TAMKANG JOURNAL OF MATHEMATICS
Volume 27, Number 4, Winter 1996

ON THE DIRECT PROBLEM AND SCATTERING DATA FOR A SINGULAR SYSTEM OF DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

A. A. DARWISH

Abstract. A system of Sturm-Liouville differential equations of order n with a density matrix function is considered. The direct problem of the considered system is studied and hence the scattering data of the problem is obtained.

Introduction

We consider the system of Sturm-Liouville differential equations of order n

$$-y'' + Q(x)y = \lambda\rho(x)y, \qquad (0 \leq x \prec \infty) \tag{1}$$

and the boundary condition

$$y'(0) = 0, \tag{2}$$

where $Q(x)$ is a self-adjoint matrix function of order n with real elements defined and continuous on $[0, \infty)$.

The condition

$$\int_0^\infty x||Q(x)||dx \prec \infty \tag{3}$$

is assumed to hold throughout in this paper.

Also, the matrix function $\rho(x)$ has the form

$$\rho(x) = \begin{cases} \alpha_n^2, & a_{n-1} \leq x \prec a_n \\ E_n, & a_n \leq x \prec \infty. \end{cases} \tag{4}$$

Received June 9, 1995.

1991 *Mathematics Subject Classification.* 46, 47, A, B, E.

Key words and phrases. Direct problem-Scattering data-System-Parsevals equation-Discontinuous coefficients.

where $\alpha_n \succ 0, a_0 = 0, a_n \neq a_{n+1}, n = 1, \dots, m-1, a_m \neq 1$ and a_n are the diagonal elements of a matrix of order $n \times n$ such that they do not coinside with the identity matrix E_n.

Denote by W_n and $L_2(0, \infty : \rho(x))$ to the set of the matrix functions $\rho(x)$ and the set of all vector functios
$f(x) = \{f_1(x), f_2(x), \dots, f_n(x)\}$ with elements in $L_2(0, \infty)$ respectivey.

In $L_2(0, \infty; \rho(x))$ we introduce the scalar product (f.g.)$- \int_0^\infty \sum_{j=1}^n f_j(x)\bar{g}_j(x)dx$ and consider that (1)-(2) arising in $L_2(0, \infty; \rho(x))$.

The problem (1)-(2) was investigated earlier in the scalar form in the papers [1.8] when $\rho(x) = E_n$ and for the case $n = 1$ this problem has been discussed in the works [2,4,5,7]. So this paper is aimed to extend those previous results.

It is well known [4] that the collection of quantities $\{S(k), -\tau_n^2, M_n,\ n = 1, \dots, m\}$ is called the scattering data of the system (1)-(2), where $S(k)$ is the scattering matrix function and M_n are nonnegative matrices of order n whose ranks coincide with the multiplicity of the eigenvalues $-\tau_n^2$ of the problem (1)-(2).

This article is aimed to study the direct problem and hence to obtain the scattering data of the problem (1)-(2).

0. Notation

Throughout this paper we use the following notation:

$-E_n$ is the unit matrix in n-dimensional Euclidean space.

$-\tilde{F}$ denotes the transposed matrix of F.

$-F^*$ is the adjoin matrix of F.

$-F^{\cdot}$ denotes the differentiation with respect to k.

$-||Q||$ is the eculidcean cnorm of Q.

1. Solutions for the system (1) and its scattering function

We shall mainly use the basic solutions that have been in [8,9].
Every n vector solution $Y(x, \lambda)$ of (1) can be written in the form of quadratic matrix of order n which satisfies the equation

$$-Y'' + Q(x)Y = \lambda\rho(x)Y, 0 \leq x \prec \infty \tag{5}$$

It is evident that the columns of any matrix solution of equation (5) are solutions of equation (1). Thus, we consider the matrix differential equation (5) with the boundary condition

$$Y'(0) = 0 \tag{6}$$

instead of (1)-(2).

Denote by

$$k = \lambda^{1/2} = \mu + i\tau \text{ and } 0 \le argk \prec \pi$$

and $\sigma(x) = \int_x^\infty \|Q(t)\|dt$; $\sigma_1(x) = \int_x^\infty t\|Q(t)\|dt$.

Let us denote by $\varphi_n(x,k)$ and $\psi_n(x,k)$ the matrix solutions of the canonical equation (5) as $x \in [a_{n-1}, a_n)$.

These solutions satisfy the following conditions

$$\begin{aligned} \varphi_n(a_{n-1},k) = 0, \quad \psi_n(a_{n-1},k) = E_n \\ \varphi_n'(a_{n-1},k) = E_n, \quad \psi_n'(a_{n-1},k) = 0 \end{aligned} \tag{7}$$

As already known [1,4], these solutions can be represented in the form

$$\varphi(x,k) = \cos ka_n(x-a_n) + \int_{a_{n-1}}^{x} A_n(x,t)\cos ka_n(t-a_n)dt,$$

$$= \psi_n(x,k) = \frac{\sin ka_n(x-a_n)}{k}a_n^{-1} + \int_{a_{n-1}}^{x} B_n(x,t)\frac{\sin ka_n(t-a_n)}{k}a_n^{-1}dt;$$

where

$$A_n(x,x) = \frac{1}{2}\int_{t_{n-1}}^{2x-a1_{n-1}} q(t)dt \qquad , \frac{\partial}{\partial t}A_n(x,t)1_{1=a_{n-1}} = 0.$$

and

$$B_n(x,x) = \frac{1}{2}\int_{a_{n-1}}^{2x-a_{n-1}} q(t)dt \qquad , B_n(2x-a_{n-1}, a_{n-1}) = 0;$$

Lemma 1. *If the condition* (3) *is satisfied, then for* $x \ge a_n$ *and* $\tau \ge 0$ *equation* (1) *has a solution* $F(x,k)$ *that can be represented in the form*

$$F(x,k) = \exp(ikx)E_n + \int_x^\infty K(x,t)\exp(ikt)dt,$$

where the kernel $K(x,t)$ *satisfies the inequality* $\|K(x,t)\| \le \frac{1}{2}\sigma(\frac{x+1}{2})\exp(\sigma_1(x))$ *and the condition* $K(x,x) = \frac{1}{2}\int_x^\infty dt$.

Moreover, if $q(x)$ *is differential, then* $K(x,t)$ *is twice differentiable and satisfies both equation* $\frac{\partial^2 k(x,t)}{\partial x^2} - \frac{\partial^2 k(x,t)}{\partial \tau^2} = q(x)K(x,t)$
and the condition $\lim_{x+t\to\infty}\frac{\partial k(x,t)}{\partial x} = \lim_{x+t\to\infty}\frac{\partial k(x,t)}{\partial t} = 0$.

The solution $F(x,k)$ is an analytic function of k in the upper half plane $\tau \prec 0$ and is continuous on the real line. This solution has the following asymptotic behaviour

$F(x,k) = \exp(ikx)[E_n + o(1)]$, $F_x'(x,k) = (ik)\exp(ixk)[E_n + o(1)]$

as $x \to \infty$ for all $\tau \ge 0$, $k \ne 0$

Also, $F(x,k) = \exp(ikx)\Big[E_n + O(\frac{1}{k})\Big]$, $F_x'(x,k) = (ik)\exp(ixk)\Big[E_n + O(\frac{1}{k})\Big]$

as $|k| \to \infty$ and for all $\tau \ge 0$.

Next, if we continue the solution $F(x,k)$ of the equation (1) $[a_{n-1}, a_n)$ thus we find the following asymptotic form

$$F(x,k) = \begin{cases} \exp(ika_n)\left[\cos ka_n(x-a_n) + \frac{i}{a_n}\sin k_{a_n}(x-a_n)\right]\left[E_n + O(\frac{1}{k})\right] & ,a_{n-1} \le x \prec a_n \\ \exp(ikx)\left[E_n + O(\frac{1}{k}\right] & ,a_n \le x \prec \infty \end{cases}$$

First we introduce the concept of the Wronskian of a pair of solutions of the system (1). Denote by $W[\varphi_n, \psi_n]$ to the Wronskian of two matrix solutions $\varphi_n(x,k)$ and $\psi_n(x,k)$ such that

$$W[\varphi_n, \psi_n] = \tilde{\varphi}_n(x,k)\psi_n'(x,k) - \tilde{\varphi}_n'(x,k)\psi_n(xk).$$

The following property can be easily shown.

Lemma 2. *The Wronskian of two matrix solutions of* (1) *does not depends on* x.

Next since the matrix solutions $F(x,k)$ *and* $F(x,-k)$ *are linearly independent as* $\tau = 0$. *Thus, we have* $\varphi_n(x,k) = F(x,k)c_1(k) + F(x,-k)c_2(k)$, *where* $c_1(k)$ *and* $c_2(k)$ *are matrices of order* n, *which we have to find. For this purpose, we have*

$$F'(a_{n-1},k)c_1(k) + F(a_{n-1},-k)c_2(k) = E_n$$

and

$$F'(a_{n-1},k)c_1(k) + F'(a_{n-1}.-k)c_2(k) = 0.$$

Multiplying the first equality from the left by $\tilde{F}'(a_{n-1},-k)$ and then the second equality by $\tilde{F}(a_{n-1},-k)$. As a result we have $c_1(k) = -\frac{1}{2ik}\tilde{F}'(a_{n-1},k)$. Similarly $c_2(k) = \frac{1}{2ik}\tilde{F}'(a_{n-1},k)$. Thus

$$\varphi_n(x,k) = \frac{1}{2ik}[F(x,-k)\tilde{F}'(a_{n-1},k) - F(x,k)\tilde{F}'(a_{n-1},-k)].$$

Since $\varphi_n'(a_{n-1},k) = 0$, it yields

$$F'(a_{n-1},-k)\tilde{F}'(a_{n-1},k) = F'(a_{n-1},k)\tilde{F}'(a_{n-1}.-k).$$

Let $\det F'(a_{n-1},k) = 0$ as $\tau = 0, k \neq 0$ thus we find a vector $\vec{v} \neq 0$ such that $F'(a_{n-1},k)\vec{v} = 0$ and $\vec{v}^* F^{*'}(a_{n-1},k) = 0$.
Evidently,

$$F^*(x,k)F'(x,k) - F^{*'}(x,k)F(x,k) = 2ikE_n.$$

Multiplying this equality from the left by $\overline{v}^*$ and the right by $\overline{v}$ to have

$$\vec{v}^*[F^*(x,k)F'(x,k) - F^{*'}(x,k)F(x,k)]\vec{v} = \vec{v}^* 2ikE_n\vec{v}.$$

Setting $x = a_{n-1}$ to get $\vec{v}^*\vec{v} = 0$. Then, if $F'(a_{n-1}, k)\vec{v} = 0$ we find $\vec{v} = 0$ which lead to a contradiction.

Thus for all $x \geq a_{n-1}$ and $\tau = 0$, $k \neq 0$ the matrix function $F'(x, k)$ is non singular. Hence

$$\varphi_n(x, k) = \frac{1}{2ik}[F(x, -k) - F(x, k)S(k)]\tilde{F}'(a_{n-1}, k).$$

where $S(k) = \tilde{F}'(a_{n-1}, -k)[\tilde{F}'(a_{n-1}, k)]^{-1}$ is called scattering matrix of the equation (5) with the initial conditions (7).

Hence:

Theorem 1. *The identity*

$$\frac{2ik\varphi_n(x, k)}{\tilde{F}'(a_{n-1}, k)} = F(x, -k) - S(k)F(x, k) \tag{12}$$

is valid for all real $k \neq 0$.
where

$$S(k) = \tilde{F}'(a_{n-1}, -k)[\tilde{F}'(a_{n-1}, k)]^{-1} \tag{13}$$

The scattering matrix $S(k)$ *satisfies the following properties:*

(i) $S(k)S^*(k) = S^*(k)S(k) = E_n$,

(ii) $S(-k) = S^*(k)$. *Here, taking into account formulas* (8), (9), (10) *and* (11) *to prove that:*

Theorem 2. *For large real* k, $|k| \to \infty$ *the following asymptotic form holds*

$$S(k) = S_0(k) + O(\frac{1}{k}), \tag{14}$$

where

$$\begin{aligned} S_0(k) = &\exp(-2ika_n)[\sin k\alpha_n(a_n - a_{n-1}) + ia_n^{-1}\cos k\alpha_n(a_n - a_{n-1})] \\ &[\sin k\alpha_n(a_n - a_{n-1}) - ia_n^{-1}\cos k\alpha_n(a_n - a_{n-1})] + O(\frac{1}{k}). \end{aligned} \tag{15}$$

2. The discrete spectrum and Parsevals equation

We consider the singular boundary value problem arising from the canonical equation (5) with the conditions (3), (4) and (6).

Theorem 2. *The necessary and sufficient conditions that* $\lambda \neq 0$ *be an eigenvalue of the problem* (5)-(6) *are* $\lambda = k^2, \tau \succ 0, det\ F'(a_{n-1}, k) \neq 0$.

They are countable in number and its limit points can lie on the real axis.

This theorem can be proved via [1,5]

Theorem 3. *All the singular points of the matrix* $[F'(a_{n-1},k)]^{-1}$ *are all simple.*

Proof. By differentiating the equation

$$-F''(x,k)+q(x)F(x,k)=k'\rho(x)F(x,k) \tag{16}$$

with respect to k, and go over the hermitian conjugates of the matrices, we have

$$-[F^*(x,k)]''+q(x)F^*(x,k)=2\bar{k}\rho(x)F^*(x,k)+\bar{k}^2F(x,k) \tag{17}$$

We multiply (16) on the left by $F^*(x,k)$ and (17) on the right by $F(x,k)$ and subtract to have

$$F^*(x,k)F''(x,k)-[F^*(x,k)]''F(x,k)=-2kF^*(x,k)F(x,k)$$

. Since the elements of $F(x,k)$ and $F(x,k)$, $F'(x,k)$ lie in $L_2(0,\infty,\rho(x))$ thus it yields

$$F^*(x,k)F'(x,k)-[F^*(x,k)]'F(x,k)=2k\int_1^\infty F^*(t,k)\rho(t)F(t,k)dt \tag{18}$$

Suppose that the point $k_0=i_{\tau_0},\tau\succ 0$ is a zero of det $F'(a_{n-1},k_0)=0$. Then there exists a non zero vector $\vec{v}$ such that

$$F'(a_{n-1},k)\vec{v}=0 \tag{19}$$

Multiplying (18) on the right by $\vec{v}$ and on the left by $\vec{v}^*$ and letting x goes to a_{n-1}, we get

$$\begin{aligned}&\vec{v}^*F^*(a_{n-1},k)F'(a_{n-1},k)\vec{v}-\vec{v}^*[F^*(a_{n-1},k)]'F(a_{n-1},k)\vec{v}\\ =&2k\int_{a_{n-1}}^\infty \vec{v}^*F^*(t,k)\rho^*(t)F(t,k)\vec{v}dt.\end{aligned} \tag{20}$$

From the behaviour of $F(x,k)$; $F^*(x,k)$ and using Mean Value theorem to have

$$\vec{v}^*[F_i^*(a_{n-1},k)]'IF'(a_{n-1},k)\vec{v}=-2k\int_{a_{n-1}}^\infty F^*(t,k)\vec{v}^*I\rho^*(t)F(t,k)\vec{v}dt\neq 0 \tag{21}$$

Suppose that $\vec{v}$ not only (19) but also the relation have

$$F'(a_{n-1},k_0)\vec{w}+F'(a_{n-1},k_0)\vec{v}=0 \tag{22}$$

Here, along the hermitian conjugate and multiplying on the right by

$IF'(a_{n-1},k_0)\vec{v}$, we have

$$\vec{w}^*[F^*(a_{n-1},k)]'IF'(a_{n-1},k)\vec{v}^*[F^*(a_{n-1},k)]'IF(a_{n-1},k)\vec{v}=0 \tag{23}$$

In view of [1] and the Wronskian of $F(x,k)$; $F^*(x,k)$ to have

$$[F^*(a_{n-1},k)]'IF(a_{n-1},k) == [F^*(a_{n-1},k)]'IF'(a_{n-1},k)=0$$

Then by (19) we have

$$\vec{w}^*[F_1(a_{n-1},k)]'IF'(a_{n-1},k)\vec{v}=0$$

Therefore (23) takes the form

$$\vec{v}^*[F^*(a_{n-1},k)]'IF'(a_{n-1},k)\vec{v}=0$$

which contradicts (21). Hence it follows from equations (19) and (22) that $\vec{v}=0$ and this complete the proof of the theorem.

Lemma 3. *When $\tau \succ 0$, the matrix function*

$$R_k(x,t)=\begin{cases} F(x,k)F'^{-1}(a_{n-1},k)I\tilde{\varphi}_n(t,k), & t\le x\\ \varphi_n(x,k)I\tilde{F}'^{-1}(a_{n-1},k)\tilde{F}(t,k), & t\ge x\end{cases} \tag{24}$$

is the kernel resolvent of the problem (5)-(6).

Proof. We can find Green functions of the problem (5)-(6) by using the method of variation of parameters and thus the resolvent on the form (24).

Lemma 4. *Suppose that the vector function $f(t)$ is finite and has a continuous derivative $L_2(0,\infty;\rho(x))$ and satisfies the boundary condition (6). Then*

$$\int_0^\infty R_k(x,t)\rho(t)f(t)dt=\frac{-f(x)}{k^2}+\frac{1}{k^2}\int_0^\infty R_k(x,t)g(t)dt.$$

where

$$g(t)=-f''(t)+Q(t)f(t)$$

Moreover, if $\tau\succ 0$ and $|k|\to\infty$, then

$$\int_0^\infty R_k(x,t)\rho(t)f(t)dt=\frac{-f(x)}{k^2}+O(\frac{1}{k^2}). \tag{25}$$

Proof. Using formula (24) to get

$$\begin{aligned}&\int_{a_{n-1}}^\infty R_k(x,t)\rho(t)f(t)dt\\ =&F'^{-1}(a_{n-1},k)\Big\{F(x,k)\Big[\int_{a_{n-1}}^x[-\frac{1}{k^2}\tilde{\varphi}''(t,k)+\frac{1}{k^2}Q(t)\tilde{\varphi}(t,k)f(t)dt\Big]\\ &+\varphi(x,k)\int_x^\infty\Big\{-\frac{1}{k^2}\tilde{F}''(x,k)+\frac{1}{k^2}Q(t)\tilde{F}(t,k)\Big\}f(t)dt\Big\}\end{aligned}$$

Integrating this identity by parts, and taking into account Riemman Lebsegue theorem it yields that $\int_{a_{n-1}}^{\infty} R_k(x,t)g(t)dt = o(1)$. Hence (25) follows directly.

The following lemma is well known:

Lemma 5. $\bar{R}_{k^2} = R_{k^{-2}}$

With the help of these lemmas we can prove the following theorem:

Theorem 4. *The following Parsevals equation is valid:*

$$\frac{1}{2\pi}\int_{a_{n-1}}^{\infty} u(x,k)u^*(t,k)dk + \sum_{n=1}^{m} u(x,i,\tau_n)u^*(t,i_{\tau_n}) = \delta(x-t)\rho^{-1}(x)E_n, \tag{26}$$

where

$$u(x,k) = F(x,-k) - S(k)F(x,k)$$

and

$$u(x,i_{\tau_n}) = M_n F(x,i_{\tau_n}), m = 1,\ldots,m.$$

such that $M_1, M_2, \ldots, M_m$ *are non negative matrices.*

Proof. Suppose that $f(x)$ satisfies the conditions of lemma 4. Thus (25) holds. Integrating both sides of (25) with respect to k over the semi-circle $\{k; |k| = r, r \succ 0\}$ in the uper-half plane k. It is evident that the integral $\int_{a_{n-1}}^{\infty} R_k(x,t)\rho(t)\ f(t)dt$ is an analytical function except the zeros $\{i_{\tau_1}, i_{\tau_2}, \ldots, i_{\tau_m}\}$ of the det $F'(a_{n-1},k)$.

Hence, upon using [3] we find that

$$\begin{aligned} f(x) =& \frac{1}{\pi i}\int_{a_{n-1}}^{\infty} k\int_{a_{n-1}}^{\infty} [R_{k+i0}(x,k) - R_{k-i0}(x,k)]\rho(t)f(t)dtdk \\ &+ \sum_{n=1}^{m} Res[2k\int_{a_{n-1}}^{\infty} R_k(x,t)\rho(t)f(t)dt]_{k=i_{\tau_n}} \end{aligned} \tag{27}$$

Next, let us compute the first quantity in the right - hand side of equation (27). By lemma 5 it yields that $R_{k-i0} = \bar{R}_{k+i0}$. Then, we can compute R_{k+i0} and thus R_{k-i0} at once. Therefore, using (24) to obtain

$$R_{k+i0}(x,t) - R_{k-i0}(x,t) = \frac{\varphi_n(x,k)2ik\varphi_n^*(t,-k)}{W(k)W^*(-k)}.$$

where $W(k) = detF'(a_{n-1}.k)$

Thus, taking into account (12) to have

$$R_{k+i0}(x,t) - R_{k-i0}(x,t) = \frac{u(x,k)u^*(t,-k)}{-2ik}.$$

where

$$u(x,k) = \frac{2ik\varphi_n(x,k)}{W^*(-k)} = F(x,-k) - S(k)F(x,k).$$

Hence

$$\frac{1}{\pi i}\int_{a_{n-1}}^{\infty} k\int_{a_{n-1}}^{\infty}[R_{k+i0}(x,k)-R_{k-i0}(x,k)]\rho(t)f(t)dtdk$$
$$=\frac{1}{2\pi}\int_{a_{n-1}}^{\infty}dk\int_{a_{n-1}}^{\infty}u(x,k)u(t,-k)\rho(t)f(t)dt. \tag{28}$$

Now we compute the second quantity on the right-hand side of (27).
From (24) we have

$$Res_{k=i_{\tau_n}}[k\int_{a_{n-1}}^{\infty}R_k(x,t)\rho(t)f(t)dt]$$
$$=2i_{\tau_n}[F(x,i_{\tau_n})P_nI\int_{a_{n-1}}^{\infty}\tilde{\varphi}(t,i_{\tau_n})\rho(t)f(t)dt+\varphi(x,i_{\tau_n})\int_x^{\infty}I\tilde{P}_n\tilde{F}(t,i_{\tau_n})\rho(t)f(t)dt] \tag{29}$$

where P_n is the residue of $F'^{-1}(a_{n-1},k)$ at $k=i_{\tau_n}$.

Since $F'(a_{n-1},k)$ is analytical function as $\tau>0$ and $F'^{-1}(a_{n-1},k)$ has a simple pole at i_{τ_n}, then the following relation is valied

$$F'(a_{n-1},k)=F'(a_{n-1},i_{\tau_n})+F'(a_{n-1},i_{\tau_n})(k-i_{\tau_n})+\cdots$$

and

$$F'^{-1}(a_{n-1},k)=\frac{P_n}{(k-i_{\tau_n})}+P_n^{(0)}+\cdots \tag{30}$$

From (30) and the relation

$$F'^{-1}(a_{n-1},k)F'^{-1}(a_{n-1},k)=F'^{-1}(a_{n-1},k)F'^{-1}(a_{n-1},k)=E_n$$

it yields that

$$E_n=\frac{F'(a_{n-1},k)P_n}{k-i_{\tau_n}}+F'(a_{n-1},k)P_n+F'(a_{n-1},k)P^{(0)}+\cdots$$

Hence

$$F'(a_{n-1},k)P_n=P_nF'(a_{n-1},k)=0 \text{ and}$$
$$F'(a_{n-1},k)P_n+F'(a_{n-1},k)P_n^{(0)}=P_nF'(a_{n-1},k)+P_n^{(0)}F'(a_{n-1},k)=E_n \tag{31}$$

Let H_n be the operator of orthogonal projection onto P_n. It is easily shown that [3] the ranks of H_n and P_n are the same and that $H_nP_k=P_n$. From (18), we have

$$F^*(x,i_{\tau_n})F'(x,i_{\tau_n})-[F^*(x,i_{\tau_n})]'F(x,i_{\tau_n})=-2k\int_x^{\infty}F^*(t,i_{\tau_n})\rho(t)F(t,i_{\tau_n})dt$$

Thus, we have for $x = a_{n-1}$ that

$$F^*(a_{n-1}, i_{\tau_n})F'(a_{n-1}, i_{\tau_n}) - [F^*(a_{n-1}, i_{\tau_n})]'F(a_{n-1}, i_{\tau_n}) \\ = -2i_{\tau_n} \int_{a_{n-1}}^{\infty} F^*(t, i_{\tau_n})\rho(t)F(t, i_{\tau_n})dt = -2i_{\tau_n} A_n. \tag{32}$$

Clearly, A_n is a positive definite matrix multiplying equation (32) on the left by P_n^* and on the right by H_n and taking into account that

$$F'(a_{n-1}, i_{\tau_n})H_n = 0 \text{ to have } P_n^* F^*(a_{n-1}, i_{\tau_n})IF(a_{n-1}, i_{\tau_n})H_n = 2i_{\tau_n} P_n^* A_n H_n \tag{33}$$

Since the matrix function $F(x, i_{\tau_n})H_n$ and $\varphi(x, i_{\tau_n})$ are the solutions of the same equation, thus we have

$$F(x, i_{\tau_n})H_n = \varphi(x, i_{\tau_n})F(a_{n-1}, i_{\tau_n})H_n. \tag{34}$$

It follows from (31) and (33) that

$$P_n^* F^{*'}(a_{n-1}, i_{\tau_n})IF(a_{n-1}, i_{\tau_n})H_n \\ =[E_n - P_n^{*(0)} F^{*'}(a_{n-1}, i_{\tau_n}) -]F(a_{n-1}, i_{\tau_n})H_n \\ =F(a_{n-1}, i_{\tau_n})H_n - P_n^{*(0)} F^{*'}(a_{n-1}, i_{\tau_n})F(a_{n-1}, i_{\tau_n})H_n = F(a_{n-1}, i_\tau)H_n.$$

Therefore, equation (33) takes the form

$$F(a_{n-1}, i_{\tau_n})H_n = 2i_{\tau_n} P_n^* A_n H_n$$

Now, from (34) we have

$$F(x, i_{\tau_n})H_n = \varphi(x, i_{\tau_n})F(a_{n-1}, i_{\tau_n})H)n \\ =2i_{\tau_n}\varphi(x, i_{\tau_n})P_n^* A_n H_n = 2i_{\tau_n}\varphi(x, i_{\tau_n})IP_n^*[H_n A_n H_n + E_n - H_n] \\ =2i_{\tau_n}\varphi(x, i_{\tau_n})IP_n^* D)n, \tag{35}$$

where

$$D_n = H_n A_n H_n + E_n - H_n.$$

Evidently, $D_n H_n = H_n D_n$ and D_n is a positive definite matrix. Thus, there exists the matrix $M_n^2 = H_n D_n^{-1} = D_n^{-1} H_n$ is nonnegative and its rank is the same rank of H_n, i.e. the multiplicity of the zeros of deal $F'(a_{n-1}, k)$.

Multiplying both sides of (35) on the left by D_n^{-1} to have $F(x, i_{\tau_n})M_n^2 = 2i_{\tau_n} \varphi(x, i_{\tau_n})P_n^*$. We multiply both sides of this formula on the right by $\tilde{F}(t, i_{\tau_n})$ to give $2i_{\tau_n}\varphi(x, i_{\tau_n})I\tilde{P}_n\tilde{F}(t, i_{\tau_n}) = F(x, i_{\tau_n})M_n^2 F^*(t, i_{\tau_n})$. Thus, it follows from (29) that

$$Res_{k=i_{\tau_n}}[2k \int_{a_{n-1}}^{\infty} R_k(x,t)\rho(t)f(t)dt] = F(x, i_{\tau_n})M_n^2 \int_{a_{n-1}}^{\infty} F^*(t, i_{\tau_n})\rho(t)f(t)dt. \tag{36}$$

Hence, by (28) and (36) we conclude that

$$f(x)=\frac{1}{2\pi}\int_{a_{n-1}}^{\infty}dk\int_{a_{n-1}}^{\infty}u(x,k)u^*(t,-k)\rho(t)f(t)dt+F(x,i_{\tau_n})M_n^2\int_{a_{n-1}}^{\infty}F^*(t,i_{\tau_n})\rho(t)f(t)dt.$$

Then, multiplying both sides of the last formula by $f(x)\rho(x)$ and integrating from a_{n-1} to ∞ to obtain Parsevals equation (26) and the theorem is proved.

We claim that the collection of quantities $\{S(k), i_{\tau_n}, M_n, n=1,\dots,m\}$ is called the scattering data of the problem (1)-(2).

Note. In the forthcoming article I plan to study the inverse scattering problem of the problem (1)-(2).

References

[1] Z. S. Agranovich, V. A. Marchenko, *The inverse scattering theory*, Eng. Tran, Gordon and Breach, New York, 1963.

[2] M. G. Gasymov, *Forward and inverse problem spectral analysis for one class equation with discontinuous coefficient*, Proceeding of the international conference for non classical methods in geophysics, Novosibirk. USSR, 1977.

[3] M. G. Gasymov, "The inverse scattering problem for a system of Dirac equation of order $2n$," *Trans. Moscow Math. Soc.*, 19(1968).

[4] V. A. Marchenko, *Sturm-Liouville operators and applications*, Birkhauser Verlag Basel, 1986.

[5] R. T. Pashaev, "Uniqueness theorem inverse problem and spectral theory for one system differential equation with discontinuous coefficient," *Dok. Akad. Nauk. SSR.* 35(10)(1979).

[6] E. C. Titchmarsh, "Some eigenfunction expansion formulaes," *Proc. London. Math. Soc.*, 11(3) (1961), 159-168.

[7] A. A. Darwish, "The inverse scattering problem for a singular boundary value problem," *Newzealand Journal of Mathematics*, 22(1993), 1-20.

[8] N. M. Gelfand, B. M. Levitan, "On the determination of differential equation by the spectral function," *Izv. Akad. Nauk. SSSR, Ser Math.*, 15(1951), 309-360(Russ).

[9] B. M. Levitan, M. G. Gasymov, "Determination of a differential equation from two spectra," *Uspeki. Mat. Nauk,* 19(1964), 3-63 (Russ).

Mathematics Department-Faculty of Science-Qatar University-State of Qatar.