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EXPANSIONS OF APPELL AND LAURICELLA FUNCTIONS

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Abstract. Expansion formulas for $F_D^{(r+1)}$ in terms of products of ${}_2F_1$ and $F_D^{(r)}$ has been obtained by Srivastava et al. [5, 3.9]. In this note we present an extension of this result. Possibility of existence of analogous formulas for Lauricella functions $F_A^{(r)}, F_B^{(r)}$ and $F_C^{(r)}$ are examined and corresponding results for Appell hypergeometric functions are also derived.

1. Introduction

The Pochhammer symbol $(\lambda)_n$ is defined by,

$$(\lambda)_n = \begin{cases} 1 & ; n = 0 \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & ; n = 1, 2, \ldots \end{cases} \tag{1.1}$$

or in terms of Gamma function,

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \qquad \lambda \neq 0, -1, -2, \ldots \tag{1.2}$$

The Gauss's hypergeometric series, ${}_2F_1\ (a, b; c; z)$ is defined as,

$${}_2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \cdot \frac{z^m}{m!}, \qquad c \neq 0, -1, -2, \ldots \tag{1.3}$$

which converges absolutely if $|z| < 1$ and $|z| = 1$, if $Re(c-a-b) > 0$ and in that case,

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{1.4}$$

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Srivastava and Goyal [5, 3.9] obtained a typical expansion formula for $F_D^{(r+1)}$ in terms of products of ${}_2F_1$ and $F_D^{(r)}$. Non-availability of such type of expansion formulas for all other Lauricella's functions and generalized hypergeometric functions leads us to application of identities in obtaining expansion formulas for all hypergeometric series. To this end we shall consider the following identities (for m and n being positive integers):

$$\sum_{k=0}^{\infty}\frac{(c-a)_k(c_1-n)_k}{(c+c_1+m)_k k!}=\frac{\Gamma(c+c_1+m)\Gamma(a+m+n)}{\Gamma(a+c_1+m)\Gamma(c+m+n)}; Re(c+c_1)\neq 0,-1,-2,\ldots; Re(a)>0, \tag{1.5}$$

$$\sum_{k=0}^{\infty}\frac{(-a_1-m)_k(a_1-n)_k}{(a)_k k!}=\frac{\Gamma(a)\Gamma(a+m+n)}{\Gamma(a+a_1+m)\Gamma(a-a_1+n)}; Re(a)>0 \tag{1.6}$$

and

$$\sum_{k=0}^{\infty}\frac{(c_1-c-n)_k(c-c_2+m)_k}{(c_1+m)_k k!}=\frac{\Gamma(c_1+m)\Gamma(c_2+n)}{\Gamma(c_1+c_2+c)\Gamma(c+m+n)}; Re(c_1)\neq 0,-1,-2,\ldots; Re(c_2)>0, \tag{1.7}$$

The proof of (1.5) follows on using Gauss theorem, if we assume $R(c+c_1)\neq 0,-1,-2,\ldots$ and $Re(a)>0$. The proof of (1.6) and (1.7) follow on similar lines.

2. Expansion Formulas for Appell Functions:

Let,

$$A_1=\frac{\Gamma(c)\Gamma(a+c_1)}{\Gamma(a)\Gamma(c+c_1)};\ A_2=\frac{\Gamma(a+a_1)\Gamma(a-a_1)}{\Gamma(a)\Gamma(a)};\ A_3=\frac{\Gamma(c_1+c_2-c)\Gamma(c)}{\Gamma(c_1)\Gamma(c_2)}$$

$$\phi_1(k)=\frac{(c-a)_k(c_1)_k}{(c+c_1)_k k!};\ \phi_2(k)=\frac{(a_1)_k(-a_1)_k}{(a)_k k!}\ \text{and}\ \phi_3(k)=\frac{(c-c_1)_k(c-c_2)_k}{(c_1)_k k!},$$

then the expansion formulas that are established in this section are,

$$F_1(a,b_1;b_2;c;z_1,z_2)$$

$$=A_1\sum_{k=0}^{\infty}\phi_1(k)\cdot{}_2F_1\left(\begin{matrix}a+c_1;\\ c+c_1+k;\end{matrix}\ z_2\right)\cdot{}_2F_1\left(\begin{matrix}1-c_1,b_2;\\ 1-c_1-k;\end{matrix}\ z_2\right);$$
$$Re(c+c_1)\neq 0,-1,-2,\ldots; Re(a)>0, \tag{2.1}$$

$$=A_2\sum_{k=0}^{\infty}\phi_2(k)\cdot F^{0:3,3}_{1:1,1}\left(\begin{matrix}-: & 1+a_1,a+a_1,b_1; & 1-a_1,a-a_1,b_2;\\ c: & 1+a_1-k; & 1-a_1-k;\end{matrix}\ z_1,z_2\right);$$
$$Re(a)>0, \tag{2.2}$$

$$=A_3\sum_{k=0}^{\infty}\phi_3(k)\cdot F^{1:2;2}_{0:2;2}\left(\begin{matrix}a: & c-c_2+k,b_1; & 1-c_1+c,b_2;\\ -: & c-c_2,c_1+k; & 1-c_1+c-k;\end{matrix}\ z_1,z_2\right);$$
$$Re(c_1)\neq 0,-1,-2,\ldots; Re(c_2)>0, \tag{2.3}$$

$$F_2(a,b_1,b_2;c_3,c_4;z_1,z_2)$$

$$=A_1\sum_{k=0}^{\infty}\phi_1(k)\cdot F^{1:2;2}_{0:2;2}\left(\begin{matrix} c: & a+c_1,b_1; & 1-c_1,b_2; \\ -: & c_1+k,c_3; & 1-c_1-k,c_4; \end{matrix}\; z_1,z_2\right);$$

$$Re(c+c_1)\neq 0,-1,-2,\ldots;Re(a)>0 \tag{2.4}$$

$$=A_2\sum_{k=0}^{\infty}\phi_2(k)\cdot {}_3F_2\left(\begin{matrix} 1+a_1,a+a_1,b_1; \\ 1+a_1-k,c_3; \end{matrix}\; z_1\right)\cdot {}_3F_2\left(\begin{matrix} 1-a_1,a-a_1,b_2; \\ 1-a_1-k,c_4; \end{matrix}\; z_2\right);$$

$$Re(a)>0 \tag{2.5}$$

$$=A_3\sum_{k=0}^{\infty}\phi_3(k)\cdot F^{2:2;2}_{0:3;3}\left(\begin{matrix} a,c: & c-c_2+k,b_1; & 1-c_1+c,b_2 \\ -: & c-c_2,c_1+k; & 1-c_1+c-k,c_2,c_4; \end{matrix}\; z_1,z_2\right);$$

$$Re(c_1)\neq 0,-1,-2,\ldots;Re(c_2)>0, \tag{2.6}$$

$$F_3(a_2,a_3,b_1,b_2;c;z_1,z_2)$$

$$=A_1\sum_{k=0}^{\infty}\phi_1(k)\cdot F^{0:3;3}_{1:1;1}\left(\begin{matrix} -: & a+c_1,a_2,b_1; & 1-c_1,a_3,b_2; \\ a: & c+c_1+k; & 1-c_1-k; \end{matrix}\; z_1z_2\right);$$

$$Re(c+c_1)\neq 0,-1,-2,\ldots;Re(a)>0, \tag{2.7}$$

$$=A_2\sum_{k=0}^{\infty}\phi_2(k)\cdot F^{0:4;4}_{2:1;1}\left(\begin{matrix} -: & 1+a_1,a+a_1,a_2,b_1; & 1-a_1,a-a_1,a_3,b_2; \\ a,c: & 1+a_1-k; & 1-a_1-k; \end{matrix}\; z_1,z_2\right);$$

$$Re(a)>0, \tag{2.8}$$

$$=A_3\sum_{k=0}^{\infty}\phi_3(k)\cdot {}_3F_2\left(\begin{matrix} c-c_2+k,a_2,b_1; \\ c-c_2,c_1+k \end{matrix}\; z_1\right)\cdot {}_3F_2\left(\begin{matrix} 1-c_1+c,a_3,b_2; \\ 1-c_1+c-k,c_2; \end{matrix}\; z_2\right);$$

$$Re(c_1)\neq 0,-1,-2,\ldots;Re(c_2)>0, \tag{2.9}$$

$$F_4(a,b,;c_3,c_4;z_1,z_2)$$

$$=A_1\sum_{k=0}^{\infty}\phi_1(k)\cdot F^{2:1;1}_{0:2;2}\left(\begin{matrix} b,c: & a+c_1; & 1-c_1; \\ -: & c+c_1+k,c_3; & 1-c_1-k,c_4; \end{matrix}\; z_1,z_2\right);$$

$$Re(c+c_1)\neq 0,-1,-2,\ldots;Re(a)>0, \tag{2.10}$$

$$=A_2\sum_{k=0}^{\infty}\phi_2(k)\cdot F^{1:2;2}_{0:2;2}\left(\begin{matrix} b: & 1+a_1,a+a_1,a_2; & 1-a_1,a-a_1,a_3; \\ -: & 1-a_1-k,c_3; & a_1-k,c_4; \end{matrix}\; z_1,z_2\right);$$

$$Re(a)>0 \tag{2.11}$$

and

$$=A_3\sum_{k=0}^{\infty}\phi_3(k)\cdot F^{3:1;1}_{0:3;3}\left(\begin{matrix} a.b.c: & c-c_2+k; & 1-c_1+c; \\ -: & c-c_2,c_1+k,c_3; & 1-c_1+c-k,c_2,c_3; \end{matrix}\; z_1,z_2\right);$$

$$Re(c_1) \neq 0, -1, -2, \ldots; Re(c_2) > 0. \tag{2.12}$$

For proof of (2.1), consider,

$$\frac{\Gamma(c)\Gamma(a+c_1)}{\Gamma(a)\Gamma(c+c_1)} \sum_{k=0}^{\infty} \frac{(c-a)_k (c_1)_k}{(c+c_1)_k k!} \cdot {}_2F_1\left(\begin{matrix} a+c_1, b_1; \\ c+c_1+k; \end{matrix} \; z_1\right) \cdot {}_2F_1\left(\begin{matrix} 1-c_1, b_2; \\ 1-c_1-k; \end{matrix} \; z_2\right)$$

$$= \frac{\Gamma(c)\Gamma(a+c_1)}{\Gamma(a)\Gamma(c+c_1)} \sum_{k=0}^{\infty} \sum_{m,n=0}^{\infty} (b_1)_m (b_2)_n \cdot \frac{z_1^m}{m!} \cdot \frac{z_2^n}{n!} \cdot \frac{(c-a)_k (c_1)_k}{(c+c_1)_k k!} \cdot \frac{(a+c_1)_m (1-c_1)_n}{(c+c_1+k)_m (1-c_{1_k})_n}.$$

Interchanging summation under the stated conditions and using the identity (1.5), the L.H.S of (2.1) is established. In a similar manner, the proof of (2.2) and (2.3) follow on using the identities (1.6) and (1.7), under the stated conditions.

3. Extension in terms of Kampé de Fériet Functions

Extension of all these formulas in terms of Kampé de Fériet's functions admit the forms,

$$F^{1:r;u}_{1:s;v}\left(\begin{matrix} a: & (b_r), (e_u); \\ c: & (d_s); (f_v); \end{matrix} \; z_1\right)$$

$$= A_1 \sum_{k=0}^{\infty} \phi_1(k) \cdot {}_{r+1}F_{s+1}\left(\begin{matrix} a+c_1, (b_r); \\ c+c_1+k, (d_s); \end{matrix} \; z_1\right) \cdot {}_{u+1}F_{v+1}\left(\begin{matrix} 1-c_1, (e_u); \\ 1-c_1-k, (f_v); \end{matrix} \; z_2\right);$$

$$Re(c+c_1) \neq 0, -1, -2, \ldots, Re(a) > 0, \tag{3.1}$$

$$= A_2 \sum_{k=0}^{\infty} \phi_2(k) \cdot F^{0:r+2;u+2}_{1:s+1.v+1}\left(\begin{matrix} a: & a+a_1, 1+a_1, (b_r); & a-a_1, 1-a_1, (c_u); \\ c: & 1+a_1-k, (d_s); & 1-a_1-k, (f_v) \end{matrix} \; z_1, z_2\right);$$

$$Re(a) > 0 \tag{3.2}$$

$$= A_3 \sum_{k=0}^{\infty} \phi_3(k) \cdot F^{1:r+1;u+1}_{0:s+2;v+2}\left(\begin{matrix} a: & c-c_2+k, (b_r); & 1-c_1+c(e_u); \\ -: & c-c_2, c_1+k, (d_s); & 1-c_1+c-k, c_2, (f_v); \end{matrix} \; z_1, z_2\right);$$

$$Re(c_1) \neq 0, -1, -2, \ldots; Re(c_2) > 0 \tag{3.3}$$

$$F^{1:r;u}_{0:s;v}\left(\begin{matrix} a: & (b_r); (e_u); \\ -: & (d_s); (f_v); \end{matrix} \; z_1, z_2\right)$$

$$= A_1 \sum_{k=0}^{\infty} \phi_1(k) \cdot F^{1:r+1;u+1}_{0:s+1,v+1}\left(\begin{matrix} c: & a+c_1, (b_r); & 1-c_1, (c_u); \\ -: & c+c_1+k, (d_s); & 1-c_1-k, (f_v); \end{matrix} \; z_1, z_2\right);$$

$$Re(c+c_1)\neq 0,-1,-2,\ldots;Re(a)>0, \tag{3.4}$$

$$=A_2\sum_{k=0}^{\infty}\phi_2(k)\cdot{}_{r+2}F_{s+1}\left(\begin{matrix}(b_r),1+a_1,a+a_1;\\ \\(d_s),1+a_1-k;\end{matrix}\ z_1\right)\cdot{}_{u+2}F_{v+1}\left(\begin{matrix}(e_u),1-a_1,a-a_1;\\ \\(f_v),1-a_1-k;\end{matrix}\ z_2\right);$$

$$Re(a)>0, \tag{3.5}$$

$$=A_3\sum_{k=0}^{\infty}\phi_3(k)\cdot F^{1:r+1;u+1}_{0:s+2;v+2}\left(\begin{matrix}a,c: & (b_r),c-c_2+k; & (e_u),1-c_1+c;\\ \\ -: & (d_s),c_1+k,c-c_2; & (f_v),1-c_1+c-k,c_2;\end{matrix}\ z_1,z_2\right);$$

$$Re(c_1)\neq 0,-1,-2,\ldots;Re(c_2)>0, \tag{3.6}$$

$$F^{0:r;u}_{1:s;v}\left(\begin{matrix}-: & (b_r); & (e_u);\\ \\ c: & (d_s); & (f_v);\end{matrix}\ z_1,z_2\right)$$

$$=A^{\infty}_{k=0}\phi_1(k)\cdot F^{0:r+1;u+1}_{1:s+1;v+1}\left(\begin{matrix}-: & a+c_1,(b_r); & 1-c_1,(e_u);\\ \\ a: & c+c_1+k,(d_s); & 1-c_1-k,(f_v);\end{matrix}\ z_1,z_2\right);$$

$$Re(c+c_2)\neq 0,-1,-2,\ldots,Re(a)>0, \tag{3.7}$$

$$=A_2\sum_{k=0}^{\infty}\phi_2(k)\cdot F^{0:r+2;u+2}_{2:s+1,v+1}\left(\begin{matrix}-: & (b_r),1+a_1,a+a_1; & (e_u),1-a,a-a_1;\\ \\ a,c: & (d_s),1+a_1-k; & (f_v),1-a_1-k:\end{matrix}\ z_1,z_2\right);$$

$$Re(a)>0 \tag{3.8}$$

and

$$=A_3\sum_{k=0}^{\infty}\phi_3(k)\cdot{}_{r+1}F_{s+2}\left(\begin{matrix}(b_r),c-c_2+k;\\ \\(d_s),c_1+k,c-c_2;\end{matrix}\ z_1\right){}_{u+1}F_{v+2}\left(\begin{matrix}(e_u),1-c_1+c;\\ \\(f_v),1-c_1+c-k,c_2;\end{matrix}\ z_2\right);$$

$$Re(c_1)\neq 0,-1,-2,\ldots;Re(c_2)>0 \tag{3.9}$$

The inverse series expansion corresponding to (3.1), (3.2) and (3.3) are given by

$${}_rF_s\left(\begin{matrix}(b_r);\\ \\(d_s)\quad;\end{matrix}\ z_1\right)\cdot{}_uF_v\left(\begin{matrix}(e_u);\\ \\(f_v);\end{matrix}\ z_2\right)$$

$$=A_1\sum_{k=0}^{\infty}\phi_1(k)\cdot F^{1:r+1;u+1}_{0;s+2;v+2}\left(\begin{matrix}a: & a+c_1,b_r; & 1-c_1(e_u);\\ \\ a: & c+c_1+k,(d_s); & 1-c_1-k,(f_v);\end{matrix}\ z_1,z_2\right);$$

$$Re(c+c_1)\neq 0,-1,-2,\ldots;Re(a)>0, \tag{3.10}$$

$$=A_2\sum_{k=0}^{\infty}\phi_2(k)\cdot F^{0:r+2;u+2}_{1:s+1,v+1}\left(\begin{matrix}-: & (b_r),1+a_1,a+a_1; & (e_u),1-a_1,a-a_1\quad;\\ \\ a: & (d_s),1+a_1-k; & (f_v),1-a_1-k:\end{matrix}\ z_1,z_2\right);$$

$$Re(a)>0 \tag{3.11}$$

and

$$=A_3\sum_{k=0}^{\infty}\phi_3(k)\cdot F^{1:r+1;u+1}_{0:s+2;v+2}\left(\begin{matrix} c: & (b_r),c-c_2+k; & (e_u)-c_1+c; \\ -: & (d_s),c_1+k,c-c_2; & (f_v),1-c_1+c-k,c_2; \end{matrix} z_1,z_2\right);$$

$$Re(c_1)\neq 0,-1,-2,\ldots;Re(c_2)>0 \tag{3.12}$$

4. Extension in terms of Lauricella's functions:

The expansion for $F_D^{(r+1)}$ given by Srivastava and Goyal [5, 3.9] is

$$F_D^{(r+1)}[a,b_1,\ldots,b_r,\beta;c;z_1,\ldots z_r,w) \tag{4.1}$$

$$=\sum_{m=0}^{\infty}\binom{\lambda-u}{m}\frac{\Gamma(u)}{\Gamma(\mu+m)\Gamma(1-m)}\cdot{}_2F_1(1,\alpha;1-m;w)F_D^{(r)}(\lambda,\beta_1,\ldots,\beta_r;\mu+m;z_1,\ldots,z_r).$$

The extension of this formula is given by:

$$F_D^{(r+1)}[a,b_1,\ldots,b_r,\beta;c;z_1,\ldots z_r,w)$$

$$=A_1\sum_{k=0}^{\infty}\phi_1(k)\cdot F_D^{(r)}(a+c_1,b_1,b_r;c+c_1+k;z_1,z_r)\cdot{}_2F_1\left(\begin{matrix}1-c_1,\beta; \\ 1-c_1-k;\end{matrix} w\right);$$

$$Re(c+c_1)\neq 0,-1,-2,\ldots;Re(a)>0. \tag{4.2}$$

Indeed this follows by expressing the R.H.S. in the form;

$$\sum_{m_1,\ldots,m_r,n}^{\infty}(b_1)_{m_1}\cdots(b_r)_{m_r}(\beta)_n\cdot\frac{z_1^{m_r}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!}\cdot\frac{w^n}{n!}\Big\{\frac{\Gamma(C)\Gamma(a+c_1)}{\Gamma(a)\Gamma(c+c_1)}$$

$$\cdot\sum_{k=0}^{\infty}\frac{(c-a)_k(c_1)_k}{(c+c_1)_k k!}\cdot\frac{(a+c_1)_{m_1+\cdots+m_r}(1-c_1)_n}{(c+c_1+k)_{m_1+\cdots+m_r}(1-c_1-k)_n}\Big\}.$$

Then summing w.r. to k by using the identity (1.5) for $m=m_1+\cdots+m_r$, after appropriate simplication, one obtains the L. H. S. of (4.2). This corresponds to Srivastava and Goyal's result (4.1) for $c_1\to 0$.

Using the identity (1.6) and (1.7) we have respectively for $F_A^{(r)}$ and $F_B^{(r)}$, the formulas:

$$F_A^{(r+1)}[a,b_1,\ldots,b_r,\beta;c_1,\ldots,c_r,v;z_1,\ldots,z_r,w)$$

$$=A_2\sum_{k=0}^{\infty}\phi_2(k)F^{2:1;\ldots;1}_{1:1;\ldots;1}\left(\begin{matrix}1+a_1,a+a_1: & b_1;\ldots & ;b_r; \\ 1+a_1: & c_1;\ldots & ;c_r;\end{matrix} z_1,\ldots,z_r\right)$$

$$\cdot{}_3F_2\left(\begin{matrix}1-a_1,a-a_1,\beta; \\ 1-a_1-k,v;\end{matrix} w\right);Re(a)>0, \tag{4.3}$$

and

$$F_B^{(r+1)}[a_1,\dots,a_r,\alpha,b_1,\dots,b_r,\beta;c;z_1,\dots,z_r,w)$$
$$=A_3\sum_{k=0}^{\infty}\phi_3(k)F^{1:2;\dots;2}_{2:0;\dots;0}\left(\begin{matrix}c-a_2+k,a_1,b_1;\dots\quad ;a_r,b_r;\\ c-c_2,c_1+k:-;\dots\quad ;-;\end{matrix}\ z_1,\dots,z_r\right)$$
$$\cdot{}_3F_2\left(\begin{matrix}1-c_1+c,\alpha,\beta;\\ 1-c_1+c-k,c_2;\end{matrix}\ w\right);Re(c_1)\neq 0,-1,-2,\dots;Re(c_2)>0. \quad (4.4)$$

However, repeated use of identify (1.6) for $F_C^{(r)}$ gives:

$$F_C^{(r+1)}[a,b;c,\dots,c_r,v;z_1,\dots,z_r,w)$$
$$=A\sum_{k,R=0}^{\infty}\phi(k,R)F^{4:0;\dots 0}_{2:1;\dots;1}\left(\begin{matrix}1+a_1,a+a_1,1+b_1,b+b_1:;\dots\quad ;-;\\ 1+a_1-k,1+b_1-R:c_1;\dots\quad ;c_r;\end{matrix}\ z_1,\dots,z_r\right)$$
$$\cdot{}_4F_3\left(\begin{matrix}1-a_1,a-a_1,1-b_1,b-b_1;\\ 1-a_1-k,1-b_1-R,v;\end{matrix}\ w\right), \quad (4.5)$$

Provided $Re(a)>0, Re(b)>0$,
where

$$A\equiv\frac{\Gamma(a+a_1)\Gamma(a-a_1)\Gamma(b+b_1)\Gamma(b-b_1)}{\Gamma(a)\Gamma(a)\Gamma(b)\Gamma(b)}\text{ and }\phi(k,R)\equiv\frac{(a_1)_k(-a_1)_k(b_1)_R-(-b_1)_R}{(a)_k(b)_Rk!R!}.$$

Similarly, by using identities (1.6) and (1.7), we get

$$F_D^{(r+1)}(a,b_1,\dots,b_r,\beta;c;z_1,\dots,z_r,w)$$
$$=B\sum_{k,R=0}^{\infty}\phi^1(k,R)\cdot{}_4F_3\left(\begin{matrix}a_{a_1},1-a_1,1+c-c_1,\beta;\\ 1-a_1-k,1+c-c_1-k,c_2;\end{matrix}\ w\right)$$
$$\cdot F^{3:1;\dots;1}_{3:0;\dots;0}\left(\begin{matrix}a+a_1,1+a_1,c-c_2+k:a_1;\cdots a_r,;\\ 1+a_1-k,c-c_2,c_1+k:-;\cdots;\end{matrix}\ z_1,\dots,z_2\right)$$
$$Re(c_1)\neq 0,-1,-2,\dots;Re(a)>0;Re(c_2)>0. \quad (4.6)$$

where

$$B\equiv\frac{\Gamma(a+a_1)\Gamma(a-a_1)\Gamma(c)\Gamma(c_1+c_2-c)}{\Gamma(a)\Gamma(a)\Gamma(c_1)\Gamma(c_2)}\text{ and }\phi^1(k,r)\equiv\frac{(a_1)_k(-a_1)_k(c_1-c)_R(c-c_2)_R}{(a)_k(c_1)_Rk!R!}.$$

The repeated use of the identity (1.5), gives

$$F_D^{(r+1)}(a,b_1,b_{r+1};c;z_1,\dots,z_{r+1})$$
$$=\frac{\Gamma(c)}{\Gamma(a)}\sum_{k_1,\dots,k_r=0}^{\infty}(c-a)_{k_1+\dots+k_r}\cdot\frac{\Gamma(a+\sum_{i=1}^r c_i)}{\Gamma(c+\sum_{i=1}^r c_i+\sum_{i=1}^r k_i)}$$
$$\cdot\prod_1^r\left\{\frac{(c_i)_{k_i}}{(k_i)!}\cdot{}_2F_1\left(\begin{matrix}1-c_i,b_i;\\ 1-c_i-k_i;\end{matrix}\ z_i\right)\right\}\cdot{}_2F_1\left(\begin{matrix}a+\sum_i^r c_i,b_{r+1};\\ c+\sum_1^r c_i+\sum_1^r k_i;\end{matrix}\ z_{r+1}\right), \quad (4.7)$$

provided $Re(a) > 0; Re(c+\sum_{i=1}^{j} c_i) \neq 0, -1, -2, \ldots; Re(a+\sum_{i=1}^{j} c_i) > 0; j = 1, 2, \ldots, r.$

It may be remarked that expansion formula for $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ analogous to $F_D^{(r)}$ that express the function in series of the same type of function are not possible. However, such symmetrical results are possible in terms of multiple series expansions if one expresses Lauricella's function in terms of Kampé de Fériet function of several variables due to Srivastava and Daoust [4]. For example,

$$\begin{aligned}
&F_A^{(r+1)}(a, b_1, \ldots, b_r, \beta; c_1, \ldots, c_r, v; z_1, \ldots z_r, w) \qquad (4.8)\\
=&F_{0:1;\ldots;1}^{1:1;\ldots;1}\left(\begin{matrix} 1: & a_1; \ldots; a_r; \beta; & \\ & & z_1, \ldots, z_r, w \\ -: & c_1; \ldots c_r; v; & \end{matrix}\right)\\
=&A_1 \sum_{k_1,\ldots,k_r}^{\infty} \Phi_1(k_1, \ldots, k_r) \cdot {}_3F_2\left(\begin{matrix} a-a_1, 1-a_1, \beta; & \\ & w \\ 1-a_1-k_r, \gamma; & \end{matrix}\right)\\
&\cdot F_{0:2;\ldots;2}^{1:2;\ldots;2}\left(\begin{matrix} 1+a_1 : 1-\gamma_1, b_1; \ ; 1-\gamma_{r-1}, b_{r-1}; a+a_1+\sum_{i=1}^{r-1}\gamma_i, b_r; & \\ & z_1, \ldots, z_r \\ - : 1-\gamma_1-k_1, c_1; \ ; 1-\gamma_{r-1}, c_{r-1}-k_{r-1}; 1+a_1-k_r+\sum_{i=1}^{r-1}\gamma_i+\sum_{i=1}^{r-1}k_i, c_r & \end{matrix}\right),
\end{aligned}$$

provided $Re(a) > 0$, $Re(1 + a_1 + \sum_{i=1}^{j} \gamma_i) \neq 0, -1, -2, \ldots; Re(a + a_1 + \sum_{i=1}^{j} \gamma_i) > 0;$ $j = 1, 2, \ldots, r-1.$

$$\Phi_1(k_1, \ldots, k_r) \equiv (1-a_1-k_r)_{k_1+\cdots+k_r} \prod_1^{r-1} \left\{\frac{(\gamma_i)_{k_i}}{(k_i)!}\right\} \cdot \frac{\Gamma(1+a_1-k_r)}{\Gamma(1+a_1-k_r+\sum_1^{r-1}\gamma_i+\sum_1^{r-1}k_i)},$$

$$A_1 \equiv \frac{\Gamma(a-a_1)\Gamma(a+a_1+\sum_1^{r-1}\gamma_i)}{\Gamma(a)\Gamma(a)}$$

Similarly, results can also be established for $F_B^{(r+1)}$ and $F_C^{(r+1)}$, but the details are omitted for reason of brevity. It may be remarked in conclusion that various permutations and combinations of identities (1.5), (1.6) and (1.7) may be taken to derive numerous other forms of formulas for these functions.

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