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ELLIPTICALLY CONTOURED MODEL AND FACTORIZATION OF WILKS' Λ: NONCENTRAL CASE

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Abstract. Kshirsagar in a series of papers, see e.g., Kshirsagar (1964, 1971), McHenry and Kshirsagar (1977), factorizes Wilks' Λ into a number of factors and finds the independent central multivariate beta densities of these factors. These factors are the Wilks' likelihood ratio test criteria for testing goodness of fit of certain canonical variables. Essentially the factors of Wilks' Λ are the factors of the determinants of certain multivariate beta distributed matrices. The Bartlett decompositions of the underlying multivariate beta distribution into independent factors, determine the distributions of these factors. The present paper generalizes Kshirsagar's (1971) normal central distribution theory to elliptically contoured model noncentral distribution theory, showing that Kshirsagar's (1971) nonnull normal theory is nonnull robust for elliptically contoured model.

1. Introduction

Following Gupta and Kabe (1979), see also Kshirsagar (1971), we assume $\mathbf{y}$ and $\mathbf{x}$ to be respectively p_1 and p_2 component (column) vectors normally distributed with zero means and an unknown $p \times p$, $p_1 + p_2 = p$, covariance matrix

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \Sigma_{11}(p_1 \times p_1). \tag{1}$$

Then the squared canonical correlation coefficients are the roots of the canonical correlation matrix $(\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2})$. For a sample of size n on $(\mathbf{y}'\ \mathbf{x}')'$, we calculate the sample dispersion matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, S_{11}(p_1 \times p_1), \tag{2}$$

and the sample quantities $S_{11} = T = W + B$, $S_{12}S_{22}^{-1}S_{21} = B$. The null hypothesis $\Sigma_{12} = 0$ is tested by the Wilks' likelihood ratio test (LRT) criterion

$$\Lambda = |W|/|T| = |I - R|, \quad R = S_{11}^{-1/2}S_{12}S_{22}^{-1}S_{21}S_{11}^{-1/2}, \tag{3}$$

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R being termed the canonical correlation matrix.

Now Kshirsagar (1971) wishes to test the null hypothesis $F\Sigma_{22}^{-1}\Sigma_{12}' = F\beta' = 0$, where $F(g \times p_2)$ is unknown. Assuming all matrices in this paper to be full rank matrices, he notes that

$$\begin{aligned} \mathrm{Var}\,(F\hat{\beta}') &= \mathrm{Var}\,(FS_{22}^{-1}S_{12}') \\ &= (n - p_2 - 1)^{-1}(F\Sigma_{22}^{-1}F') \oplus (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}), \end{aligned} \tag{4}$$

and that

$$\underset{F}{\mathrm{Max}}\, \mathrm{tr}\, \mathrm{Var}\,(F\hat{\beta}') = (\lambda_1 + \cdots + \lambda_g)\, \mathrm{tr}(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}), \tag{5}$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_{p_2}$ are the roots of Σ_{22}^{-1}. Thus testing $F = F_0$ (specified) amounts to the testing that certain linear functions of $\mathbf{x}$ are the canonical variables of $\mathbf{x}$. Note that $E(F\mathbf{y}|\mathbf{x}) = F\Sigma_{22}^{-1}\Sigma_{21}\mathbf{x} = F\beta'\mathbf{x}$, and hence testing $F\beta' = 0$, amounts to the testing that the first g canonical correlation coefficients are zero, i.e, some g linear combinations of $\mathbf{y}$ cannot be predicated by any linear combinations of $\mathbf{x}$.

For this testing purpose, Kshirsagar (1971) sets

$$F_0\hat{\beta}' = F_0S_{22}^{-1}S_{12}' = C_{xg}',\ C_{xg}C_{gg}^{-1/2} = Z,\ F_0S_{22}^{-1}F_0' = C_{gg}, \tag{6}$$

$$\Lambda = \Lambda_1\Lambda_2\Lambda_3,\ L = B - ZZ', \tag{7}$$

$$\Lambda_1 = |C_{gg} - C_{xg}'T^{-1}C_{xg}|/|C_{gg}|, \tag{8}$$

$$\Lambda_2 = |\Gamma L\Gamma'||C_{gg}|/|\Gamma T\Gamma'||C_{gg} - \Gamma T\Gamma'|, \tag{9}$$

$$\Lambda_3 = |L||\Gamma C_{xg}|/|T||\Gamma L\Gamma'|, \tag{10}$$

where $\Gamma' = T^{-1}C_{xg}$. An alternative factorization of Λ is

$$\Lambda = \Lambda_1\Lambda_4\Lambda_5, \tag{11}$$

$$\Lambda_4 = |L||C_{xg}'L^{-1}C_{xg}|/|T||\Gamma C_{xg}|, \tag{12}$$

$$\Lambda_5 = |C_{gg}||\Gamma C_{xg}|/|C_{gg} - C_{xg}'T^{-1}C_{xg}||C_{xg}'L^{-1}C_{xg}|. \tag{13}$$

Now if $ABC = 0$, and rank A = rank AB, then $BC = 0$; if $ABC = 0$, and rank$B$ = rank BC, then $AB = 0$. It follow that $\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = 0$ implies $\Sigma_{12} = 0$, and under this condition Kshirsagar (1971) derives the independent central multivariate beta densities of $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5$, (see also R. D. Gupta and Kabe, 1979). Unfortunately, the simultaneous noncentral distribution theory of $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$, and Λ_5 becomes very involved. We derive the joint noncentral density of Λ_2 and Λ_3 or that of Λ_4 and Λ_5 under the condition that $F_0\Sigma_{22}^{-1}\Sigma_{12}' = 0$, although the distribution theory is not completely solved unless we assume that $F_0\Sigma_{22}^{-1}\Sigma_{12}' \neq 0$.

The upper triangular Bartlett decomposition of a symmetric positive definite matrix M is

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} W_{11}^{1/2} & G_{12} \\ 0 & M_{22}^{1/2} \end{bmatrix} \begin{bmatrix} W_{11}^{1/2} & 0 \\ G_{21} & M_{22}^{1/2} \end{bmatrix} = UU', \tag{14}$$

where $W_{11} = M_{11} - M_{12}M_{22}^{-1}M_{21}$, and $M_{21} = M_{22}^{1/2}G_{21}$.

The lower triangular Bartlett decomposition of M is

$$M = \begin{bmatrix} M_{11}^{1/2} & 0 \\ G_{21} & W_{22}^{1/2} \end{bmatrix} \begin{bmatrix} M_{11}^{1/2} & G_{12} \\ 0 & W_{22}^{1/2} \end{bmatrix} = LL', \tag{15}$$

where $W_{22} = M_{22} - M_{21}M_{11}^{-1}M_{12}$, and $M_{12} = M_{11}^{1/2}G_{12}$.

The Bartlett decompositions of $(I-M)$ are obvious and are used by Kshirsagar (1971), and R. D. Gupta and Kabe (1979), to derive the densities of $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5$ respectively to be $\Lambda(n-p_1, g, p_1)$, $\Lambda(n, g, p_2 - g)$, $\Lambda(n-2g, p_1 - g, p_2 - g)$, $\Lambda(n-g, p_1 - g, p_2 - g)$, $\Lambda(n-p+g, g, p_2)$, where $\Lambda(n-q, p, q-g)$ denotes the density of $p \times p$ positive definite symmetric M

$$\phi(M) = K|M|^{\frac{1}{2}(n-q-p-1)}|I-M|^{\frac{1}{2}(q-g-p-1)},$$

with K denoting the normalizing constants of density functions in this paper.

An n-component vector $\mathbf{y}$ is said to have an n-variate elliptically contoured model (ECM) density if its characteristic function $g(\mathbf{t})$ is of the form

$$g(\mathbf{t}) = \exp\{i\mathbf{t}'\mu\}\Psi(\mathbf{t}'\Sigma\mathbf{t}), \quad E(\mathbf{y}) = \mu. \tag{16}$$

Thus e.g.,

$$\phi(\mathbf{y}) = K\exp\{-[(\mathbf{y}-\mu)'\Sigma^{-1}(\mathbf{y}-\mu)]^{1/2}\},$$

has the characteristic function

$$g(\mathbf{t}) = K\exp\{it'\mu\}(1+\mathbf{t}'\Sigma\mathbf{t})^{-\frac{1}{2}(n+1)}.$$

In case $\mathbf{y}$ is complex, then

$$g(\mathbf{t}) = K\exp\{i\overline{\mathbf{t}}'\mu\}(1+\overline{\mathbf{t}}'\Sigma\mathbf{t})^{-\frac{1}{2}(2n+1)}$$

and

$$\phi(\mathbf{y}) = K\exp\{-(\overline{\mathbf{y}-\mu})'\Sigma^{-1}(\mathbf{y}-\mu)\}.$$

Krishnaiah and Lin (1986) list several examples of complex symmetric ECM densities (see also Gupta and Varga, 1993). Anderson, Fang, and Hsu (1986) derive and prove the null robustness of normal theory Wilks' Λ for the ECM. The present paper shows that Kshirsagar's (1971) normal nonnull Wilks' Λ theory is nonnull robust for the ECM.

In the next section we study some aspects of the noncentral canonical correlation matrix distribution theory for the ECM, and the densities of $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5$ are derived in Section 3.

2. Canonical Correlation Theory

Before we proceed with the canonical correlations distribution theory, we record the following known results. If $\mathbf{y}$ has n-components, $-\infty < \mathbf{y} < \infty$, then

$$\int_{\mathbf{y}'A\mathbf{y}=u} f(\mathbf{y}'A\mathbf{y}) \exp\{\mathbf{d}'\mathbf{y}\} d\mathbf{y} = K|A|^{-\frac{1}{2}} f(u) u^{\frac{1}{2}(n-2)} {}_0F_1\Big(\frac{1}{2}n; \frac{1}{4}u\mathbf{d}'A^{-1}\mathbf{d}\Big). \tag{17}$$

$$\int \exp\left\{-\frac{1}{2}\operatorname{tr} PG\right\} |G|^{\frac{1}{2}(n-p-1)} (\mu' G\mu)^r dG = K2^r\Gamma\Big(\frac{1}{2}n + r\Big)(\mu' P^{-1}\mu)^r, G(p \times p). \tag{18}$$

For any absolutely continuous function g, we know that

$$\left[g\left[\frac{d}{d\theta}\right]\exp\{\theta x\}\right]_{\theta=0} = g(x),$$

where we also assume

$$\left[\exp\left\{-\Big(\frac{d}{d\theta}\Big)^{\frac{1}{2}}\right\}\exp\{\theta x\}\right]_{\theta=0} = \exp\{-(x)^{\frac{1}{2}}\},$$

a result useful for translating normal theory results to similar results for ECM (16).

Now Anderson, Fang, and Hsu (1986) note that the density of any function $h(G)$ of the Wishart matrix G is nonnull robust for the ECM if $h(\theta G) = h(G)$ for every scalar θ. Obviously $h(G) = R$ of (3) is such a function.

Given the Wishart density of a $p \times p$ positive definite symmetric matrix G to be

$$\phi(G) = K \exp\left\{-\frac{1}{2}\operatorname{tr} PG\right\} |G|^{\frac{1}{2}(n-p-1)},$$

where for convenience

$$P^{-1} = \rho = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{bmatrix},$$

denotes the population correlation matrix, the following results of the normal correlation theory are known.

The squared sample multiple correlation coefficient $R^2_{1\cdot 2\ldots p}$ is defined by

$$\mathbf{g}'(1) G_{22}^{-1} \mathbf{g}_{(1)} = g_{11} R^2_{1\cdot 2\ldots p'}$$

where

$$G = \begin{bmatrix} g_{11} & \mathbf{g}'_{(1)} \\ \mathbf{g}_{(1)} & G_{22} \end{bmatrix}, \ G_{22}((p-1) \times (p-1)),$$

and conformably the population squared multiple correlation coefficient is

$$\rho^2_{1\cdot 2\ldots p} = \rho'(1)\rho_{22}^{-1}\rho_{(1)} = \mathbf{p}'(1)P_{22}^{-1}\mathbf{p}_{(1)}/p_{11},$$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1p} \\ p_{21} & p_{22} & \cdots & p_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ p_{p1} & p_{p2} & \cdots & p_{pp} \end{bmatrix} = \begin{bmatrix} p_{11} & \mathbf{p}'(1) \\ \mathbf{p}_{(1)} & P_{22} \end{bmatrix}.$$

The density of $R^2_{1\cdot 2\ldots p}$ is

$$\begin{aligned} \phi(R^2_{1\cdot 2\ldots p}) = K(1-\rho^2_{1\cdot 2\ldots p})^{\frac{1}{2}n}(1-R^2_{1\cdot 2\ldots p})^{\frac{1}{2}(n-p-1)}(R^2_{1\cdot 2\ldots p})^{\frac{1}{2}(p-3)} \\ \cdot {}_2F_1\left[\frac{1}{2}n, \frac{1}{2}n; \frac{1}{2}(p-1); R^2_{1\cdot 2\ldots p}\rho^2_{1\cdot 2\ldots p}\right]. \end{aligned} \tag{19}$$

When $p = 2$, the result (19) reduces to the density of the squared sample correlation coefficient r^2, ρ^2 being the population counterpart

$$\phi(r^2) = K(1-\rho^2)^{\frac{1}{2}n}(1-r^2)^{\frac{1}{2}(n-3)}(r^2)^{-\frac{1}{2}}{}_2F_1\left(\frac{1}{2}n, \frac{1}{2}n; \frac{1}{2}; r^2\rho^2\right). \tag{20}$$

Recently Mathai (1981) sets

$$\begin{aligned} R &= G_{11}^{-1/2}G_{12}G_{22}^{-1}G_{21}G_{11}^{-1/2}, G_{22}(p_2 \times p_2), \qquad p_1 + p_2 = p, \\ \rho &= p_{11}^{-1/2}P_{12}P_{22}^{-1}P_{21}p_{11}^{-1/2}, P_{22}(p_2 \times p_2), \qquad p_1 + p_2 = p, \end{aligned}$$

and finds the density of the canonical correlation R to be

$$\phi(R) = K|I-\rho|^{\frac{1}{2}n}|R|^{\frac{1}{2}(p_2-p_1-1)}|I-R|^{\frac{1}{2}(n-p-1)} \cdot {}_2F_1\left[\frac{1}{2}n, \frac{1}{2}n; \frac{1}{2}p_2; \rho R\right]. \tag{21}$$

The partial canonical correlation matrix is defined by

$$R = R_{p_{11}p_{12}\cdot p_1+1,\ldots,p} = D_{11}^{-1/2}D_{12}D_{22}^{-1}D_{21}D_{11}^{-1/2}, \tag{22}$$

where

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = G_{11} - G_{12}G_{22}^{-1}G_{21}, \qquad p_{11} + p_{12} = p_1, \qquad p_1 + p_2 = p,$$

G_{11} is $p_1 \times p_1$, and D_{11} is $p_{11} \times p_{11}$. It follows from (21) that the density of R of (22) is

$$\phi(R) = K|I-\rho|^{\frac{1}{2}n}|R|^{\frac{1}{2}(p_{12}-p_{11}-1)}|I-R|^{\frac{1}{2}(n-p-1)} \cdot {}_2F_1\left(\frac{1}{2}(n-p_2), \frac{1}{2}(n-p_2); \frac{1}{2}p_{12}; \rho R\right), \tag{23}$$

where ρ in (23) is defined by

$$\rho = \Delta_{11}^{-1/2}\Delta_{12}\Delta_{22}^{-1}\Delta_{21}\Delta_{11}^{-1/2}, \qquad \Delta = \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix},$$

with $\Delta = P_{11} - P_{12}P_{22}^{-1}P_{21}$, P_{11} is $p_1 \times p_1$ and Δ_{11} is $p_{11} \times p_{11}$.

Form (23), with $p_{11} = 1$, $p_{12} = 1$, $p_2 = (p-2)$, the density of the squared partial sample correlation coefficent $R^2_{12\cdot 3\ldots p}$ is

$$\begin{aligned}\phi(R^2_{12\cdot 3\ldots p}) &= K(1-\rho^2_{12\cdot 3\ldots p})^{\frac{1}{2}n}(R^2_{12\cdot 3\ldots p})^{-\frac{1}{2}}(1-R^2_{12\cdot 3\ldots p})^{\frac{1}{2}(n-p-1)} \\ &\quad \cdot {}_2F_1\left[\frac{1}{2}(n-p+2), \frac{1}{2}(n-p+2); \frac{1}{2}; \rho^2_{12\cdot 3\ldots p}R^2_{12\cdot 3\ldots p}\right],\end{aligned}$$

where $\rho^2_{12\cdot 3\ldots p}$ is the population counterpart of $R^2_{12\cdot 3\ldots p}$.

Now before we proceed with the derivation of (19) for the ECM, we derive (20) to clarify our methodology. Obviously the ECM Wishart density of $G(2\times 2)$ is

$$\begin{aligned}\phi(G) &= Kg(\operatorname{tr} PG)|G|^{\frac{1}{2}(n-3)} \\ &= K\left[g\left(\frac{d}{d\theta}\right)\exp\{\operatorname{tr}(\theta P)G\}\right]_{\theta=0}|G|^{\frac{1}{2}(n-3)},\end{aligned} \tag{24}$$

i.e., following Anderson, Fang, and Hsu (1986) we simply change P to θP. Next setting $\theta = -\frac{1}{2}\alpha$ we write (24) as

$$\begin{aligned}\phi(g_{11}, g_{22}, r^2) &= Kg\left(\frac{d}{d\theta}\right)(g_{11}g_{22})^{\frac{1}{2}(n-2)}(1-r^2)^{\frac{1}{2}(n-3)} \\ &\quad \cdot \exp\left\{-\frac{1}{2}\alpha(p_{11}g_{11} - 2p_{12}r\sqrt{g_{11}g_{22}} + p_{22}g_{22})\right\}.\end{aligned} \tag{25}$$

Now in (25) set $p_{11}g_{11} = u\exp\{-v\}$, $p_{22}g_{22} = u\exp\{v\}$, and noting that the Jacobian of this transformation is

$$J(g_{11}, g_{22}; u, v) = 2(p_{11}p_{22})^{-1},$$

we find that

$$\phi(u, v, r^2) = Kg\left[\frac{d}{d\theta}\right]u^{n-1}(1-r^2)^{\frac{1}{2}(n-3)}\exp\left\{-\frac{1}{2}\alpha u(\cosh v - \rho r)\right\},$$

i.e.,

$$\begin{aligned}\phi(u, r^2) &= Kg\left[\frac{d}{d\theta}\right]\exp\{\theta u\}u^{n-1}\int_0^\infty(\cosh v - \rho r)^{-n}dv \\ &= Kg(u)u^{n-1}\psi(r^2),\end{aligned} \tag{26}$$

where $\psi(r^2)$ denotes the density (20).

Evidently, the result (26) states that

$$\begin{aligned}\phi(r^2) &= Kg\left[\frac{d}{d\theta}\right]\int_R \exp\{\theta \operatorname{tr} PG\}|G|^{\frac{1}{2}(n-3)}dG \\ &= Kg\left[\frac{d}{d\theta}\right]\int_0^\infty \exp\{\theta u\}u^{n-1}du\psi(r^2) \\ &= K\int_0^\infty g(u)u^{n-1}du\psi(r^2), \qquad (27)\end{aligned}$$

the region R of integration is defined by the equation $g_{11}r^2 = g_{12}g_{22}^{-1}g_{12}$.

It follows now from (27) that the density of the squared multiple correlation coefficient $R^2_{1\cdot 2\ldots p}$ is

$$\begin{aligned}\phi(R^2_{1\cdot R\ldots p}) = Kg\left[\frac{d}{d\theta}\right]\int_R \exp\left\{-\frac{1}{2}\alpha p_{11} - \alpha\mathbf{p}(1)'\mathbf{g}_{(1)} - \frac{1}{2}\alpha \operatorname{tr} P_{22}G_{22}\right\} \\ \cdot(g_{11} - \mathbf{g}(1)'G_{22}^{-1}\mathbf{g}(1))^{\frac{1}{2}(n-p-1)}|G_{22}|^{\frac{1}{2}(n-p-1)}dG, \qquad (28)\end{aligned}$$

where the region R of integration is defined by

$$\mathbf{g}(1)'G_{22}^{-1}\mathbf{g}(1) = g_{11}R^2_{1\cdot 2\ldots p}.$$

We first use (17) and integrate out $\mathbf{g}(1)$ from (28) to obtain

$$\begin{aligned}\phi(R^2_{1\cdot 2\ldots p}) = Kg\left(\frac{d}{d\theta}\right)\int \exp\left\{-\frac{1}{2}\alpha p_{11}g_{11} - \frac{1}{2}\alpha \operatorname{tr} P_{22}G_{22}\right\} \\ \cdot g_{11}^{\frac{1}{2}(n+2r)-1}|G_{22}|^{\frac{1}{2}(n-p)}(1-R^2_{1\cdot 2\ldots p})^{\frac{1}{2}(n-p-1)}(R^2_{1\cdot 2\ldots p})^{\frac{1}{2}(p-3)} \\ \cdot {}_0F_1\left[\frac{1}{2}(p-1); \frac{1}{4}R^2_{1\cdot 2\ldots p}(\alpha\mathbf{p}(1)'G_{22}\mathbf{p}(1)\alpha)\right).\end{aligned}$$

The next step is to use (18) and first integrate out G_{22}, and then integrate out g_{11}, to find that

$$\begin{aligned}\phi(R^2_{1\cdot 2\ldots p}) &= Kg\left(\frac{d}{d\theta}\right)\alpha^{-\frac{1}{2}pn}\psi(R^2_{1\cdot 2\ldots p}) \\ &= Kg\left(\frac{d}{d\theta}\right)\int_0^\infty e^{-\frac{1}{2}\alpha u}u^{\frac{1}{2}pn-1}du\psi(R^2_{1\cdot 2\ldots p}) \\ &= K\int_0^\infty g(u)u^{\frac{1}{2}np-1}du\psi(R^2_{1\cdot 2\ldots p}), \qquad (29)\end{aligned}$$

where $\psi(R^2_{1\cdot 2\ldots p})$ denotes the density (19). Note that when $p = 2$, (29) yields (27).

The density (21) follows from (29) by noting that

$$\begin{aligned}\phi(R) &= K\int_A g(\operatorname{tr} PG)|G|^{\frac{1}{2}(n-p-1)}dG \\ &= K\int_0^\infty g(u)u^{\frac{1}{2}np-1}du\psi(R),\end{aligned}$$

where $\psi(R)$ is the density (21), and the region A of integration is defined by $G_{11}^{1/2} R G_{11}^{1/2} = G_{12} G_{22}^{-1} G_{21}$.

We now proceed with the derivation of the densitites of $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4, \Lambda_5$. These are derived under the condition that $F_0 P_{12}{}' = 0$, i.e., $F_0 P_{22}^{-1} P_{12}' = 0$.

3. Wilks' Λ Factorization Theory

The simultaneous reduction of the quadratic form $S_{12} S_{22}^{-1} S_{21} = B$, and the linear form $F_0 S_{22}^{-1} S_{12}' = C_{xg}'$yields the result

$$B = S_{12} S_{22}^{-1} S_{12}' = C_{xg} (F_0 S_{22}^{-1} F_0')^{-1} C_{xg}' + L = ZZ' + L,$$

and hence

$$|T - B| = |T - ZZ' - L| = |T - ZZ'||I - M|, \tag{30}$$

where M is defined by $L = (T - ZZ')^{\frac{1}{2}} M (T - ZZ')^{\frac{1}{2}}$. Again, setting $ZZ' = T^{\frac{1}{2}} \Delta T^{\frac{1}{2}}$, we reduce (30) to the identity

$$|I - R||T| = |T||I - \Delta||I - M|, \tag{31}$$

i.e., to the factorization of $|I - R|$ to be

$$|I - R| = |I - \Delta||I - M|. \tag{32}$$

The matrix factorization corresponding to (32) is

$$T - B = T - ZZ' - L = (T - ZZ')^{\frac{1}{2}} (I - M)(T - ZZ')^{\frac{1}{2}}.$$

The next two factorizations of (32) are

$$|I - R| = \Lambda = |I - \Delta||I - M_{11}||I - M_{22} - M_{21}(I - M_{11})^{-1} M_{12}| = \Lambda_1 \Lambda_2 \Lambda_3$$

and

$$|I - R| = \Lambda = |I - \Delta||I - M_{22}||I - M_{11} - M_{12}(I - M_{22})^{-1} M_{21}| = \Lambda_1 \Lambda_4 \Lambda_5.$$

Obviously, from (31) and (20), it follows that the joint density of Δ and M is

$$\begin{aligned} \phi(\Delta, M) = K |I - M|^{\frac{1}{2}(n-p-1)} |M|^{\frac{1}{2}(p_2 - p_1 - g - 1)} |I - \Delta|^{\frac{1}{2}(n - p_1 - g - 1)} \\ \cdot |\Delta|^{\frac{1}{2}(p_1 - g - 1)} {}_2F_1 \left(\frac{1}{2} n, \frac{1}{2} n, \frac{1}{2}(p_2 - g); \rho(I - \Delta) M \right), \end{aligned} \tag{33}$$

where Δ is $g \times g$, M is $p_1 \times p_1$, M_{11} is $g \times g$ and M_{22} is $(p_1 - g) \times (p_1 - g)$.

Kshirsagar (1971) shows that Λ_1 is useful for testing the hypothesis that the first g canonical correlation coefficients are zero. He terms Λ_2 and Λ_5 to be direction factors and Λ_3 and Λ_4 to be collinearity factors.

By using the known integral

$$\int |M|^{a-\frac{1}{2}(p+1)}|I-M|^{b-\frac{1}{2}(p+1)}{}_2F_1(\alpha,\beta;\delta;MT)dM = B_p(a,b){}_3F_2(\alpha,\beta,a;\delta,a+b;T),$$

the marginal densities of Δ and M can be derived from (33).
However, it appears difficult to derive the marginal densities of Λ_2, Λ_3, Λ_4, Λ_5.

The equivalence of the conditions $F_0P'_{12} = 0$ and $F_0P_{22}^{-1}P'_{12} = 0$, can be seen by choosing Σ to be of the form

$$\Sigma = \begin{bmatrix} I_{p_1} & \rho & 0 \\ \rho & I_{p_1} & 0 \\ 0 & 0 & I_{p-2p_1} \end{bmatrix}, \text{ where } \rho = \mathrm{diag}(\rho_1,\ldots,\rho_g,0,\ldots,0).$$

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