})$. So the relation between $\sigma(f)$ and $J(f)$ is $\sigma(f) = J(f) + 2\sum_{i=0}^{n-2} \phi(u_i, u_{i+1})$.

**Definition 3.** A radio $k$-labeling $f$ is said to be *alternating labelling* if two consecutive colored vertices are in different branches of $P_n$ with respect to a centroid of $P_n$.

**Observation 1.** From above discussion, we can observe the following points

(a) $\phi(u, v) \geq 0$, equality hold if $u$ and $v$ are in different branch or the centroid $S \in \{u, v\}$.

(b) $\sigma(f) \geq J(f) \geq 0$, equality hold if $f$ is an alternating labelling.

(c) $L_S(u_0) + L_S(u_{n-1}) \geq 1$.

(d) If $n$ is odd and $L_S(u_0) + L_S(u_{n-1}) = 1$, then at least one of $\left(\frac{n-1}{2}\right)$-level vertices is in $\{u_2, u_3, \ldots, u_{n-2}\}$.

In the following we give a general lower bound for radio number of $P_n$ in terms of first colored vertex $u_0$, last colored vertex $u_{n-1}$, weight $w(P_n)$ and total jump $J(f)$.

**Theorem 1.** Let $P_n$ be an $n$-vertex path and $f$ be any radio labelling of $P_n$ with first and last colored vertices $u_0$ and $u_{n-1}$, respectively. Then

$$\text{span}_f(P_n) \geq n(n-1) - 2w(P_n) + f(u_0) + L_S(u_0) + L_S(u_{n-1}) + J(f),$$

where $J(f)$ is total jump in $P_n$ under the radio labelling $f$.

**Proof:** From Equation (3.2), above result follows immediately.

The above theorem immediately gives the lower bound of $\mathsf{rn}(P_n)$ when $n$ is even.

**Theorem 2.** Let $P_n$ be a path of even number of vertices $n$. Then $\mathsf{rn}(P_n) \geq \frac{n^2}{2} - n + 1$.

**Proof:** From Lemma 1, $w(P_n) = \frac{n^2}{2}$. Also $L_S(u_0) + L_S(u_{n-1}) \geq 1$, $f(u_0) \geq 0$ and $J(f) \geq 0$. Thus from Theorem 1, by simple calculation, above result follows immediately.

In the next lemma, we determine the jump from $u_i$ to $u_{i+1}$ and then $u_{i+1}$ to $u_{i+2}$.

**Lemma 2.** If $u_i$ and $u_{i+2}$ are in the same branch of $P_n$ and $u_{i+1}$ is in a different branch of $P_n$, then

$$J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq \max\{2L(u_{i+1}) + 2\phi(u_i, u_{i+2}) - n, 0\}.$$

**Proof:** $f(u_{i+1}) - f(u_i) = n - d(u_i, u_{i+1}) + J_f(u_i, u_{i+1}) = n - L(u_i) - L(u_{i+1}) + 2\phi(u_i, u_{i+1})$ and $f(u_{i+2}) - f(u_{i+1}) = n - d(u_{i+1}, u_{i+2}) + J_f(u_{i+1}, u_{i+2}) = n - L(u_{i+1}) - L(u_{i+2}) + 2\phi(u_{i+1}, u_{i+2})$. Summing up we get

$$\begin{aligned} f(u_{i+2}) - f(u_i) = 2n - L(u_i) - L(u_{i+2}) - 2L(u_{i+1}) \\ + J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \end{aligned}$$

where $J_f(u_t, u_{t+1}) = J_f(u_t, u_{t+1}) + 2\phi(u_t, u_{t+1})$ for $t = i, i+1$. On the other hand, since $f$ is a radio labeling, we have

$$\begin{aligned} f(u_{i+2}) - f(u_i) &\geq n - d(u_i, u_{i+2}) \\ &= n - L(u_i) - L(u_{i+2}) + 2\phi(u_i, u_{i+2}). \end{aligned}$$

Combining the two expressions above, we get $J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq 2L(u_{i+1}) + 2\phi(u_i, u_{i+2}) - n$. Since the value $J_f(u_t, u_{t+1}) \geq 0$ for $t = i, i+1$, the result follows immediately.

From here to onwards, the *highest level vertices with respect to a centroid* $S$ we mean the vertices which are $\left\lfloor \frac{n}{2} \right\rfloor$ distances from the centroid $S$.

**Observation 2.** If $u_i$ and $u_{i+2}$ are in the same branch of $P_n$ and $u_{i+1}$ is in a different branch of $P_n$, then from Lemma 2, we can observe the following points

(a) *Jump* is due to highest level vertices.

(b) If $n$ is even and the vertex $u_{i+1}$ is in highest level, then $J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq 2$ when $S \notin \{u_i, u_{i+2}\}$.

(c) If $n$ is odd and the vertex $u_{i+1}$ is in highest level, then $J_f(u_i, u_{i+1}) + J_f(u_{i+1}, u_{i+2}) \geq 1$ when $S \notin \{u_i, u_{i+2}\}$.

(d) If $f$ is not an alternating labelling, then $\sigma(f) \geq 2$ because there exist vertices $u_t$ and $u_{t+1}$ which are in same branch i.e., $\phi(u_t, u_{t+1}) \geq 1$.

Notice that, *jump* is due to highest level vertices. When $n$ is odd and $L_S(u_0)+L_S(u_{n-1}) = 1$, from Observation 1($d$) and Lemma 2, we have $J(f) \geq 1$ if $f$ is alternating labelling. Again if $f$ is not alternating then $\sigma(f) \geq 2$. Thus in each case whether $f$ is alternating or not, $L_S(u_0) + L_S(u_{n-1}) + \sigma(f) \geq 2$ if $n$ is odd. Using this fact, following we give a theorem on lower bound of $rn(P_n)$ when $n$ is odd.

**Theorem 3.** Let $P_n$ be a path of odd number of vertices $n$. Then $\text{span}_f(P_n) \geq \frac{(n-1)^2}{2} + 2$.

**Proof :** Let $f$ be any radio labelling of $P_n$. Using Lemma 1, the weight of $P_n$ is $w(P_n) = \frac{n^2-1}{4}$. Then from Theorem 1, $\text{span}_f(G) \geq \frac{(n-1)^2}{2} + f(u_0) + L_S(u_0) + L_S(u_{n-1}) + \sigma(f) \geq \frac{(n-1)^2}{2} + 2$.

The lower bound given in Theorem 2 and Theorem 3 are coincide with the exact value of $\mathsf{rn}(P_n)$ which has been shown in Section 5.

## 4 Lower bound of radio $k$-chromatic number of path $P_n$ with $k \leq n-2$

**Definition 4.** Let $f : E \to F$ be a mapping from a set $E$ to a set $F$. For a set $A \subset E$, we call the mapping $f|_A : A \to F$ as the *restriction of $f$ on $A$*.

**Lemma 3.** For an $n$-vertex path $P_n$, $rc_k(P_n) \geq \mathsf{rn}(P_{k+1})$ for any sub-path $P_{k+1}$ of $P_n$ with $k < n-1$.

**Proof:** Let $f$ be a radio $k$-labelling of $P_n$. Here the diameter of $P_{k+1}$ is $k$ with $k < d$. Thus $V(P_{k+1}) \subset V(P_n)$. Let $g = f|_{V(P_{k+1})}$ be the restriction of $f$ on $V(P_{k+1})$. Then $\text{span}_f(P_n) \geqslant \text{span}_g(P_{k+1})$ and this is true for any radio $k$-labelling of $P_n$ and its restriction $g = f|_{V(P_{k+1})}$. Since the diameter of $P_{k+1}$ is $k$, we obtain the required result.

Following we give a general lower bound for $rc_k(P_n)$ where $k \leq n-2$.

**Theorem 4.** For an $n$-vertex path $P_n$ and a positive integer $k \leq n-2$,

$$rc_k(P_n) \geq \begin{cases} \frac{k^2}{2} + 2, & \text{if } k \text{ is even;} \\ \frac{(k+1)^2}{2} - k, & \text{if } k \text{ is odd.} \end{cases}$$

**Proof:** Consider a sub-path $P_{k+1}$ of $P_n$. Now find the radio number of $P_{k+1}$, using Theorem 2 and Theorem 3. Then using Lemma 2, we get the result by simple calculation.

Now we study some special types of radio $k$-labelling for $k = n-2$, $k = n-3$ separately and try to improve the general lower bound for them.

**Definition 5.** A radio labelling $f$ of $P_n$ is said to be an *optimal radio labelling*, if

$$\text{span}_f(P_n) = \begin{cases} \frac{n^2}{2} - n + 1, & \text{if } n \text{ is even;} \\ \frac{(n-1)^2}{2} + 2, & \text{if } n \text{ is odd.} \end{cases}$$

The following lemma give that the position of first and last colored vertices for an optimal radio labelling $f$.

**Lemma 4.** Let $f$ be an optimal radio labelling of $P_n$. Then $f$ has following properties:

(a) The labelling $f$ is alternating labelling.

(b) First and last colored vertices are adjacent and one of them must be a centroid of $P_n$. Moreover, when $n$ is even then first and last colored vertices are the centroids of $P_n$

**Theorem 5.** For an $n$-vertex path $P_n$,

$$ac(P_n) \geq \begin{cases} \frac{n^2}{2} - 2n + 4, & \text{if } n \text{ is even;} \\ \frac{(n-1)^2}{2} - n + 3, & \text{if } n \text{ is odd.} \end{cases}$$

**Proof:** *Case-I: $n$ is even.* In this case $k = n - 2$ is also even. Hence from Theorem 4 result follows immediately.

*Case-II: $n$ is odd.* In this case $k = n - 2$ is odd. There are two $(k+1)$-vertex sub-paths $P^1_{k+1}$ and $P^2_{k+1}$ of $P_n$. Two sub-paths $P^1_{k+1}$ and $P^2_{k+1}$ are obtained from $P_n$ by removing a vertex from right end and left end of $P_n$, respectively. Let $f$ be any radio $k$-labelling of $P_n$. For each $i \in \{1, 2\}$, let $g_i : V\left(P^i_{k+1}\right) \to \{0, 1, 2, \ldots\}$ be a restriction of $f$ on $P^i_{k+1}$. So $g_i$ is a radio labelling of $P^i_{k+1}$. Here $\text{span}_f(P_n) \geq \max_i \text{span}_{g_i}\left(P^i_{k+1}\right)$.

**Our claim:** All $g_i$ are not optimal and there exist one $g_t$, $t \in \{1, 2\}$ such that $\text{span}_{g_t} \geq rn\left(P^t_{k+1}\right) + 1$.

If possible, let both $g_1$ and $g_2$ are optimal. So both $g_1$ and $g_2$ satisfy the properties of Lemma 4. With out loss of generality we consider the first and last colored vertices $u_0$, $u_{n-1}$ be the centroids of $P^1_{k+1}$. Then $\{u_0, u_{n-1}\} \neq \{C_1, C_2\}$, where $C_i$ denote the centroids of $P^2_{k+1}$. Thus from Lemma 4, we get $g_2$ is not optimal, which is a contradiction.

**Theorem 6.** For an $n$-vertex path $P_n$,

$$ac'(P_n) \geq \begin{cases} \frac{n^2}{2} - 3n + 7, & \text{if } n \text{ is even;} \\ \frac{(n-1)^2}{2} - 2n + 7, & \text{if } n \text{ is odd.} \end{cases}$$

where $ac'(P_n)$ denotes the nearly antipodal number of $P_n$.

**Proof:** Let $k = n - 3$. Then there are three $(k+1)$-vertex sub-paths $P^i_{k+1}$, $i = 1, 2, 3$ of $P_n$. Let $P^1_{k+1}$ be the sub-path removing two consecutive vertices from right end of $P_n$; $P^2_{k+1}$ be the sub-path removing two end vertices of $P_n$; and $P^3_{k+1}$ be the sub-path removing two consecutive vertices from left end of $P_n$. Let $f$ be any radio $k$-labelling of $P_n$. For each $i \in \{1, 2, 3\}$, let $g_i : V\left(P^i_{k+1}\right) \to \{0, 1, 2, \ldots\}$ be a restriction of $f$ on $P^i_{k+1}$. So $g_i$ is a radio labelling of $P^i_{k+1}$ for each $i \in \{1, 2, 3\}$. Here $\text{span}_f(P_n) \geq \max_i \text{span}_{g_i}\left(P^i_{k+1}\right)$.

**Our claim:** All $g_i$ are not optimal and there exist one $g_t$, $t \in \{1, 2, 3\}$ whose span

$$\text{span}_{g_t}\left(P^t_{k+1}\right) \geq \begin{cases} rn\left(P^t_{k+1}\right) + 1, & \text{if } k \text{ is even;} \\ rn\left(P^t_{k+1}\right) + 2, & \text{if } k \text{ is odd.} \end{cases}$$

*Case-I : $n$ is odd.* In this case three centroid $S_i$, $i = 1, 2, 3$ of $P^i_{k+1}$ are consecutive vertices of $P_n$. So it is not possible for all the restriction function $g_i$ to satisfy the properties of Lemma 4. So there exist one $g_t$, $t \in \{1, 2, 3\}$ which is not optimal and $\text{span}_{g_t}(P^t_{k+1}) \geq \frac{k^2}{2} + 3$.

*Case-II : n is even.* Since $k+1 = n-2$ is even, so each $t \in \{1,2,3\}$ the path $P_{k+1}^t$ has exactly two centroids and let them $C_1^t$ and $C_2^t$. For each $t \in \{1,2,3\}$, define $C^t = \{C_1^t, C_2^t\}$. If $d(u_0, u_{n-1}) \geq 3$, then $L_S(u_0) + L_S(u_{n-1}) \geq 3$ for all $S \in \cup_{t=1}^3 C^t$. Thus in this case $\text{span}_{g_t}\left(P_{k+1}^t\right) \geq rn\left(P_{k+1}^t\right) + 2$ for each $t \in \{1,2,3\}$. Again if $d(u_0, u_{n-1}) \leq 2$, using Lemma 1(*e*) either $\text{span}_{g_1}\left(P_{k+1}^1\right) \geq rn\left(P_{k+1}^1\right) + 2$ or $\text{span}_{g_3}\left(P_{k+1}^3\right) \geq rn\left(P_{k+1}^3\right) + 2$.

So there exist one $g_t$, $t \in \{1,2,3\}$ whose $\text{span}_{g_t} \geq \frac{(k+1)^2}{2} - k + 2$.

# 5 Radio $k$-chromatic number of $P_n$ for $k = n-1, n-2, n-3$

Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of an $n$-vertex path $P_n$. In this section, we give the exact value of $\mathsf{rn}(P_n)$, $ac(P_n)$ and $ac'(P_n)$. The lower bound of $rn(P_n)$, $ac(P_n)$ and $ac'(P_n)$ are given in Theorems 2, 3, 5, and 6 respectively. To prove equality, our task is to find a radio $k$-labelling $f$ of $P_n$ with the span as specified in their respective lower bound theorem. Reader can easily verify that the following defined labellings $f$ satisfy radio $k$-condition and are optimal.

## 5.1 A radio labelling of $P_n$

Define a mapping $f : V(P_n) \to \{0,1,2,\ldots\}$ as follows:
**Case I : $n$ is even.** $f(v_i) = \frac{n}{2} + (i-1)(n-1)$, $1 \leq i \leq \frac{n}{2}$; $f\left(v_{\frac{n}{2}+1+j}\right) = j(n-1)$, $1 \leq j \leq \frac{n}{2} - 1$.
**Case I : $n$ is odd.** $f(v_1) = \frac{n+1}{2}$; $f\left(v_{\frac{n-1}{2}-i}\right) = n(i+1) + 2$, $0 \leq i \leq \frac{n-1}{2} - 2$; $f\left(v_{\frac{n+1}{2}}\right) = 0$; $f\left(v_{\frac{n-1}{2}+j}\right) = f(v_j) + \frac{n+1}{2}$, $2 \leq j \leq \frac{n-1}{2}$; $f(v_n) = \frac{n+1}{2} + 1$.

## 5.2 An antipodal labelling of $P_n$

Define a mapping $f : V(P_n) \to \{0,1,2,\ldots\}$ as follows:
**Case I : $n$ is even.** $f(v_1) = \frac{n}{2} - 1$; $f(v_i) = (n-1)(\frac{n}{2} - i) + 1$, $2 \leq i \leq \frac{n}{2} - 1$; $f\left(v_{\frac{n}{2}}\right) = \frac{n^2}{2} - 2n + 4$; $f\left(v_{\frac{n}{2}+j}\right) = \frac{n^2}{2} - 2n + 4 - f\left(u_{\frac{n}{2}+1-j}\right)$, $1 \leq j \leq \frac{n}{2}$.
**Case II : $n$ is odd.** $f(v_1) = \frac{3(n-1)}{2} + 1$; $f(v_2) = \frac{n-1}{2}$; $f(v_i) = i(n-2) - \frac{n-1}{2} + 2$, $3 \leq i \leq \frac{n-1}{2}$; $f\left(v_{\frac{n+1}{2}}\right) = n$; $f\left(v_{\frac{n+1}{2}+1}\right) = 0$; $f\left(v_{\frac{n-1}{2}+j}\right) = j(n-2) - n + 3$, $3 \leq j \leq \frac{n-1}{2}$; $f(v_n) = 1$.

## 5.3 An nearly antipodal labelling of $P_n$

Define a mapping $f : V(P_n) \to \{0,1,2,\ldots\}$ as follows:
**Case I : $n$ is even.** $f(v_1) = \frac{n}{2} - 2$; $f\left(v_{\frac{n}{2}-1-i}\right) = (i+1)(n-3) + 1$, $0 \leq i \leq \frac{n}{2} - 3$; $f\left(v_{\frac{n}{2}}\right) = \frac{n^2}{2} - 3n + 7$; $f\left(v_{\frac{n}{2}+1}\right) = 0$; $f\left(v_{\frac{n}{2}-1-j}\right) = \frac{n^2}{2} + j(n-3)$, $0 \leq j \leq \frac{n}{2} - 3$; $f(v_n) = \frac{n^2}{2} - \frac{7n}{2} + 9$.
**Case II : $n$ is odd.** $f(v_1) = \frac{n-1}{2}$; $f(v_2) = \frac{(n-1)^2}{2} - \frac{5(n-1)}{2} + 6$; $f(v_{3+i}) = \frac{3(n-1)}{2} + i(n-2)$, $0 \leq i \leq \frac{n-1}{2} - 4$; $f\left(v_{\frac{n-1}{2}+j}\right) = (n-2)j$, $0 \leq j \leq 1$; $f\left(v_{\frac{n-1}{2}+2}\right) = \frac{(n-1)^2}{2} - 2n + 7$; $f\left(v_{\frac{n-1}{2}+3+l}\right) = (n-2)(l+1) - 1$, $0 \leq l \leq \frac{n-1}{2} - 4$; $f(v_{n-1+m}) = \frac{n-1}{2} + m(n-2) - 1$, $0 \leq m \leq 1$.
Following theorem is the summery of radio $k$-chromatic number for each $k \in \{n-1, n-2, n-3\}$.

**Theorem 7.** Let $P_n$ be an $n$-vertex path $P_n$ and $l \in \{1,2,3\}$. Then radio $(n-l)$-chromatic number is given by

$$rc_{n-l}(P_n) = \begin{cases} \frac{n^2}{2} - l.n + 3l - 2, & \text{if } n \text{ is even;} \\ \frac{(n-1)^2}{2} - (l-1)(n-1) + 3\left\lfloor \frac{l}{3} \right\rfloor + 2, & \text{if } n \text{ is odd.} \end{cases}$$

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