ON A SUBRING OF PRIME RING WITH DERIVATION

CHEN-TE YEN

Abstract. Let $R$ be a noncommutative prime ring of characteristic not 2, and let $d$ be a nonzero derivation of $R$. We prove that the subring $V$ of $R$ generated by all $[d(x), y], x, y \in R$ contains a nonzero two-sided ideal of $R$.

I. Introduction

Throughout the note, $R$ will represent an associative ring. An additive mapping $d$ of $R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y$ in $R$. Let $R$ be a prime ring with center $Z$. We shall denote the commutator by $[x, y] = xy - yx$ for all $x, y$ in $R$. Posner [2] proved the

Lemma 1[2]. Let $R$ be a noncommutative prime ring, and let $d$ be a nonzero derivation of $R$. Then the subring of $R$ generated by all $[d(x), x], x \in R$ is not contained in $Z$.

Lemma 2[2]. Let $R$ be a prime ring of characteristic not 2. If there exist derivations $d$ and $g$ of $R$ such that $gd$ is a derivation of $R$, then either $d = 0$ or $g = 0$.

Recently, M. Brešar and J. Vukman showed the

Theorem A[1]. Let $R$ be a noncommutative prime ring of characteristic not 2, and let $d$ be nonzero derivation of $R$. Then $U$, the subring of $R$ generated by all $[d(x), x], x \in R$, contains a nonzero left ideal of $R$ and a nonzero right ideal of $R$.

There is an open question [1]: is it possible to generalize Theorem A by proving that $U$ contains a nonzero two-sided ideal? A linearization of this assumption gives
\[ [d(x), y] + [d(y), x] \in U \] for all \( x, y \) in \( R \). In the note, we prove that this question is true under the stronger hypothesis \([d(x), y] \in U\) for all \( x, y \) in \( R \).

2. Result

**Theorem.** Let \( R \) be a noncommutative prime ring of characteristic not 2, and let \( d \) be a nonzero derivation of \( R \). Then \( V \), the subring of \( R \) generated by all \([d(x), y], x, y \in R\), contains a nonzero two-sided ideal of \( R \).

**Proof.** By the assumption, we get
\[ [d(x), y] \in V \text{ for all } x, y \text{ in } R. \]
(1)
Replacing \( y \) by \( yz \) in (1), we have \([d(x), yz] \in V\) and so
\[ [d(x), y]z + y[d(x), z] \in V \text{ for all } x, y, z \text{ in } R. \]
(2)
Then with \( y = v \in V \) and \( z = u \in V \) in (2) respectively, and using (1), and noting that \( V \) is a subring of \( R \), we obtain
\[ [d(x), v]z \in V \text{ for all } x, z \in R \text{ and } v \in V. \]
(3)
and
\[ y[d(x), u] \in V \text{ for all } x, y \in R \text{ and } u \in V. \]
(4)
Replacing \( y \) by \( yt \) in (2), we have \([d(x), yt]z + yt[d(x), z] \in V\) and so
\[ [d(x), y]tz + y[d(x), t]z + yt[d(x), z] \in V \text{ for all } x, y, t, z \in R. \]
(5)
For all \( y, z, w \in R \), and \( u \in V \), by (3) and (4) we get \([d(w), u]z \in V \) and \( y[d(w), u] \in V \).
Replacing \( t \) by \([d(w), u] \) in (5), and applying these and (1), we obtain
\[ y[d(x), [d(w), u]]z \in V \text{ for all } x, y, z, w \in R \text{ and } u \in V. \]
(6)
We suppose that the Theorem is not true. Thus \( V \) does not contain nonzero two-sided ideals of \( R \). Hence (6) implies
\[ [d(x), [d(w), u]] = 0 \text{ for all } x, w \in R \text{ and } u \in V. \]
(7)
Assume that \( w \in R \) and \( u \in V \). Let \( g \) be the inner derivation defined by \( g(x) = [x, [d(w), u]] \), for all \( x \) in \( R \). Then using (7), we get \( gd(x) = 0 \). Thus \( gd = 0 \). By Lemma 2 and \( d \neq 0 \), this implies \( g = 0 \). Hence \([d(w), u] \in Z\). Applying this and (3), we have that \([d(w), u]R \) is an ideal of \( R \) and \([d(w), u]R \subseteq V \). Therefore, we obtain
\[ [d(w), u] = 0 \text{ for all } w \in R \text{ and } u \in V. \]
(8)
Let \( h \) be the inner derivation defined by \( h(w) = [w, u] \) for all \( w \in R \). Then by (8), we get \( hd(w) = 0 \). Thus \( hd = 0 \). By Lemma 2 and \( d \neq 0 \), this implies \( h = 0 \). Hence \( u \in Z \). Thus, \( V \subseteq Z \) which contradicts Lemma 1. This completes the proof of the Theorem.
References


Department of Mathematics, Chung Yuan University, Chung Li, Taiwan, 320, Republic of China.