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ON A SUBRING OF PRIME RING WITH DERIVATION

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Abstract. Let R be a noncommutative prime ring of characteristic not 2, and let d be a nonzero derivation of R. We prove that the subring V of R generated by all $[d(x), y], x, y \in R$ contains a nonzero two-sided ideal of R.

I. Introduction

Throughout the note, R will represent an associative ring. An additive mapping d of R is called a derivation if $d(xy) = d(x)y + xd(y)$ hold for all x, y in R. Let R be a prime ring with center Z. We shall denote the commutator by $[x, y] = xy - yx$ for all x, y in R. Posner [2] proved the

Lemma 1[2]. *Let R be a noncommutative prime ring, and let d be a nonzero derivation of R. Then the subring of R generated by all $[d(x), x], x \in R$ is not contained in Z.*

Lemma 2[2]. *Let R be a prime ring of characteristic not 2. If there exist derivations d and g of R such that gd is a derivation of R, then either $d = 0$ or $g = 0$.*

Recently, M. Brešar and J. Vukman showed the

Theorem A[1]. *Let R be a noncommutative prime ring of characteristic not 2, and let d be nonzero derivation of R. Then U, the subring of R generated by all $[d(x), x], x \in R$, contains a nonzero left ideal of R and a nonzero right ideal of R.*

There is an open question [1]: is it possible to generalize Theorem A by proving that U contains a nonzero two-sided ideal ? A linearization of this assumption gives

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$[d(x), y] + [d(y), x] \in U$ for all x, y in R. In the note, we prove that this question is true under the stronger hypothesis $[d(x), y] \in U$ for all x, y in R.

2. Result

Theorem. *Let R be a noncommutative prime ring of characteristic not 2, and let d be a nonzero derivation of R. Then V, the subring of R generated by all $[d(x), y], x, y \in R$, contains a nonzero two-sided ideal of R.*

Proof. By the assumption, we get

$$[d(x), y] \in V \text{ for all } x, y \text{ in } R. \quad (1)$$

Replacing y by yz in (1), we have $[d(x), yz] \in V$ and so

$$[d(x), y]z + y[d(x), z] \in V \text{ for all } x, y, z \text{ in } R. \quad (2)$$

Then with $y = v \in V$ and $z = u \in V$ in (2) respectively, and using (1), and noting that V is a subring of R, we obtain

$$[d(x), v]z \in V \text{ for all } x, z \in R \text{ and } v \in V. \quad (3)$$

and

$$y[d(x), u] \in V \text{ for all } x, y \in R \text{ and } u \in V. \quad (4)$$

Replacing y by yt in (2), we have $[d(x), yt]z + yt[d(x), z] \in V$ and so

$$[d(x), y]tz + y[d(x), t]z + yt[d(x), z] \in V \text{ for all } x, y, t, z \in R. \quad (5)$$

For all $y, z, w \in R$, and $u \in V$, by (3) and (4) we get $[d(w), u]z \in V$ and $y[d(w), u] \in V$. Replacing t by $[d(w), u]$ in (5), and applying these and (1), we obtain

$$y[d(x), [d(w), u]]z \in V \text{ for all } x, y, z, w \in R \text{ and } u \in V. \quad (6)$$

We suppose that the Theorem is not true. Thus V does not contain nonzero two-sided ideals of R. Hence (6) implies

$$[d(x), [d(w), u]] = 0 \text{ for all } x, w \in R \text{ and } u \in V. \quad (7)$$

Assume that $w \in R$ and $u \in V$. Let g be the inner derivation defined by $g(x) = [x, [d(w), u]]$, for all x in R. Then using (7), we get $gd(x) = 0$. Thus $gd = 0$. By Lemma 2 and $d \neq 0$, this implies $g = 0$. Hence $[d(w), u] \in Z$. Applying this and (3), we have that $[d(w), u]R$ is an ideal of R and $[d(w), u]R \subseteq V$. Therefore, we obtain

$$[d(w), u] = 0 \text{ for all } w \in R \text{ and } u \in V. \quad (8)$$

Let h be the inner derivation defined by $h(w) = [w, u]$ for all $w \in R$. Then by (8), we get $hd(w) = 0$. Thus $hd = 0$. By Lemma 2 and $d \neq 0$, this implies $h = 0$. Hence $u \in Z$. Thus, $V \subseteq Z$ which contradicts Lemma 1. This completes the proof of the Theorem.

References

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