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ASYMPTOTIC SOLUTIONS OF NONDIAGONAL LINEAR DIFFERENCE SYSTEMS

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Abstract. This paper, relying on dichotomic properties of the matrix difference system $W(n+1)=A(n)W(n)A^{-1}(n)$, gives conditions under which a perturbed system $y(n+1)=(A(n)+B(n))y(n)$, by means of a nonautonomous change of variables $y(n)=S(n)x(n)$, can be reduced to the form $x(n+1)=A(n)x(n)$. From this, a theory of asymptotic integration of the perturbed system follows, where the linear system $x(n+1)=A(n)x(n)$ is nondiagonal. As a consequence of these results, we prove that the diagonal system $x(n+1)=\Lambda(n)x(n)$ has a Levinson dichotomy iff system $W(n+1)=\Lambda(n)W(n)\Lambda^{-1}(n)$ has an ordinary dichotomy.

1. Introduction

In this paper we are concerned with the linear difference system

$$x(n+1)=A(n)x(n), \qquad n=1,2,\ldots, \tag{1}$$

where $(A(n))_{n=1}^{\infty}$ is a sequence of $r\times r$ invertible matrices. This last condition is assumed for all linear difference systems considered in this paper. The fundamental matrix of Sys. (1) is defined by

$$\Phi(n)=\prod_{m=1}^{n-1}A(m)=A(n-1)\cdots A(2)A(1), \Phi(1)=I,$$

where I is the identity matrix. The explicit calculation of $\Phi(n)$ is a difficult matter, unless $A(n)$ have a simple structure. For this reason an important problem in the theory of difference systems is concerned with the asymptotic integration of the perturbed system

$$y(n+1)=[A(n)+B(n)]y(n), \tag{2}$$

i.e. the representation of the fundamental matrix $\Psi(n)$ of the Sys. (2) into the form

$$\Psi(n)=(I+\rho(n))\Phi(n)$$

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where $\rho(n)$ is a "small" sequence [19, 3, 10, 9, 2]. The aim of this paper is to propose a method for solving this problem. We will show the existence of a change of variable

$$y(n) = (I + H(n))x(n), \tag{3}$$

reducing Sys. (2) to the form (1). This method leads to the problem of existence of solutions of the equation.

$$H(n+1) = A(n)H(n)A^{-1}(n) + B(n)A^{-1}(n) + B(n)H(n)A^{-1}(n) \tag{4}$$

vanishing at infinity. This will be obtained from the dichotomic and trichotomic properties of the linear system

$$W(n+1) = [(A^{-1}(n))^T \otimes A(n)]W(n), \tag{5}$$

called in the sequel the tensor equation associated to the equation (1) (for short, the tensor equation). In (5), C^T denotes the transpose matrix of C and $A \otimes B$ is the tensorial product of matrices A and B [7]. The main hypothesis we assume in the paper is the existence of an (h,k)-dichotomy for Sys. (5). This notion has shown its versatility in problems of asymptotic integration [14, 16, 17, 18, 20, 21]. In this paper we attempt to show that Sys. (5) is important in the study of asymptotic integration of Sys. (2). For instance, we prove that the diagonal system

$$x(n+1) = \Lambda(n)x(n), \Lambda(n) = \mathrm{diag}\{\lambda_1(n), \lambda_2(n), \ldots, \lambda_r(n)\}. \tag{6}$$

with a Levinson dichotomy [5] is equivalent to the system

$$W(n+1) = [\Lambda^{-1}(n))^T \otimes \Lambda(n)]W(n), \tag{7}$$

having an ordinary dichotomy. This connection between a Levinson dichotomy and ordinary dichotomies is new in the literature and we think is a notable result.

The study of the dichotomic properties of Sys. (7) has other advantages. In general Sys. (7) may have (h,k)-dichotomies allowing asymptotic formulas of the form

$$\Psi(n) = (I + o(h(n)))\Phi(n),$$

where $h(n)$ may have the property $\lim_{n\to\infty} h(n) = 0$. Concretely, a result of our research is the following.

Theorem A. *Let us assume that Sys.* (5) *has an ordinary dichotomy and*

$$\sum_{n=1}^{\infty} |A^{-1}(n)||B(n)| < \infty, \tag{8}$$

then there exists a change of variable $X(n) = S(n)Y(n)$ *with the property* $\lim_{n\to\infty} S(n) = I$*, reducing Sys.* (2) *to Sys.* (1) *and the fundamental matrix of Sys.* (2) *allows the asymptotic representation*

$$\Psi(n) = (I + o(1)\Phi(n).$$

Since ordinary dichotomies occur frequently in systems of the form (5) this result has a general character.

Although there are several asymptotic results asking a dichotomy condition on the Sys. (1) and an integrable condition for the sequence $\{B(n)\}$, none of them has the three following notable characteristics: (i) it is valid for general matrices A; (ii) it is needed only an ordinary dichotomy for Sys. (1); (iii) the asymptotic formula obtained is the best possible.

Perhaps, as a known result, the best asymptotic representation of the fundamental matrix of Sys. (2) is the discrete version of the Levinson's Theorem [5], given by Benzaid-Lutz [2], but only valid for almost diagonal systems of the form

$$y(n+1) = [\Lambda(n) + B(n)]y(n). \tag{9}$$

Theorem B. *If Sys.* (6) *has a Levinson dichotomy, and*

$$\sum_{n=1}^{\infty} |\Lambda^{-1}(n)||B(n)| < \infty.$$

then $\Psi(n)$ *has the asymptotic formula*

$$\Psi(n) = (I + o(1)) \prod_{m=1}^{n-1} \Lambda(m),$$

where $o(1)$ *is a sequence satisfying* $\lim_{n\to\infty} o(1)(n) = 0$.

This theorem is an extension of the Levinson asymptotic theorem for ordinary differential equations [5] to the discrete equation (9). Its proof, as well as a theory of asymptotic integration of Sys. (9), can be found in [2]. These results essentially rely on [6] and therefore its fundamental hypothesis is the dichotomic character of the coefficients of the diagonal Sys. (6) described by the discrete Levinson dichotomy [13-20, 22, 23]. Despite of the importance of the results exposed in [2], they do not give a method to study similar questions in Sys. (2) for a general nondiagonal Sys. (1).

In section 6, using the equivalence of a Levinson and an ordinary dichotomies, we will show that Theorem B follows from Theorem A. Consequently, Theorem A is a generalization of the Levinson's asymptotic theorem.

Finally, in section 7, emphasizing that our theory is developed for nondiagonal systems, we show that our results can deal with problems of asymptotic integration of Sys. (2) not considered under any hypotheses of theorems of papers [2, 3, 10].

2. Notations and Preliminaries

Let K denote the field of scalars R of C; $K^{r\times p}$ is the set of $r \times p$ matrices with coefficients of field K; we abbreviate $K^r = K^{r\times 1}$. Let $A = (a_{ij}) \in K^{r\times p}$, $B = (b_{ij}) \in$

$K^{h\times k}$. The tensor product $A\otimes B$ is defined as the block matrix

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1p}B \\ a_{21}B & a_{22}B & \cdots & a_{2p}B \\ \cdots & \cdots & \cdots\cdots \\ a_{r1}B & a_{r_2}B & \cdots & a_{rp}B \end{bmatrix}$$

If $A\in K^{r\times p}$ then by A_j we denote the j-column of matrix A. For a $A\in K^{r\times p}$ we denote by $\mathrm{vec}(A)$ the rp-vector [8] defined by

$$\mathrm{vec}(A)=\begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}\in K^{rp}.$$

Clearly the function $\mathrm{vec}(A)$ is an isomorphysm between linear spaces $K^{r\times p}$ and K^{rp}. A remarkable property linking the tensor product and the linear application vec is given by the Roth identity [22]

$$\mathrm{vec}(ABC)=(C^T\otimes A)\mathrm{vec}(B), \tag{10}$$

By $|A|$ we will denote the Frobenious norm of matrix A

$$|A|=\left\{\sum_{1\le i,j\le r}|a_{ij}|^2\right\}^{1/2},$$

for which we have the properties

$$|AB|\le|A||B|,\ \ |A|=|\mathrm{vec}(A)|,|A\otimes B|=|A||B|. \tag{11}$$

In this paper the letters h and k denote sequences $h,k:N\to(0,+\infty)$, where $N=\{1,2,3,\ldots,\ldots\}$. The following normed spaces will be frequently used

$$\ell_h^1[K^{r\times p}=\{B:N\to K^{r\times p};\sum_{n=1}^{\infty}H(n)^{-1}|B(n)|<\infty\},$$

and

$$\ell_h^\infty[K^{r\times p}]=\{B:N\to K^{r\times p};\mathrm{sup}_{n\ge1}|h(n)^{-1}B(n)|<\infty\}.$$

For these spaces, respectively, we define the norms

$$|B|_h^1=\sum_{n=1}^{\infty}h(n)^{-1}|B(n)|,|B|_h^\infty=\mathrm{sup}_{n\ge1}|h(n)^{-1}B(n)|.$$

Standard arguments show that $(\ell_h^1[K^{r\times p}]_r|_h^1)$ and $(\ell_h^\infty[K^{r\times p}],\cdot||_h^\infty)$ are complete spaces. Frequently, these spaces will be abbreviated by ℓ_h^1 and ℓ_h^∞.

3. Discrete Dichotomies

The following dichotomic notion was introduced in [20, 21] (see also [4, 11, 12, 13, 15, 18] for the theory and applications to ordinary differential equations).

Definition 1. Let $C : N \to K^{r\times r}$.

$$x(n+1) = C(n)x(n), \quad x(n) \in K^r, \tag{12}$$

has an (h, k)-dichotomy iff there exists a projection matrix P such that

$$\begin{aligned} |\Psi(n)P\Psi^{-1}(m)| &\le Kh(n)h(m)^{-1}, \qquad n \ge m, \\ |\Psi(n)(1-P)\Psi^{-1}(m)| &\le Kk(n)k(m)^{-1}, \qquad m \ge n, \end{aligned} \tag{13}$$

where Ψ is a fundamental matrix of (12) and K is a positive constant. An (h, h)-dichotomy will be called an h-dichotomy.

From the above definition we observe that if $h(n) = k(n) = 1$, then the 1-dichotomies coincide with ordinary dichotomies. The (ρ^n, ρ^{-n})-dichotomies, $0 < \rho < 1$ corresponds to exponential dichotomies [1]. The notion of (h, k)-dichotomy involves many situations that cannot be studied by ordinary or exponential dichotomies. This may be appreciated even in the study of Sys. (1) when $A(n)$ is constant [14, 20, 21].

Definition 2. We say that an (h, k) dichotomy is uniformly compensated iff for a positive constant C we have

$$h(n)h(m)^{-1} \le Ck(n)k(m)^{-1}, \qquad n \ge m \ge 1. \tag{14}$$

If Sys. (1) has an (h, k)-dichotomy, then, simultaneously, this system has an h and a k-dichotomy, both with the same projection. These considerations are important in the theory of asymptotic integration of discrete systems [14, 20 ,21]. Our aim is not the study of asymptotic equivalence of difference systems, but the existence of certain solutions vanishing at infinity. Although, (h, k) dichotomies will frequently appear, we will handle them as an h-dichotomy in order to use the following property

$$\lim_{n\to\infty} h(n)^{-1}\Psi(n)P = 0, \tag{15}$$

a property that can always be obtained by modifying P in (13) [15].

The statement of our results require some notations. Let

$$\alpha(n) = h(n)h(n+1)^{-1}, B_h[0, \delta] = \{x \in \ell_h^\infty ; |x|_h^\infty \le \delta\}.$$

Lemma 1. *Consider the system*

$$y(n+1) = C(n)y(n) + f(n, y(n)), \qquad n \ge 1, \tag{16}$$

where (12) *has an h-dichotomy and the function* $f : N \times K^{r\times p} \to$, *has the following properties*

$$\textbf{C1:} \qquad \alpha(n)f(n,0) \in \ell_h^1[K^{r\times p}],$$
$$\textbf{C2:} \qquad |f(n,x) - f(n,y)| \le \rho(n)|x-y)|, \ \ \alpha\rho \in \ell_1^1[K].$$

Then for an n_0 *sufficiently large, Eq.* (16) *has a solution*

$$y : N_{n_0} \to K^{r\times p}, \quad N_{n_0} = \{n_0+1, n_0+2, \ldots\},$$

belonging to $B_h[0, 2^{-1}]$ *and satisfying*

$$\lim_{n\to\infty} h(n)^{-1}y(n) = 0. \tag{17}$$

Proof. As we have pointed out, we can take the projection P satisfying (15) We will abbreviate our text if we introduce the Green function

$$\Gamma(n,m) = \begin{cases} \Psi(n)P\Psi^{-1}(m), & \text{if } n \ge m, \\ -\Psi(n)(I-P)\Psi^{-1}(m), & \text{if } n > m, \end{cases}$$

From definition (13) and (14) we have the estimate

$$|h(n)^{-1}\Gamma(n,m+1)| \le KC\alpha(m)h(m)^{-1}, \tag{18}$$

for any values of m and n. Let us consider the following operator

$$\mathcal{D}(y)(n) = \sum_{m=n_0}^{\infty} \Gamma(n,m+1)f(m,y(m)).$$

we obtain from **C2** and (18) the estimate

$$|h(n)^{-1}\mathcal{D}(y)(n)| \le KC\{|\alpha f(\cdot,0)|_h^1 + |\alpha\rho|_1^1|y|_h^\infty\}.$$

From assumptions **C1** and **C2** we obtain for a large n_0

$$\sup_{n\ge n_0}|h(n)^{-1}\mathcal{D}(y)(n)| \le 2^{-1}.$$

Thus $\mathcal{D} : \ell_h^\infty[K^{r\times r}] \to \ell_h^\infty[K^{r\times r}]$. By similar tokens we obtain

$$|\mathcal{D}(x) - D(y)|_h^\infty \le KC\sum_{m=n_0}^{\infty} \alpha(m)\rho(m)|x-y|_h^\infty.$$

For a large n_0, condition **C2** implies $KC\sum_{m=n_0}^{\infty}\alpha(m)\rho(m) < 1$; henceforth, the contraction $\mathcal{D}$ has a unique fixed point y in the ball $B_h[0,2^{-1}]$. A straightforward calculation shows that y satisfies the Eq. (16). Because y is a fixed point of operator $\mathcal{D}$, we can write

$$y(n) = \left[\sum_{m=n_0}^{N-1} + \sum_{m=N}^{n-1} - \sum_{m=n}^{\infty}\right] \Gamma(n, m+1) f(m, y(m)), \qquad n_0 \leq N < n.$$

Since

$$\sum_{m=N}^{n-1} h(n)^{-1} |\Gamma(n, m+1)||f(m, y(m))| \leq KC \sum_{m=N}^{\infty} (h(m)^{-1}(\alpha(m) f(m, 0) + \alpha(m)\rho(m)|y|_h^{\infty}),$$

and

$$\sum_{m=N}^{\infty} h(n)^{-1} |\Gamma(n, m+1)||f(m, y(m))| \leq KC \sum_{m=N}^{\infty} (h(m)^{-1}(\alpha(m) f(m, 0) + \alpha(m)\rho(m)|y|_h^{\infty}),$$

then given $\varepsilon > 0$, there exists an integer N, such that $n > N$ implies

$$y(n) = \sum_{m=n_0}^{N-1} \Gamma(n, m+1) f(m, y(m))| + h(n) O(\varepsilon), |O(\varepsilon)| \leq \varepsilon.$$

Now, from (15) we can fix $n_0(\varepsilon) > N$ such that for $n > n_0(\varepsilon)$ we have

$$|h(n)^{-1} \sum_{m=n_0}^{N-1} \Gamma(n, m+1) f(m, y(m))| < \varepsilon.$$

Thus, for any ε, there exists an $n_0(\varepsilon)$, such that $n > n_0(\varepsilon)$ implies

$$|h^{-1}(n) y(n)| < \varepsilon.$$

4. Discrete Trichotomies

We need a result similar to Lemma 1 for Eq. (12) having an (h, k)-trichotomy.

Definition 3. We shall say that Sys. (12) has an (h, k)-trichotomy iff there exist three projection matrices P_s, P_u, P_c such that $P_s P_u = P_u P_s = 0$, $P_c = I - P_s - P_u$, such that

$$\begin{aligned} &|\Phi(n) P_s \Phi^{-1}(m)| \leq K h(n) h(m)^{-1}, \qquad n \geq m, \\ &|\Phi(n) P_u \Phi^{-1}(m)| \leq K k(n) k(m)^{-1}, \qquad m \geq n \\ &|\Phi(n) P_c \Phi^{-1}(m)| \leq K, \quad \text{for any } m, n. \end{aligned} \tag{19}$$

We shall deal with uniformly compensated trichotomies in the following sense

Definition 4. We shall say that the (h, k)-trichotomy (19) is uniformly compensated iff each pair of sequences (h, k) and $(h, 1)$ satisfy (14).

Similarly to Lemma 1 we can prove the following

Lemma 2. *Assume that Sys.* (12) *has a uniformly compensated (h, k)-trichotomy. Under conditions* **C1** *and* **C2***, for an n_0 large, Eq.* (16) *has a solution*

$$y : N_{n_0} \to K^{r\times p},\ \ N_{n_0} = \{n_0, n_0 + 1, n_0 + 2, \ldots\},$$

belonging to $B_h[0, 2^{-1}]$. In addition, if

C3 : $$\lim_{n\to\infty} h(n)^{-1}\Psi(n)P_s = 0,$$

then this solution satisfies (17).

Proof. Let us consider the operator

$$\begin{aligned}\mathcal{T}(y)(n) = &\sum_{m=n_0}^{n-1} \Psi(n)P_s\Psi^{-1}(m + 1)f(m, y(m)) - \sum_{m=n}^{\infty} \Psi(n)P_u\Psi^{-1}(m + 1)f(m, y(m))\\ &- \sum_{m=n}^{\infty} \Psi(n)P_c\Psi^{-1}(m + 1)f(m, y(m)).\end{aligned}$$

Using the compensation of the pairs (h, k) and $(h, 1)$ we obtain

$$|h(n)^{-1}\mathcal{T}(y)(n)| \le 2KC^2 \sum_{m=n_0}^{\infty} (h(m)^{-1}\alpha(m)|f(m, 0) + \alpha(m)\rho(m)|y|_h^\infty).$$

The properties **C1** and **C2** say that the right hand side of the this last inequality is less than 2^{-1} if n_0 is sufficiently large. Moreover,

$$|h(n)^{-1}(\mathcal{T}(x)(n) - \mathcal{T}(y)(n))| \le 2KC^2 \sum_{m=n_0}^{\infty} \alpha(m)\rho(m)|x - y|_h^\infty.$$

This estimate shows that $\mathcal{T}$ is a contraction if n_0 is large. Henceforth $\mathcal{T}$ has a fixed point in $B_h]0, 2^{-1}]$. Analogous to the proof of Lemma 1, we may prove that condition **C3** implies that this fixed point satisfies property (17).

5. Reduction of the Perturbed Equation

Performing the change of variable (3) in (2) we obtain

$$(I+H(n+1))(y(n+1)-A(n)y(n)) = [(A(n)+B(n))(I+H(n))-(I+H(n+1))A(n)]y(n).$$

The reduction of Sys. (2) to Sys. (1) requires that the sequence $H(n)$ be a solution of Eq. (4) and $I+H(n)$ be invertible for each n. To solve this problem, previously denoting $W(n) = \mathrm{vec}(H(n))$, we vectorize equation (4) [8]. Using the property (10) we obtain

$$\begin{aligned} W(n+1) = {} & [(A^{-1}(n))^T \otimes A(n)]W(n) + [(A^{-1}(n))^T \otimes B(n)]\mathrm{vec}(I) \\ & +[(A^{-1}(n))^T \otimes B(n)]W(n). \end{aligned} \tag{20}$$

Equations (4) and (20) are quivalent; each of them is called the reducing equation of Sys. (2). In the forthcoming subsections we will study (20) under different dichotomic properties of tensor equation (5).

5.1. The tensor equation as an (h, k)-dichotomy

Observe that Eq. (5) can exhibit interesting dichotomic situations, not necessarily characterized by an ordinary dichotomy.

Theorem 1. *If the difference equation* (5) *has a uniformly compensated* (h, k)*-dichotomy, the sequence* h *is bounded and*

C4 :
$$\sum_{m=1}^{\infty} h(m+1)^{-1}|A^{-1}\|B(m)| < \infty,$$

then there exists a change of variable $y(n) = (I + H(n))x(n)$ *reducing Eq.* (2) *to* (1). *Moreover, Sys.* (2) *has a fundamental matrix* $\tilde{\Psi}$ *satisfying*

$$\tilde{\Psi}(n) = (I + o(h(n))) \prod_{m=n_0}^{n-1} A(m). \tag{21}$$

Proof. The compensation of the pair (h, k) yields the definition of an h-dichotomy for Sys. (5), with a projection satisfying (15). From the boundedness of h and **C4** we obtain

$$\sum_{m=1}^{\infty} \max(h(m)^{-1}, 1)\alpha(m)|A^{-1}(m)\|B(m)| < \infty.$$

This property implies conditions **C1** and **C2** of Lemma 1 for

$$f(n, W(n)) = [(A^{-1}(n))^T \otimes B(n)]\mathrm{vec}(I) + [(A^{-1}(n))^T \otimes B(n)]W(n).$$

According to Lemma 1, Eq. (20) has solution

$$W : N_{n_0} \to K^{r^2}, |W|_h^\infty \le 2^{-1},$$

such that

$$\lim_{n\to\infty} h(n)^{-1}W(n) = 0.$$

Since $h(n)$ is bounded, then $\lim_{n\to\infty} W(n) = 0$. Let $H = \mathrm{vec}^{-1}(W)$. From (11) we have $\lim_{n\to\infty} H(n) = 0$; this implies that $I + H(n)$ is invertible for $n \geq n_0$, for a large n_0. Therefore, (20) implies that the change of variable (3) reduces (2) to the form (1) and the matrix

$$\tilde{\Psi}(n) = (I + H(n))^{-1} \prod_{m=n_0}^{n-1} A(n).$$

is a fundamental matrix of (2). From $\lim_{n\to\infty} h(n)^{-1}H(n) = 0$ we deduce the asymptotic formula (21).

For $h(n) = 1$, **C4** coincides with condition (8). Therefore, Theorem B follows as a corollary from Theorem 1.

5.2. The tensor equation has an (h,k)-trichotomy

Frequently the tensor equation (5) will have a trichotomic behavior.

Theorem 2. *Assume that Sys.* (5) *has a uniformly compensated* (h,k) *trichotomy and the projection* P_s *satisfies* **C3**. *Then under condition* **C4**, *there exists a change of variables* (3) *reducing Sys.* (2) *to* (1). *Moreover, Sys.* (2) *has a fundamental matrix* $\tilde{\Psi}$ *with the asymptotic representation* (21).

Proof. This follows from Lemma 2.

An important application of this theorem is given by the exponential trichotomy $(h(n), k(n)) = (\rho^n, \rho^{-n})$, where ρ is a fixed number and $0 < \rho < 1$. If $C3$ and $C4$ are satisfied for $h(n) = \rho_1^n$, $\rho < \rho_1 < 1$. In this case, from Theorem 2 we obtain that Sys. (2) allows a fundamental matrix $\tilde{\Psi}(n)$ satisfying

$$\tilde{\Psi}(n) = (I + o(\rho_1^n)) \prod_{m=n_0}^{n-1} A(m), \qquad \rho < \rho_1 < 1.$$

In particular, let us consider Sys. (1) with $A(n) = A_0 = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let $\rho > 0$ such that

$$\max\{\frac{|\lambda_i|}{|\lambda_j|}\frac{|\lambda_i|}{|\lambda_j|} < 1\} < \rho < 1.$$

It is easy to verify that system:

$$W(n+1) = [(A_0^{-1})^T \otimes A_0]W(n),$$

has a (ρ^n, ρ^{-n})-trichotomy. For system $y(n+1) = [A_0 + B(n)]y(n)$, the corresponding asymptotic formula for the fundamental matrix of this system has the form $\tilde{\Psi}(n) = (I + o(\rho_1^n))A_0^{n-1}$, if $\sum_{m=1}^{\infty} \rho_1^{-m}|B(m)|, \infty$, $\rho < \rho_1 < 1$.

6. Diagonal Systems

Definition 5. We say that the Sys. (6) has a Levinson dichotomy iff each pair of indexes (i, j), $1 \le i,\, j \le r$ satisfies either

$$\mathbf{L1}: \qquad \prod_{n=1}^{k} \frac{|\lambda_i(n)|}{|\lambda_j(n)|} \to 0, \text{ as } k \to \infty; \text{ and } \prod_{n=k_1}^{k_2} \frac{|\lambda_i(n)|}{|\lambda_j(n)|} \le M, \quad 0 \le k_1 \le k_2,$$

or

$$\mathbf{L2}: \qquad \prod_{n=k_1}^{k_2} \frac{|\lambda_i(n)|}{|\lambda_j(n)|} \ge M^{-1}, \qquad 0 \le k_1 \le k_2,$$

where M is some positive constant.

Let us define

$$h_i(n) = \prod_{m=1}^{n-1} |\lambda_i(m)|.$$

Using these notations, the definition of a Levinson dichotomy can be accomplished with the following conditions $\mathbf{L1}'$ and $\mathbf{L2}'$ instead of $\mathbf{L1}$ and $\mathbf{L2}$

$$\mathbf{L1}' \qquad \frac{h_i(n)}{h_j(n)} \to 0, \text{ as } n \to \infty; \text{ and } \frac{h_i(n)h_j(m)}{h_j(n)h_i(m)} \le M, \qquad 0 \le m \le n,$$

or

$$\mathbf{L2}' \qquad \frac{h_i(n)}{h_j(n)} \frac{h_j(m)}{h_i(m)} \ge M^{-1}, \qquad 0 \le m \le n.$$

Let us consider a fixed index i and define the projection matrix

$$P_i = \operatorname{diag}\{p_i^1, p_i^2, \dots, p_i^r\}$$

as follows: $p_i^j = 1$ if (i, j) satisfies $L1'$ and $p_i^j = 0$ if (i, j) satisfies $L2'$.

Proposition 1. *If Sys.* (6) *has Levinson dichotomy, then for each index i, Sys.* (6) *has an h_i-dichotomy satisfying the asymptotic condition*

$$\lim_{n\to\infty} h_i(n)^{-1} \Phi(n) P_i = 0.$$

We will show that Sys. (6) has Levinson dichotomy iff Sys. (7) has an ordinary dichotomy. Sys. (7) has the following fundamental matrix

$$T(n) = \prod_{m=1}^{n-1} \operatorname{diag}\left\{\frac{\lambda_1}{\lambda_1}, \frac{\lambda_2}{\lambda_1}, \dots, \frac{\lambda_r}{\lambda_1}, \dots, \frac{\lambda_1}{\lambda_r}, \frac{\lambda_2}{\lambda_r}, \dots, \frac{\lambda_r}{\lambda_r}\right\}(m).$$

Let us define

$$P = \operatorname{diag}\{P_1, P_2, \dots, P_r\}$$

Theorem 3. *The diagonal Sys.* (6) *has a Levinson dichotomy iff the Sys.* (7) *has an ordinary dichotomy with a diagonal projection matrix Q satisfying the asymptotic condition*

$$\lim_{n\to\infty} T(n)Q = 0. \tag{22}$$

Proof. From the definitions of $T(n)$, the definition of projection P and conditions $L1'$ and $L2'$, we obtain

$$\begin{aligned} |T(n)T^{-1}(m)P| &\le K, \qquad n \ge m,\\ |T(n)T^{-1}(m)(I-P)| &\le K, \qquad m \ge n, \end{aligned}$$

for K a positive constant. The definition of projections P_i implies the asymptotic property

$$\lim_{n\to\infty} T(n)P = 0.$$

Let us assume that Sys. (7) has an ordinary dichotomy with a diagonal projection $Q = \operatorname{diag}(\alpha_{11}, \alpha_{20}, \alpha_{r1}, \alpha_{12}, \alpha_{22}, \dots, \alpha_{r2}, \dots, \alpha_{1r}, \alpha_{2r}, \dots, \alpha_{rr})$, satisfying the asymptotic property (22). Then those indexes (i,j) for which $\alpha_{ij} = 1$ satisfy the condition $\mathbf{L1'}$ and those indexes (i,j) for which $\alpha_{ij} = 0$ satisfy the condition $\mathbf{L2'}$.

This last result says that Theorem A follows from Theorem B.

In section 7, it will be shown that Sys. (7) can have an (h,k)-trichotomy. In such case, for Sys. (9) we obtain the following

Theorem 4. *Let us assume that Sys.* (7) *has a uniformly compensated (h,k)-trichotomy, with h bounded, defined by projections, P_s, P_u and P_c such that $\lim_{t\to\infty} h(n)^{-1}T(n)P_c = 0$. If*

$$\sum_{m=1}^{\infty} h(m)^{-1} |\Lambda^{-1}(m)| |B(m)| < \infty, \tag{23}$$

then Sys. (9) *has a fundamental matrix $\Psi(n)$ with the asymptotic formula*

$$\Psi(n) = (I + o(h(n))) \prod_{m=1}^{n-1} \Lambda(n).$$

The hypothesis (23) imposes on the coefficient $B(n)$ more severe conditions than those of Theorem A, but the obtained asymptotic result is more precise (see Example 2 in the next section).

7. Examples

The first example of this section shows that the method of the tensor equation allows to obtain the asymptotic integration of Sys. (2), not satisfying the conditions required in [2, 3, 10]. The second establishes, that even under conditions of Theorem A, this method gives more precise formulas of asymptotic integration. The reader will appreciate in both examples that the use of (h, k)-dichotomies yields estimates that cannot be obtained, if we restrict our analysis to ordinary and exponential dichotomies only.

7.1. A system with no Levinson conditions

Let us consider the system

$$y(n+1) = \left[\begin{pmatrix} 1/2 & \mu(n) \\ 0 & 2 \end{pmatrix} + B(n)\right] y(n), \qquad n \geq 1, \tag{24}$$

where we assume that $\mu(n)$ is a bounded sequence. We consider that Sys. (24) is a perturbation of system

$$x(n+1) = \begin{pmatrix} 1/2 & \mu(n) \\ 0 & 2 \end{pmatrix} x(n). \tag{25}$$

The fundamental matrix of Sys. (25) is

$$\Phi(n) = \begin{pmatrix} 1/2^{n-1} & \nu(n-1) \\ 0 & 2^{n-1} \end{pmatrix},$$

whee $\nu(n)$ is defined by means of the recurrence relation

$$\nu(n) = 2^{-1}\nu(n-1) + 2^{n-1}\mu(n), \qquad \nu(0) = 0.$$

Assuming that $|\mu(n)| \leq M$, for all n, we obtain

$$|\nu(n)| \leq M 2^n. \tag{26}$$

For the projection matrix $P = \operatorname{diag}\{1, 0\}$, we obtain

$$|\Phi(n)P\Phi^{-1}(m)| = 2^{m,n} | \begin{pmatrix} 1 & \frac{\nu(m-1)}{2^{m-1}} \\ 0 & 0 \end{pmatrix} |.$$

Because of the estimate (26) we have

$$|\Phi(n)P\Phi^{-1}(m) \leq K 2^{m-n}, \qquad \forall n, \forall m. \tag{27}$$

Similarly

$$|\Phi(n)(I-P)\Phi^{-1}(m) \leq K 2^{n-m}, \qquad \forall n, \forall m. \tag{28}$$

Let us consider the tensor product

$$(A^{-1}(n))^T \otimes A(n) = \begin{bmatrix} 2A(n) & O \\ -\mu(n)A(n) & 2^{-1}A(n) \end{bmatrix}, \ A(n) = \begin{pmatrix} 1/2 & \mu(n) \\ 0 & 2 \end{pmatrix},$$

where O denotes the 2×2 null matrix. In this case Sys. (5) has the fundamental matrix

$$T(n)=\begin{bmatrix} 2^{n-1}\Phi(n) & O \\ \nu(n-1)\Phi(n) & 2^{1-n}\Phi(n)\end{bmatrix}.$$

and

$$T^{-1}(n)=\begin{bmatrix} 2^{1-n}\Phi^{-1}(n) & O \\ -\nu(n-1)\Phi^{-1}(n) & 2^{1-n}\Phi^{-1}(n)\end{bmatrix},$$

Let us consider the 4×4 projection matrices

$$P_s=\operatorname{diag}\{O,P\},\ P_u=\operatorname{diag}\{I-P,O\},\ P_c=\operatorname{diag}\{P,I-P\},$$

where $P=\operatorname{diag}\{1,0\}$. Then we have

$$T(n)P_sT^{-1}(m)=\begin{bmatrix} O & O \\ 2^{1-n}\nu(m-1)\Phi(n)P\Phi^{-1}(m) & \Phi(n)P\Phi^{-1}(m)\end{bmatrix}.$$

From (26) and (27) we obtain for some constant M

$$|T(n)P_sT^{-1}(m)|\le M2^{m-n},\qquad n\ge m. \tag{29}$$

The estimate (28) implies similar calculations, namely

$$|T(n)P_uT^{-1}(m)|\le M2^{n-m},\qquad m\ge n. \tag{30}$$

Working with projection P_c we obtain

$$T(n)P_cT^{-1}(m)=\begin{bmatrix} 2^{n-m}\Phi(n)P\Phi^{-1}(m) & O \\ V(n,m) & 2^{m-n}\Phi(n)P\Phi^{-1}(m)\end{bmatrix},$$

where

$$V(n,m)=-2^{1-m}\nu(n-1)\Phi(n)P\Phi^{-1}(m)+2^{1-n}\Phi(n)(I-P)\Phi^{-1}(m).$$

From (27) and (28) we obtain

$$|T(n)P_cT^{-1}(m)|\le M,\qquad \forall m,n. \tag{31}$$

Thus, Sya. (25) has an ordinary dichotomy with projection $Q=P_s+P_c$, since (29), (30) imply $|T(n)QT^{-1}(m)|\le M2^{m-n}$, $n\ge m$, and (31) implies $|T(n)(I-Q)T^{-1}(m)|\le M2^{m-n}$, $m\ge n$. Therefore, from the corollary of Theorem 1, Sys. (2) has a fundamental matrix satisfying (21), if the sequence $B(n)$ satisfies (8). This result does not follow from Theorem A nor from the results of papers [2, 3, 10]. Let $2^{-1}<\rho<1$, then estimates (29)-(31) say that Sys. (25) has a $(\rho^n,2^n)$-trichotomy with projections P_s, P_u and P_c, satisfying property **C3**. According to Theorem 4, Sys. (24) has the asymptotic integration

$$\tilde\Psi(n)=(I+o(\rho^n))\prod_{m=n_0}^{n-1}A(n),$$

if **C4** is satisfied for $h(n) = \rho^n$.

7.2. An example under Levinson conditions

Let us consider Sys. (6) with

$$y(n+1) = (\mathrm{diag}\{1/n, n\} + B(n))y(n), \tag{32}$$

where the correspoinding. Sys. (1) had the fundamental matrix

$$\Phi(n) = \mathrm{diag}\{\frac{1}{(n-1)!}, (n-1)!\},$$

and Eq. (5) has the fundamental matrix

$$T(n) = \mathrm{diag}(1, ((n-1)!)^2, ((n-1)!)^{-2}, 1).$$

According to Theorem A, if $nB(n) \in \ell^1$, Sys. (32) has the following asymptotic

$$\Psi(n) = (I + o(1))\Phi(n) \tag{33}$$

This formula does not reflect the properties of the sequence $B(n)$; for a sequence $B(n)$ satisfying $(n-1)!B(n) \in \ell^1$ or $((n-1)!)^2 B(n) \in \ell^1$, Theorem A gives the same asymptotic formula (33). The method of the tensor equation in conjunction with the notion of (h,k)-dichotomies make possible to make explicit the difference between these sequences in the asymptotic integration of Sys. (32).

We will use the definition of projection matrices P_s, P_u, P_c given in the first example. We can prove the following estimates

$$\begin{aligned}
|T(n)P_sT^{-1}(m)| &\le K\Big(\frac{(m-1)!}{(n-1)!}\Big)^2, \qquad n \ge m,\\
|T(n)P_uT^{-1}(m)| &\le K\Big(\frac{(n-1)!}{(m-1)!}\Big)^2, \qquad m \ge n,\\
|T(n)P_cT^{-1}(m)| &\le K, \ \text{for all } m, \text{ and } n.
\end{aligned}$$

These estimates and Theorem 3 imply that Eq. (5) has an (1,1) trichotomy. Then for Sys. (32), (33) is valid if $nB(n) \in \ell^1$. On the other hand, Eq. (5) has $(((n-1)!)^{-1}, (n-1)!)$-trichotomy. If $n!B(n) \in \ell^1$, then the fundamental matrix of Sys. (32) has the asymptotic formula

$$\Psi(n) = (I + o(1/n-1)!))\mathrm{diag}\{\frac{1}{(n-1)!}, (n-1)!\}.$$

Finally, Eq. (5) has a $(((n-1)!)^{-2}, ((n-1)!)^2)$-trichotomy. If $((n-1)!)^2 B(n) \in \ell^1$, then the fundamental matrix of Sys. (32) has the asymptotic formula

$$\Psi(n) = (I + o(1/((n-1)!)^2))\mathrm{diag}\{\frac{1}{(n-1)!}, (n-1)!\}.$$

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