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ON TRAPEZOID INEQUALITY VIA A GRÜSS TYPE RESULT AND APPLICATIONS

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Abstract. In this paper, we point out a Grüss type inequality and apply it for special means (logarithmic mean, identric mean etc $\cdots$) and in Numerical analysis in connection with the classical trapezoid formula.

1. Introduction

In 1935, G. Grüss (see for example [1, p. 296]), proved the following integral ineqality which gives an approximation for the integral of a product in terms of the product of integrals:

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be two integrable mappings so that $\varphi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a, b]$, where φ, Φ, γ, Γ are real numbers. Then we have:*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right| \le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \quad (1.1)$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ can not be replaced by a smaller one.

For a simple proof of this fact as well as for extensions, generalizations, discrete variants and other associated material, see [1, p. 296], and the papers [2]-[7] where further references are given.

In this paper, we point out a different Grüss type inequality and apply it for special means (logarithmic mean, identric mean, etc $\cdots$) and in Numerical Analysis in connection with the classical trapezoid formula.

2. A Grüss' Type Inequality

We start with the following result of Grüss' type.

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Theorem 2. *Let $f, g : [a,b] \to \mathbb{R}$ be two integrable mappings. Then we have the following Grüss' type inequality.*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right|$$
$$\leq \frac{1}{b-a}\int_a^b \left|\left(f(x) - \frac{1}{b-a}\int_a^b f(y)dy\right) \cdot \left(g(x) - \frac{1}{b-a}\int_a^b g(y)dy\right)\right| dx. \quad (2.1)$$

The inequality (2.1) *is sharp.*

Proof. First of all, let us observe that

$$\begin{aligned}
I &:= \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(y)dy\right) \cdot \left(g(x) - \frac{1}{b-a}\int_a^b g(y)dy\right) dx \\
&= \frac{1}{b-a}\int_a^b \left(f(x)g(x) - g(x) \cdot \frac{1}{b-a}\int_a^b f(y)dy - f(x) \cdot \frac{1}{b-a}\int_a^b g(y)dy\right. \\
&\quad \left.+\frac{1}{b-a}\int_a^b f(y)dy \cdot \frac{1}{b-a}\int_a^b g(y)dy\right)dx \\
&= \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b g(x)dx \cdot \frac{1}{b-a}\int_a^b f(y)dy \\
&\quad -\frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(y)dy + \frac{1}{b-a}\int_a^b f(y)dy \cdot \frac{1}{b-a}\int_a^b g(y)dy \\
&= \frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b g(x)dx \cdot \frac{1}{b-a}\int_a^b f(x)dx.
\end{aligned}$$

On the other hand, by the use of modulus properties, we have

$$|I| \leq \frac{1}{b-a}\int_a^b \left|\left(f(x) - \frac{1}{b-a}\int_a^b f(y)dy\right) \cdot \left(g(x) - \frac{1}{b-a}\int_a^b g(y)dy\right)\right| dx$$

and the inequality (2.1) is proved.

Choosing $f(x) = g(x) = sgn(x - \frac{a+b}{2})$, the equality is satisfied in (2.1).

The following corollaries follow immediately:

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) having the first derivative $f' : (a,b) \to \mathbb{R}$ bounded on (a,b). Then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4} \sup_{x\in(a,b)} \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right|. \quad (2.2)$$

Proof. A simple integration by parts gives that:

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(x)dx = \int_a^b \left(x - \frac{a+b}{2}\right)f'(x)dx. \quad (2.3)$$

Applying the inequality (2.1) we find that:

$$\left|\int_a^b \frac{1}{b-a}\Big(x-\frac{a+b}{2}\Big)f'(x)dx - \frac{1}{b-a}\int_a^b \Big(x-\frac{a+b}{2}\Big)dx \cdot \frac{1}{b-a}\int_a^b f'(x)dx\right|$$
$$\le \frac{1}{b-a}\int_a^b \left|\left(x-\frac{a+b}{2}-\frac{1}{b-a}\int_a^b \Big(y-\frac{a+b}{2}\Big)dy\right)\cdot\left(f'(x)-\frac{1}{b-a}\int_a^b f'(y)dy\right)\right|dx.$$

As

$$\int_a^b \Big(x-\frac{a+b}{2}\Big)dx = 0,$$

we obtain

$$\begin{aligned}\left|\int_a^b \Big(x-\frac{a+b}{2}\Big)f'(x)dx\right| &\le \int_a^b \left|\Big(x-\frac{a+b}{2}\Big)\Big(f'(x)-\frac{f(b)-f(a)}{b-a}\Big)\right|dx\\ &\le \sup_{x\in(a,b)}\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|\int_a^b\left|x-\frac{a+b}{2}\right|dx\\ &= \frac{(b-a)^2}{4}\sup_{x\in(a,b)}\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|. \end{aligned}\tag{2.4}$$

Now, using the equality (2.3), the inequality (2.4) becomes the desired result (2.2).

Corollary 2. *Suppose $p,q>1$ satisfy $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) having the first derivative $f':(a,b)\to\mathbb{R}$ being q-integrable on (a,b). Then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{1}{2}\Big(\frac{b-a}{p+1}\Big)^{\frac{1}{p}}\left(\int_a^b\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|^q dx\right)^{\frac{1}{q}}. \tag{2.5}$$

Proof. Using Hölder's inequality, we have that:

$$\begin{aligned}&\int_a^b \left|\Big(x-\frac{a+b}{2}\Big)\Big(f'(x)-\frac{f(b)-f(a)}{b-a}\Big)\right|dx\\ &\le \left(\int_a^b\left|x-\frac{a+b}{2}\right|^p dx\right)^{\frac{1}{p}}\left(\int_a^b\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|^q dx\right)^{\frac{1}{q}}\\ &= \frac{(b-a)^{\frac{1}{p}+1}}{2(p+1)^{\frac{1}{p}}}\left(\int_a^b\left|f'(x)-\frac{f(b)-f(a)}{b-a}\right|^q dx\right)^{\frac{1}{q}}.\end{aligned}$$

Now, using the first part of (2.4) and the identity (2.3), we obtain the desired result (2.5).

The following result also holds.

Corollary 3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) and suppose that $f' : (a, b) \to \mathbb{R}$ is integrable on (a, b). Then we have the inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \frac{1}{2}\int_a^b \left| f'(x) - \frac{f(b)-f(a)}{b-a} \right| dx. \tag{2.6}$$

Proof. We have

$$\begin{aligned} &\int_a^b \left| \left(x - \frac{a+b}{2}\right)\left(f'(x) - \frac{f(b)-f(a)}{b-a}\right) \right| dx \\ &\le \sup_{x\in(a,b)} \left| x - \frac{a+b}{2} \right| \cdot \int_a^b \left| f'(x) - \frac{f(b)-f(a)}{b-a} \right| dx \\ &= \frac{b-a}{2}\int_a^b \left| f'(x) - \frac{f(b)-f(a)}{b-a} \right| dx. \end{aligned}$$

Using the first part of (2.4) and the identity (2.3), we obtain the desired result (2.6).

3. Applications for Some Special Means

In this section we shall refer to the following special means

(a) *The arithmetic mean*

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \ge 0;$$

(b) *The geometric mean*

$$G = G(a, b) := \sqrt{ab}, \quad a, b \ge 0;$$

(c) *The harmonic mean*

$$H = H(a, b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad a, b > 0;$$

(d) *The logarithmic mean*

$$L = L(a, b) := \begin{cases} a \text{ if } b = a \\ \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \quad a, b > 0; \end{cases}$$

(e) *The identric mean*

$$I = I(a, b) := \begin{cases} a \text{ if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \text{ if } b \neq a, \quad a, b > 0; \end{cases}$$

(f) *The p-logarithmic mean*

$$L_p = L_p(a, b) := \begin{cases} a \text{ if } b = a \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} \text{ if } b \neq a, \quad a, b > 0 \end{cases}$$

where $p \in \mathbb{R}\backslash\{-1,0\}$.

It is well known that

$$H \le G \le L \le I \le A \tag{3.1}$$

and the mapping L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 := I$ and $L_{-1} =: L$.

I. Now, let us consider inequality (2.2), with $f : (0,\infty) \to \mathbb{R}$ defined by $f(x) = x^r, r \in \mathbb{R}\backslash\{0,-1\}$. Then for $0 < a < b$, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4} \sup_{x\in(a,b)} \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right|, \tag{3.2}$$

where f is as in Corollary 1.

1. Consider the mapping $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^r$, $r \in \mathbb{R}\backslash\{-1,0\}$. Then for $0 < a < b$, we have

$$\frac{f(a)+f(b)}{2} = A(a^r, b^r),$$
$$\frac{1}{b-a}\int_a^b f(x)dx = L_r^r(a,b),$$
$$f'(x) - \frac{f(b)-f(a)}{b-a} = rx^{r-1} - rL_{r-1}^{r-1} = r(x^{r-1} - L_{r-1}^{r-1}),$$

and by the inequality 3.2 we obtain:

$$|A(a^r, b^r) - L_r^r(a,b)| \le \frac{|r|(b-a)}{4} \sup_{x\in(a,b)} |x^{r-1} - L_{r-1}^{r-1}|. \tag{3.3}$$

2. Consider the mapping $f : (0,\infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Then for $0 < a < b$, we have

$$\frac{f(a)+f(b)}{2} = \frac{A(a,b)}{G^2(a,b)},$$
$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{L(a,b)},$$
$$f'(x) - \frac{f(b)-f(a)}{b-a} = -\frac{1}{x^2} + \frac{1}{ab},$$
$$\sup_{x\in(a,b)} \left|f'(x) - \frac{f(b)-f(a)}{b-a}\right| = \frac{(b-a)}{a^2 b},$$

and by the inequality (3.2) we obtain

$$\left|\frac{A}{G^2} - \frac{1}{L}\right| \le \frac{(b-a)^2}{4aG^2}$$

which is equivalent to

$$0 \le LA - G^2 \le \frac{(b-a)^2}{4a} L. \tag{3.4}$$

3. Consider the mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln x$. Then for $0 < a < b$, we have

$$\frac{f(a) + f(b)}{2} = \ln G,$$
$$\frac{1}{b-a}\int_a^b f(x)dx = \ln I,$$
$$f'(x) - \frac{f(b) - f(a)}{b-a} = \frac{1}{x} - \frac{1}{L},$$
$$\sup_{x\in(a,b)}\left|f'(x) - \frac{f(b) - f(a)}{b-a}\right| = \frac{1}{a} - \frac{1}{L} = \frac{L-a}{aL}$$

and by the inequality (3.2) we obtain

$$|\ln G - \ln I| \leq \frac{(b-a)}{4}\cdot\left(\frac{L-a}{aL}\right)$$

which is equivalent to:

$$1 \leq \frac{I}{G} \leq \exp\left(\frac{(b-a)}{4}\cdot\left(\frac{L-a}{aL}\right)\right). \tag{3.5}$$

II. Now, let us consider inequality (2.6) with $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^r$, $r \in \mathbb{R}\backslash\{0, -1\}$. Then for $0 < a < b$ we have

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2}\int_a^b\left|f'(x) - \frac{f(b) - f(a)}{b-a}\right|dx. \tag{3.6}$$

1. Consider the mapping $f : (0, \infty) \to \mathbb{R} f(x) = x^r$, $r \in \mathbb{R}\backslash\{0, -1\}$ and $0 < a < b$. Then

$$\int_a^b\left|f'(x) - \frac{f(b) - f(a)}{b-a}\right|dx = |r|\int_a^b |x^{r-1} - L_{r-1}^{r-1}|dx.$$

For simplicity, let assume that $r > 1$. Then

$$\begin{aligned}
\int_a^b |x^{r-1} - L_{r-1}^{r-1}|dx &= \int_a^{L_{r-1}} (L_{r-1}^{r-1} - x^{r-1})dx + \int_{L_{r-1}}^b (x^{r-1} - L_{r-1}^{r-1})dx \\
&= L_{r-1}^{r-1}(L_{r-1} - a) - \frac{x^r}{r}\bigg|_a^{L_{r-1}} + \frac{x^r}{r}\bigg|_{L_{r-1}}^b - (b - L_{r-1})L_{r-1}^{r-1} \\
&= L_{r-1}^{r-1} - aL_{r-1}^{r-1} - \frac{L_{r-1}^r - a^r}{r} + \frac{b^r - L_{r-1}^r}{r} - (b - L_{r-1})L_{r-1}^{r-1} \\
&= \frac{b^r + a^r}{r} - L_{r-1}^{r-1}(a + b) - \frac{2L_{r-1}^r}{r} + 2L_{r-1}^r \\
&= \frac{2}{r}[A(a^r, b^r) - rL_{r-1}^{r-1}A + L_{r-1}^r(r-1)]
\end{aligned}$$

and by the inequality (3.6) we obtain

$$0 \le A(a^r, b^r) - L_r^r(a,b) \le [A(a^r, b^r) - rL_{r-1}^{r-1}A + L_{r-1}^r(r-1)] \tag{3.7}$$

or

$$rL_{r-1}^{r-1}A \le L_r^r(a,b) + (r-1)L_{r-1}^r. \tag{3.8}$$

Similar results can be obtained for $r \le 1$, $r \ne 0, -1$.
We shall omit the details.

2. Consider the mapping $f(a,b) \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Then for $0 < a < b$ we have:

$$\begin{aligned}&\int_a^b \left| \frac{-1}{x^2} + \frac{1}{ab} \right| dx = \frac{1}{ab} \int_a^b \left| 1 - \frac{ab}{x^2} \right| dx \\ &= \frac{1}{ab} \left[\int_a^{\sqrt{ab}} \Big(\frac{ab}{x^2} - 1 \Big) dx + \int_{\sqrt{ab}}^b \Big(1 - \frac{ab}{x^2} \Big) dx \right] = \frac{4}{G^2(A-G)}\end{aligned}$$

and by inequality (3.6) we obtain:

$$\left| \frac{A}{G^2} - \frac{1}{L} \right| \le \frac{2(A-G)}{G^2}$$

i.e.,

$$0 \le AL - G^2 \le 2L(A-G) \tag{3.9}$$

or equivalenty:

$$2LG \le G^2 + AL, \tag{3.10}$$

which is a very interesting inequality amongst A, L and G.

3. Consider the mapping $f : (a,b) \to \mathbb{R}$, $f(x) = \ln x$. Then for $0 < a < b$, we have:

$$\begin{aligned}\frac{1}{2} \int_a^b \left| \frac{1}{x} - \frac{1}{L} \right| dx &= \int_a^b \frac{|x-L|}{xL} dx = \int_a^L \Big(\frac{1}{x} - \frac{1}{L} \Big) dx + \int_L^b \Big(\frac{1}{L} - \frac{1}{x} \Big) dx \\ &= \frac{A-L}{L} + \ln L - \ln G\end{aligned}$$

and then by the inequality (3.6) we obtain.

$$|\ln G - \ln I| \le \ln L - \ln G + \frac{A-L}{L} = \ln \Big[\Big(\frac{L}{G} \Big) \exp \Big(\frac{A-L}{L} \Big) \Big]$$

i.e.,

$$1 \le \frac{I}{G} \le \frac{L}{G} \exp \Big(\frac{A-L}{L} \Big) \tag{3.11}$$

which implies

$$\frac{G}{L} \le \frac{I}{L} \le \exp \Big(\frac{A-L}{L} \Big). \tag{3.12}$$

4. Applications for the Trapezoid Formula

In this section we shall assume that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping whose derivative satisfies the following condition:

$$|f(b) - f(a) - (b-a)f'(x)| \le \Omega(b-a)^2, \quad \Omega > 0 \tag{4.1}$$

for all $a, b \in I$ and x between a and b.

If f' is M-lipschitzian, i.e.,

$$|f'(u) - f'(v)| \le M|u-v|, \quad M > 0$$

then

$$\begin{aligned}|f(b) - f(a) - (b-a)f'(x)| &= |f'(c) - f'(x)||b-a| \\ &\le M|b-a||c-x| \le M(b-a)^2\end{aligned}$$

where c is between a and b, too. Consequently, the mappings having the first derivative lipschitzian satisfy the condition (4.1).

The following trapezoid formula holds.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ satisfies the above conditon (4.1) on (a,b). If $I_h : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ is a partition of $[a,b]$ and $h_i = x_{i+1} - x_i$, $i = 0, \ldots, n-1$, then we have:*

$$\int_a^b f(t)dt = A_{T,I_h}(f) + R_{T,I_h}(f) \tag{4.2}$$

where

$$A_{T,I_h}(f) = \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} h_i \tag{4.3}$$

and the remainder $R_{T,I_h}(f)$ satisfies the estimation:

$$|R_{T,I_h}(f)| \le \frac{\Omega}{4} \sum_{i=0}^{n-1} h_i^3. \tag{4.4}$$

Proof. Corollary 1 applied to the interval $[x_i, x_{i+1}]$ gives:

$$\begin{aligned}&\left|(x_{i+1} - x_i)\frac{f(x_i) + f(x_{i+1})}{2} - \int_{x_i}^{x_{i+1}} f(t)dt\right| \\ &\le \frac{(x_{i+1} - x_i)^2}{4} \sup_{x \in (x_i, x_{i+1})} \left|f'(x) - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}\right| \\ &\le \frac{\Omega(x_{i+1} - x_i)^3}{4}\end{aligned}$$

i.e.,

$$\left|\frac{f(x_i)+f(x_{i+1})}{2}h_i - \int_{x_i}^{x_{i+1}} f(t)dt\right| \leq \frac{\Omega h_i^3}{4} \text{ for all } i = 0, \ldots, n-1.$$

Summing the above inequality and using the generalized triangle inequality, we obtain the approximation (4.2) and the remainder satisfies the estimation (4.4).

Remark 1. We have obtained in this way a trapezoid formula for a class larger than the class $C^2[a,b]$, for which the usual trpezoid formula holds with the remainder term satisfiying

$$|R_{T,I_h}(f)| \leq \frac{||f''||_\infty}{12} \sum_{i=0}^{n-1} h_i^3$$

where $\|f''\|_\infty = \sup_{t\in(a,b)} |f''(t)| < \infty$.

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