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Convergence theorems for Suzuki generalized nonexpansive mappings in Banach spaces

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Abstract. In this paper, we study a new iterative scheme to approximate fixed point of Suzuki nonexpansive type mappings in Banach space. We also prove some weak and strong theorems for Suzuki nonexpansive type mappings. Numerical example is given to show the efficiency of new iteration process. The results obtained in this paper improve the recent ones announced by Thakur et al. [17], Ullah and Arschad [18].

Keywords. Uniformly convex Banach Spaces, Suzuki generalized nonexpansive mapping, fixed point, iteration schemes

1 Introduction

Once the existence of a fixed point of some mapping is established, an algorithm to calculate the value of the fixed point is desired. Many iterative processes have been developed to approximate fixed point. The well-known Banach contraction theorem use Picard iteration process [13] for approximation of fixed point. Some of the other well-known iterative processes Mann [10], Ishikawa [8], Noor [11], Agarwal et al [2], Abbas and Nazir [1], Picard-S [7], Thakur et al [17], Ullah and Arshad [18] and so on. Speed of convergence play important for an iteration process to be preferred on another iteration process. In 2016, Thakur et al [17] introduced the following iteration process: for arbitrary $x_0 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = (1-b_n)x_n + b_n T x_n, \\ y_n = T((1-a_n)x_n + a_n z_n), \\ x_{n+1} = T y_n, \end{cases} \tag{1.1}$$

for all $n \geq 0$, where $\{a_n\}, \{b_n\} \in (0,1)$. They showed with help of numerical example that their new iteration process converge faster than Picard, Mann, Ishikawa, Noor, Agarwal et al, Abbas and Nazir iteration process for the class of Suzuki generalized nonexpansive mappings. Later, in 2018 Ullah and Arshad [18] introduced the following iteration process:for arbitrary $x_0 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = (1-a_n)x_n + a_n T x_n, \\ y_n = T z_n, \\ x_{n+1} = T y_n, \end{cases} \tag{1.2}$$

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for all $n \geq 0$, where $\{a_n\}, \{b_n\} \in (0,1)$. Using their new three-step iteration process,they proved some weak and strong convergence theorems for Suzuki generalized nonexpansive mappings. Numerically they also compared the speed of convergence of their iteration process with Agarwal et al [2] and Picard S-iteration [7] for given an example of Suzuki generalized nonexpansive mappings.

Motivated by above, in this paper, we introduce a new iteration scheme and prove some weak and strong convergence theorems for Suzuki generalized nonexpansive mappings in uniformly convex Banach spaces. We also provide an example of a Suzuki generalized nonexpansive mapping and compare the speed of convergence of our new iteration process with Thakur et al [17] iteration process and Ullah ve Arshad [18] iteration process.

2 Preliminaries

Let X be a real Banach space and K a nonempty subset of X, and $T : K \to K$ be a mapping. A point $x \in K$ is called a fixed point of $T : K \to K$ if $x = Tx$. We denote $F(T)$ the set of fixed points of T. A mapping $T : K \to K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$. T is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for all $x \in K$ and $p \in F(T)$. It is now well-known that the set $F(T)$ is nonempty if T acting on nonempty closed bounded convex subset of a uniformly convex Banach space (see, Browder [3], Gohde [6] and Kirk [9]).

In 2008, Suzuki [16] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called (C) *condition*. Let K be a nonempty convex subset of a Banach space X, a mapping $T : K \to K$ is satisfy condition (C) if for all $x, y \in K$, $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ implies $\|Tx - Ty\| \leq \|x - y\|$.

Suzuki [16] showed that the mapping satisfying *condition* (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. The mapping satisfy condition (C) is called Suzuki generalized nonexpansive mapping.

The following definitions will be needed in proving our main results.

A Banach space X will be said to be uniformly convex [4] if to each ε, $\varepsilon \in (0, 2]$, there corresponds a $\delta(\varepsilon) > 0$ such that the conditions $\|x\| = \|y\| = 1$, $\|x-y\| \geq \varepsilon$ imply $\frac{\|x+y\|}{2} \leq 1-\delta(\varepsilon)$.

Recall that a Banach space X is said to satisfy *Opial's condition* [12] if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ converges weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

In what follows, we shall make use of the following definitions, lemmas and propositions.

Proposition 2.1. *(i) If T is nonexpansive then T satisfies condition (C) [[16], Proposition 1],*

(ii) If T satisfies condition (C) and has a fixed point , then T is a quasi-nonexpansive mapping [[16], Proposition 2],

(iii) If T satisfies condition (C), then $\|x - T(y)\| \leq 3\|T(x) - x\| + \|x - y\|$ for all $x, y \in K$ [[16], Lemma 7].

Lemma 2.1. *([16]) Let T be a mapping on a subset K of a Banach space X with Opial's condition. Assume that T satisfies condition (C). If $\{x_n\}$ converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tp = p$. That is, $I - T$ (I is identity mapping)is demiclosed at zero.*

Lemma 2.2. *([16]) Let T be a mapping on a weakly compact convex subset K of uniformly convex Banach space X. Assume that T satisfies condition (C), then T has a fixed point.*

Let $\{x_n\}$ be a bounded sequence in a Banach space X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

The asymptotic radius of $\{x_n\}$ relative to K is defined by

$$r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}.$$

The asymptotic center of $\{x_n\}$ relative to K is the set

$$A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}.$$

It is known that, in uniformly convex Banach space, $A(K, \{x_n\})$ consists of exactly one-point [5].

Lemma 2.3. *([14]) Suppose that X is a uniformly convex Banach space and $0 < k \leq t_n \leq m < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequence of X such that $\limsup_{n\to\infty}\|x_n\| \leq r$, $\limsup_{n\to\infty}\|y_n\| \leq r$ and $\limsup_{n\to\infty}\|t_n x_n + (1 - t_n)y_n\| = r$ hold for $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Definition 1. ([15]) Let $\{u_n\}$ in K be a given sequence. $T : K \to X$ with the nonempty fixed point set $F(T)$ in K is said to satisfy *condition(A)* with respect to the $\{u_n\}$ if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $\|u_n - Tu_n\| \geq f(d(u_n, F(T))$ for all $n \geq 1$.

3 Convergence of new iterative scheme for Suzuki generalized nonexpansive mappings

In this section, we prove weak and strong convergence theorems for a new iterative scheme (3.1) of Suzuki generalized nonexpansive mappings in uniformly convex Banach space. We first introduce a new iterative scheme, defined as: for arbitrary $x_0 \in K$ construct a sequence $\{x_n\}$ by

$$\begin{cases} z_n = (1 - c_n)x_n + c_n T x_n, \\ y_n = T((1 - b_n)Tx_n + b_n T z_n), \\ x_{n+1} = (1 - a_n)T(Ty_n) + a_n T(Tz_n), \end{cases} \tag{3.1}$$

for all $n \geq 0$, where $\{a_n\}, \{b_n\}, \{c_n\} \in (0, 1)$. We now establish the following result:

Lemma 3.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X , T be a mapping satisfying condition (C) with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in K$, $\{x_n\}$ be a sequence generated by (3.1), then we have, $\lim_{n\to\infty} \|x_n - p\|$ exits for any $p \in F(T)$.*

Proof. For any $p \in F(T)$, and $x \in K$, since for T satisfy condition (C), $\frac{1}{2}\|p-Tp\| = 0 \leq \|p-x\|$ implies that $\|Tp-Tx\| \leq \|p-x\|$. Then we show that T is a quasi-nonexpansive mapping.

Now, by Proposition 2.1(ii) and using (3.1), we have,

$$\begin{aligned} \|z_n - p\| &= \|(1-c_n)x_n + c_nTx_n - p\| \\ &\leq \|(1-c_n)(x_n-p) + c_n(Tx_n-p)\| \\ &\leq (1-c_n)\|x_n-p\| + c_n\|x_n-p\| \\ &\leq \|x_n-p\|. \end{aligned} \tag{3.2}$$

Using (3.1), (3.2) together with Proposition 2.1(ii), we get

$$\begin{aligned} \|y_n - p\| &= \|T((1-b_n)Tx_n + b_nTz_n) - p\| \\ &\leq \|((1-b_n)Tx_n + b_nTz_n - p)\| \\ &= \|(1-b_n)(Tx_n-p) + b_n(Tz_n-p)\| \\ &\leq (1-b_n)\|Tx_n-p\| + b_n\|Tz_n-p\| \\ &\leq (1-b_n)\|x_n-p\| + b_n\|z_n-p\| \\ &\leq (1-b_n)\|x_n-p\| + b_n\|x_n-p\| \\ &\leq \|x_n-p\|. \end{aligned} \tag{3.3}$$

By using (3.1), (3.2) , (3.3) together with Proposition 2.1(ii), we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|(((1-a_n)T(Ty_n) + a_nT(Tz_n)) - p)\| \\ &= \|(1-a_n)(T(Ty_n)-p) + a_n(T(Tz_n)-p)\| \\ &\leq (1-a_n)\|Ty_n-p\| + a_n\|Tz_n-p\| \\ &\leq (1-a_n)\|y_n-p\| + a_n\|z_n-p\| \\ &\leq (1-a_n)\|x_n-p\| + a_n\|x_n-p\| \\ &\leq \|x_n-p\|. \end{aligned}$$

This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for p is a fixed point of T. It follows that $\lim_{n\to\infty} \|x_n - p\|$ exits. □

Theorem 3.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X. T is a mapping satisfying condition (C) with $F(T) \neq \emptyset$. p is a fixed point of T and let $\{x_n\}$ be a sequence in K defined by (3.1) with $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ real sequences in $(0,1)$, then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exits and $\{x_n\}$ is bounded. Put $\lim_{n\to\infty} \|x_n - p\| = r$. From (3.2) and (3.3), we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| \leq r$$

and

$$\limsup_{n\to\infty}\|y_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| \le r$$

and also we have

$$\limsup_{n\to\infty}\|Tx_n - p\| \le \limsup_{n\to\infty}\|x_n - p\| \le r.$$

Then by Proposition 2.1(ii), (3.2) and (3.3) , also we have,

$$\begin{aligned}\limsup_{n\to\infty}\|T(Tz_n) - p\| &\le \limsup_{n\to\infty}\|Tz_n - p\| \\ &\le \limsup_{n\to\infty}\|z_n - p\| \\ &\le \limsup_{n\to\infty}\|x_n - p\| \le r \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\limsup_{n\to\infty}\|T(Ty_n) - p\| &\le \limsup_{n\to\infty}\|Ty_n - p\| \\ &\le \limsup_{n\to\infty}\|y_n - p\| \\ &\le \limsup_{n\to\infty}\|x_n - p\| \le r. \end{aligned} \tag{3.5}$$

On the other hand,

$$\begin{aligned} r &= \limsup_{n\to\infty}\|x_{n+1} - p\| \\ &= \limsup_{n\to\infty}\|((1-a_n)T(Ty_n) + a_nT(Tz_n)) - p\| \\ &= \limsup_{n\to\infty}\|(1-a_n)(T(Ty_n) - p) + a_n(T(Tz_n) - p)\|. \end{aligned}$$

From (3.4), (3.5) and Lemma 2.3, we get

$$\limsup_{n\to\infty}\|T(Tz_n) - T(Ty_n)\| = 0. \tag{3.6}$$

Now

$$\begin{aligned}\|x_{n+1} - p\| &= \|((1-a_n)T(Ty_n) + a_nT(Tz_n)) - p\| \\ &= \|(T(Ty_n) - p) + a_n(T(Tz_n) - T(Ty_n))\| \\ &\le \|T(Ty_n) - p\| + a_n\|T(Tz_n) - T(Ty_n)\|. \end{aligned}$$

Making $n \longrightarrow \infty$ and from (3.6) we get

$$r = \limsup_{n\to\infty}\|x_{n+1} - p\| \le \|T(Ty_n) - p\|.$$

So by from (3.5) we have

$$\limsup_{n\to\infty}\|T(Ty_n) - p\| = r.$$

Then

$$\|T(Ty_n) - p\| \le \|T(Tz_n) - T(Ty_n)\| + \|T(Tz_n) - p\|$$

$$\leq \quad \|T(Tz_n) - T(Ty_n)\| + \|z_n - p\|.$$

Making $n \longrightarrow \infty$ and from (3.6) we get

$$r \leq \limsup_{n\to\infty} \|z_n - p\|.$$

Thus we have

$$r = \lim_{n\to\infty} \|z_n - p\|.$$

Thus

$$r = \lim_{n\to\infty} \|z_n - p\| = \lim_{n\to\infty} \|(1 - c_n)(x_n - p) + c_n(Tx_n - p)\|.$$

By Lemma 2.3 we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Let $p \in A(K, \{x_n\})$. By Proposition 2.1, we have,

$$\begin{aligned} r(Tp, \{x_n\}) = \limsup_{n\to\infty} \|x_n - Tp\| \quad &\leq \quad \limsup_{n\to\infty}(3\|Tx_n - x_n\| + \|x_n - p\| + \|p - Tp\|) \\ &\leq \quad \limsup_{n\to\infty}\|x_n - p\| = r(p, \{x_n\}). \end{aligned}$$

This implies that for $Tp = p \in A(K, \{x_n\})$. Since X is uniformly Banach space, $A(K, \{x_n\})$ is singleton, hence $Tp = p$. This completes the proof. $\square$

Next, we prove the following theorems of fixed points of mappings with condition (C).

Theorem 3.2. *Let T be a mapping on a compact convex subset K of a Banach space X. Assume that T satisfy condition (C). Assume that $p \in F(T)$ is a fixed point of T and let $\{x_n\}$ be a sequence in K defined by (3.1) with $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ real sequences in $(0, 1)$. Then $\{x_n\}$ converges to a fixed point of T.*

Proof. By Theorem 3.1, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $p \in K$ such that $\{x_{n_k}\}$ converges p. By Proposition 2.1, we have $\|x_{n_k} - Tp\| \leq 3\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\|$ for all $k \geq 0$. Then $\{x_{n_k}\}$ converges Tp. This implies that $Tp = p$, i.e. $p \in F(T)$. Also $\lim_{n\to\infty} \|x_n - p\|$ exists by Lemma 3.1, thus $\{x_n\}$ converges to p. $\square$

Theorem 3.3. *Let K be a weakly compact convex subset of a Banach space X. Let T be a mapping on K Assume that T satisfy condition (C). Assume that $p \in F(T)$ is a fixed point of T and let $\{x_n\}$ be a sequence in K defined by (3.1) where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0, 1)$ and satisfy the conditions of Theorem 3.1. Then T has a fixed point.*

Proof. By Theorem 3.1, we have $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Define a continuous convex function $f : K \to [0,\infty)$ by $f(x) = \limsup_{n\to\infty}\|x_n - x\|$ for all $x \in K$. Since K is weakly compact and f is convex, bounded from below and lower semicontinuous, we define the nonempty set $M = \{p \in K : f(p) \leq f(x), x \in K\}$. Now if $p \in M$, then by Proposition 2.1, $\|x_n - Tp\| \leq 3\|Tx_n - x_n\| + \|x_n - p\|$, we have $f(Tp) \leq f(p)$, that is $Tp \in M$. Then we have $f(Tp) = f(p)$. Then the asymptotic center $A(K, \{x_n\})$ is T-invariant. Then we have $Tp = p$. □

In the next result, we prove our strong convergence theorem as follows.

Theorem 3.4. *Let X be a real uniformly convex Banach space and K be a nonempty compact convex subset of X and T be a mapping satisfying condition (C). Assume that $p \in F(T)$ is a fixed point of T and let $\{x_n\}$ be a sequence in K defined by (3.1) where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0,1)$ and satisfy the conditions of Theorem 3.1. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. $F(T) \neq \emptyset$, so by Theorem 3.1, we have $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$. Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \longrightarrow p$ as $k \to \infty$ for $p \in K$. Then we have

$$\|x_{n_k} - Tp\| \quad \leq \quad 3\|Tx_{n_k} - x_{n_k}\| + \|x_{n_k} - p\| \text{ for all } k \geq 0.$$

Letting $k \to \infty$, we get $Tp = p$, $p \in F(T)$. $\lim_{n\to\infty}\|x_n - p\|$ exists for every $p \in F(T)$, so $\{x_n\}$ converges strongly to a fixed point of T. □

Theorem 3.5. *Let the conditions of Theorem 3.1 be satisfied. Also if T satisfies condition (A) and $F(T) \neq \emptyset$, then $\{x_n\}$ defined by (3.1) converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, we have $\lim_{n\to\infty}\|x_n - p\|$ exists and so $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Also by Theorem 3.1, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. It follows from condition (A) that $\lim_{n\to\infty} f(d(x_n, F(T)) \leq \lim_{n\to\infty}\|x_n - Tx_n\|$. That is, $\lim_{n\to\infty} f(d(x_n, F(T)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus , we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\{y_k\} \subset F(T)$ such that $\|x_{n_k} - y_k\| < \frac{1}{2^k}$ for all $k \in \mathbb{N}$. We can easily show that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, therefore $p \in F(T)$ and $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty}\|x_n - p\|$ exists, we have $x_n \to p \in F(T)$. The proof is completed. □

Finally, we prove the weak convergence of the iterative scheme (3.1) for Suzuki generalized nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

Theorem 3.6. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. and T be a mapping satisfying condition (C) with $F(T) \neq \emptyset$. Assume that $p \in F(T)$ is a fixed point of T and let $\{x_n\}$ be a sequence in K defined by (3.1) where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $(0,1)$ and satisfy the conditions of Theorem 3.1. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since $F(T) \neq \emptyset$, it follows from Theorem 3.1 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. For , let q_1, q_2 be weak limit of subsequence $\{x_{n_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ respectively. By $\lim_{n\to\infty} \|x_n - Tx_n\|$ and $I - T$ is demiclosed with respect to zero , therefore we obtain $Tq_1 = q_1$. Again in the same manner, we can $Tq_2 = q_2$. Next we prove the uniqueness. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist. For suppose that $q_1 \neq q_2$, then by the Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q_1\| &= \lim_{j\to\infty} \|x_{n_j} - q_1\| < \lim_{j\to\infty} \|x_{n_j} - q_2\| = \lim_{n\to\infty} \|x_n - q_2\| \\
&= \lim_{k\to\infty} \|x_{n_k} - q_2\| < \lim_{k\to\infty} \|x_{n_k} - q_1\| = \lim_{n\to\infty} \|x_n - q_1\|
\end{aligned}$$

which is contraction. So, $q_1 = q_2$. Therefore $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof. □

Now we construct an example of Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 1. Define a mapping $T : [0, 1] \to [0, 1]$ by $Tx = \begin{cases} 1 - x, & [0, \frac{1}{6}) \\ \frac{x+5}{6}, & [\frac{1}{6}, 1]. \end{cases}$

Need to prove that T is a Suzuki generalized nonexpansive mapping but not nonexpansive. If $x = \frac{3}{19}, y = \frac{1}{6}$ we see that

$$\begin{aligned}
\|x - y\| &= \|\frac{3}{19} - \frac{1}{6}\| = \frac{1}{114} = 0.008719298 \\
\|Tx - Ty\| &= \|1 - x - \frac{y+5}{6}\| = \|1 - \frac{3}{19} - \frac{\frac{1}{6}+5}{6}\| = \frac{13}{684} = 0.019005848.
\end{aligned}$$

Since $\|Tx - Ty\| > \|x - y\|$ then T is not nonexpansive mapping.

To verify that T is Suzuki generalized nonexpansive mapping, consider the following cases:

Case I: Let $x \in [0, \frac{1}{6})$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|x - (1 - x)\| = \frac{1-2x}{2} \in (\frac{1}{6}, \frac{1}{2}]$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ we must have $\frac{1-2x}{2} \leq y - x, \frac{1}{2} - x + x \leq y, i.e., \frac{1}{2} \leq y$, hence $y \in [\frac{1}{2}, 1]$. We have

$$\begin{aligned}
\|x - y\| &= |x - y| > |\frac{1}{6} - \frac{1}{2}| = \frac{1}{3} \\
\|Tx - Ty\| &= \|(1 - x) - \frac{y+5}{6}\| = \|\frac{6 - 6x - y - 5}{6}\| = \|\frac{1 - 6x - y}{6}\| < \frac{1}{6}.
\end{aligned}$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$.

Case II: Let $x \in [\frac{1}{6}, 1]$, then $\frac{1}{2}\|x - Tx\| = \frac{1}{2}\|x - \frac{x+5}{6}\| = \frac{5-5x}{12} \in [0, \frac{25}{42}]$. For $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ we must have $\frac{5-5x}{12} \leq |y - x|$, which gives two possibilities:

(a) Let $x < y$, then $\frac{5-5x}{12} \leq y - x \Rightarrow y \geq \frac{5+7x}{12} \Rightarrow y \in [\frac{37}{42}, 1] \subset [\frac{1}{6}, 1]$. So

$$\|Tx - Ty\| = \|\frac{x+5}{6} - \frac{y+5}{6}\| = \frac{1}{6}\|x - y\| \leq \|x - y\|.$$

Hence $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$.

(b) Let $x > y$, then $\frac{5-5x}{12} \leq x - y \Rightarrow y \leq x - \frac{5-5x}{12} \Rightarrow y \leq \frac{17x-5}{12} \Rightarrow y \in [-\frac{13}{72}, 1]$.Since $y \in [0, 1]$, so $y \leq \frac{17x-5}{12} \Rightarrow x \geq \frac{12y+5}{17} \Rightarrow x \in [\frac{5}{17}, 1]$. Now $x \in [\frac{5}{17}, 1]$ and $y \in [\frac{1}{6}, 1]$ is already

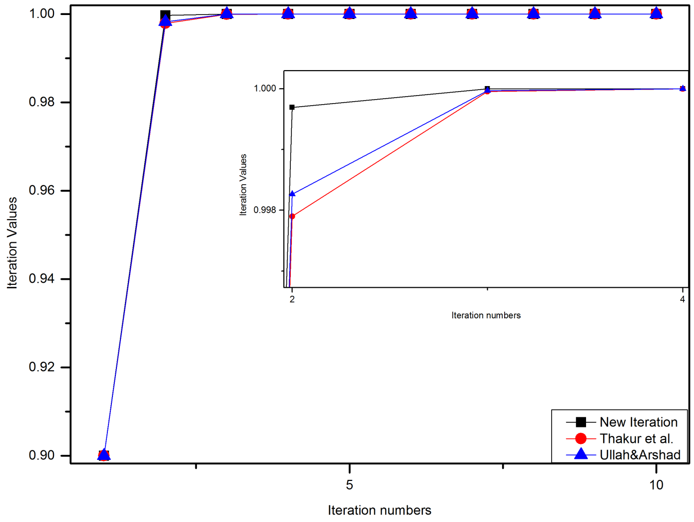

Figure 1: Comparison among different iteration processes for Example 1 with initial guess $x_0 = 0.9$.

included in case (a), so we verify $x \in [\frac{5}{17}, 1]$ and $y \in [0, \frac{1}{6})$. Now, consider $x \in [\frac{5}{17}, 1]$ and $y \in [0, \frac{1}{6})$. Then

$$\|Tx - Ty\| = \|\frac{x+5}{6} - (1-y)\| = \|\frac{x+6y-1}{6}\|.$$

For convenience, first consider $x \in [\frac{5}{17}, \frac{1}{2}]$ and $y \in [0, \frac{1}{6})$, $\|Tx - Ty\| = |\frac{x+5}{6}| \leq \frac{1}{12}$ and $\|x - y\| > \frac{13}{102}$. Hence $\|Tx - Ty\| \leq \|x - y\|$, i.e., T satisfies condition (C).

Finally consider $x \in [\frac{1}{2}, 1]$,$y \in [0, \frac{1}{6})$, then $\|Tx - Ty\| = |\frac{x+6y-1}{6}| \leq \frac{1}{6}$ and $\|x - y\| > \frac{1}{3}$. Hence $\|Tx - Ty\| \leq \|x - y\|$.

So $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$.

Numerical results: We compare convergence behavior of our iteration process with Thakur et al iteration process and Ullah and Arshad iteration process using Example 1. We set $x_0 = 0.9$ and for all $n \geq 0$ $a_n = 0,85$, $b_n = 0,65$, $c_n = 0,45$. Graphic representation is given in Figure 1. The Figure 1 is shown that our new iteration process converges to $x^* = 1$ faster than Thakur et al [17] and Ullah and Arschad [18] iteration processes.

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