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ON A CLASS OF PÓLYA PROPERTY PRESERVING OPERATORS

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Abstract. In this paper, we show that every continuous linear operator from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ has an integral representation with a kernel function $M(z, w, \zeta)$. We give two sufficient conditions on $M(z, w, \zeta)$ to ensure that its corresponding operator preserves Pólya property. We also prove that a continuous linear operator from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ either preserves the Pólya property for all functions with that property or does not preserve the Pólya property for any function.

Introduction

Let Ω_w and Ω_z be simply connected domains in C. Let $m(w, z)$ be holomorphic on $\Omega_w \times \Omega_z$. Then m is said to have the Pólya property with respect to z on Ω_z if and only if for all simple closed contours $\Gamma \subseteq \Omega_z$, if f is analytic on Γ and if $\int_\Gamma m(w, z) f(z) dz \equiv 0$, then f has an analytic continuation to the Jordan region enclosed by Γ. An equivalent definition for m to have the Pólya property with respect to z on Ω_z is that for all simple closed contours $\Gamma \subseteq \Omega_z$, if F is analytic outside and on Γ with $F(\infty) = 0$ and if $\int_\Gamma m(w, z) F(z) \equiv 0$, then $F \equiv 0$.

In [4], Chang introduces the concept of Pólya property for holomorphic functions of two variables to study the uniqueness problem for entire functions of exponential type. Using functions which have the Pólya property, he obtains stronger and more general results on the uniqueness problem than those of Gelfond [10], Buck [3] and DeMar [7,8,9]. In [5], Chang further utilizes functions which have the Pólya property as generating kernel in formal generating relations to obtain the necessary and sufficient condition for an entire function to have a unique expansion in a series of various polynomials such as Appell polynomials, Boas and Buck polynomials, generalized Appell polynomials, generalized Sheffer polynomials etc. His approach not only produces known results in stronger form and gives precise conditions on the growth of the coefficients but also provides new insight into the problem of finding causes of multiple expansions of entire functions.

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In [6], Chang proves a theorem which shows that the uniqueness problem for entire functions of exponential type is equivalent to the approximation problem for analytic functions. Thus to each function which has the Pólya property there corresponds an approximation theorem for analytic functions and conversely. It has thus been shown that holomorphic functions which have the Pólya property are closely related to (1) uniqueness problem for entire functions of exponential type, (2) unique expansion of entire functions and (3) the approximations to analytic functions. Clearly, there is a need to find new functions which have the Pólya property.

It would be nice if one could give a simple test in terms of analytic properties to apply to a holomorphic function of two variables to determine whether or not it has the Pólya property on a given domain. Since there is not such a test and we are able to show that few functions have the Pólya property is to find operators which map functions known to have the Pólya property to new functions with the Pólya property.

Köthe [11] shows that every continuous linear operator $T : H(\Omega_\eta) \to H(\Omega_z)$ has an integral representation $T(f)(z) = \frac{1}{2\pi i}\int_\Gamma f(\eta)M(\eta, z)d\eta$. In this paper, we first imitate Köthe's proof to extend his result to continuous linear operators $T : H(\Omega_w \times \Omega_z) \to H(\Omega_\eta \times \Omega_\zeta)$. We then show that every continuous linear operator from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ has an integral representation with a kernel function $M(z, w, \zeta)$. We give two sufficient conditions on $M(z, w, \zeta)$ so that its corresponding operator maps functions with the Pólya property on Ω_z to functions with the Pólya property on Ω_ζ. We call these operators Pólya property preserving and prove that $T : H(\Omega_w \times \Omega_z) \to H(\Omega_w \times \Omega_\zeta)$ either preserves the Pólya property for all functions with that property or does not preserve the Pólya property for any function. Finally, we illustrate the application of the result by an example.

The space $H(\Omega_w \times \Omega_z)$.

Let Ω_w and Ω_z be simply connected domains in the complex plane. Let $H(\Omega_w \times \Omega_z)$ denote the space of all functions holomorphic on $\Omega_w \times \Omega_z$. Then $H(\Omega_w \times \Omega_z)$ is a complex linear space. Let $\Omega_{w1} \subset \Omega_{w2} \subset \cdots \subset \Omega_{wn} \subset \cdots$ be a sequence of simply connected subdomains of Ω_w and $\Omega_{z1} \subset \Omega_{z2} \subset \cdots \subset \Omega_{zn} \subset \cdots$ be another sequence of simply connected subdomains of Ω_z such that these two sequences have the following properties: (1) $\overline{\Omega}_{wn} \subset \Omega_{w(n+1)}$ and $\overline{\Omega}_{zn} \subset \Omega_{z(n+1)}$ for $n = 1, 2, \ldots$ (2) each $\overline{\Omega}_{wn}$ and each $\overline{\Omega}_{zn}$ is bounded by a simple closed contour Γ_{wn} and Γ_{zn} respectively (3) $\bigcup_{n=1}^\infty \Omega_{wn} = \Omega_w, \bigcup_{n=1}^\infty \Omega_{zn} = \Omega_z$ and $\bigcup_{n=1}^\infty (\Omega_{wn} \times \Omega_{zn}) = \Omega_w \times \Omega_z$. Let $D_n = \Omega_{wn} \times \Omega_{zn}$ for $n = 1, 2 \ldots$. Define a sequence of norms $\| \, \|_n$ for $H(\Omega_w \times \Omega_z)$ by $\|f\|_n = \sup_{(w,z)\in\overline{D}_n} |f(w, z)|$. Then $H(\Omega_w \times \Omega_z)$ becomes a locally convex space if we define the topology τ on $H(\Omega_w \times \Omega_z)$ with $\|f\|_n < \varepsilon$, $n = 1, 2, \ldots$ and $\varepsilon > 0$ arbitrary, as the base of neighborhoods of 0. Since $H(\Omega_w \times \Omega_z)$ has a countable base of neighborhoods of 0, $H(\Omega_w \times \Omega_z)$ is metrizable [12, p.163]. Let $f(w, z)$ be holomorphic on $\Omega_w \times \Omega_z$. Then we have, by the Cauchy integral formula for one variable, that

$$f(w,z) = \frac{1}{(2\pi i)^2} \int_{\Gamma_\zeta} \int_{\Gamma_\eta} \frac{f(\eta,\zeta)}{(\eta - w)(\zeta - z)} d\eta d\zeta$$

where Γ_η and Γ_ζ are the boundaries of Ω_{wn} and Ω_{zn} respectively for some n. It follows immediately from this Cauchy integral formula for product domains that Weierstrass' theorem holds for holomorphic functions of two variables on product domains. We conclude that $H(\Omega_w \times \Omega_z)$ is complete and hence a Fréchet space.

The space $B(\overline{D}_w \times \overline{D}_z)$ and $LH(\overline{D}_w \times \overline{D}_z)$

Let D_w and D_z be Jordan domains in the complex plane. Let $B(\overline{D}_w \times \overline{D}_z)$ denote the space of all functions which are holomorphic on $D_w \times D_z$ and continuous on $\overline{D}_w \times \overline{D}_z$. Then $B(\overline{D}_w \times \overline{D}_z)$ is a Banach space with norm $\| \ \|$ defined by $\|f\| = \sup_{(w,z)\in E} |f(w,z)|$ where $E = \overline{D}_w \times \overline{D}_z$. If $f(w,z)$ is a function holomorphic on an open set containing $\overline{D}_w \times \overline{D}_z$, then f is said to be locally holomorphic on $\overline{D}_w \times \overline{D}_z$. Two such functions f, g are said to be equivalent if there is an open set D containing $\overline{D}_w \times \overline{D}_z$ such that $f = g$ on D. We call a class of functions equivalent to f a locally holomorphic function on $\overline{D}_w \times \overline{D}_z$ and denote by $[f]$ the equivalence class to which f belongs. For convenience we may sometimes simply write f instead of $[f]$. We let $LH(\overline{D}_w \times \overline{D}_z)$ denote the space of all equivalence classes of functions locally holomorphic on $\overline{D}_w \times \overline{D}_z$.

Theorem 1. *Let D_w and D_z be Jordan domains. Then $LH(\overline{D}_w \times \overline{D}_z)$ is dense in $B(\overline{D}_w \times \overline{D}_z)$.*

Proof. We first suppose that D_w and D_z are the unit disks. Let $f(w,z)$ be in $B(\overline{D}_w \times \overline{D}_z)$ Let $\{t_n\}$ be a sequence in $(0,1)$ with $t_n \to 1$ as $n \to \infty$. Define a sequence $\{f_n\}$ of functions in $LH(\overline{D}_w \times \overline{D}_z)$ as follows: $f_n(w,z) = f(t_n w, t_n z)$. Then f_n converges to f in the norm of $B(\overline{D}_w \times \overline{D}_z)$. Now suppose that D_w and D_z are Jordan domains. Let $U_\eta = \{\eta \in C \mid |\eta| < 1\}$ and $U_\zeta = \{\zeta \in C \mid |\zeta| < 1\}$. Let ϕ_1 be a conformal mapping from D_w onto U_η and ϕ_2 another conformal mapping from D_z onto U_ζ. Since D_w and D_z are Jordan domains, we can extend ϕ_1 to a homeomorphism form $\overline{D}_w$ onto $\overline{U}_\eta$ and ϕ_2 to a homeomorphism from $\overline{D}_z$ onto $\overline{U}_\zeta$. Let Ψ_1 be the inverse of ϕ_1 and Ψ_2 the inverse of ϕ_2. Let $f(w,z) \in B(\overline{D}_w \times \overline{D}_z)$. Then $f(\Psi_1(\eta), \Psi_2(\zeta)) \in B(\overline{U}_\eta \times \overline{U}_\zeta)$. Let $\{t_n\}$ be as in the above. Define a sequence $\{f_n\}$ of functions in $LH(\overline{U}_\eta \times \overline{U}_\zeta)$ as follows: $f_n(\Psi_1(\eta), \Psi_2(\zeta)) = f(\Psi_1(t_n, \eta), \Psi_2(t_n\zeta))$. Then $f_n(\Psi_1(\eta), \Psi_2(\zeta))$ converges to $f(\Psi_1(\eta), \Psi_2(\zeta))$ in the norm of $B(\overline{U}_\eta \times \overline{U}_\zeta)$. Since $\eta = \phi_1(w)$, $\zeta = \phi_2(z)$ and $f_n(\Psi_1(\eta), \Psi_2(\zeta)) \in LH(\overline{U}_\eta \times \overline{U}_\zeta)$, $f_n(w,z) = f(\Psi_1(t_n\phi_1(w)), \Psi_2(t_n\phi_2(z)) \in LH(\overline{D}_w \times \overline{D}_z)$ and $f_n(w,z)$ converges to $f(w,z)$ in the norm of $B(\overline{D}_w \times \overline{D}_z)$.

The dual space of $H(\Omega_w \times \Omega_z)$

Let L be a continuous linear functional on $H(\Omega_w \times \Omega_z)$. We shall show that there exist n and M such that $|L(f)| \le M\|f\|_n$ for all $f \in H(\Omega_w \times \Omega_z)$. Since L is continuous on $H(\Omega_w \times \Omega_z)$, it is continuous at 0. Hence for every $\varepsilon > 0$ there exist $\|\ \|_n$ and $\delta > 0$ such that $|L(f)| \le \varepsilon$ for all f in $H(\Omega_w \times \Omega_z)$ with $\|f\|_n \le \delta$. Let $\lambda > 0$ be such that $\lambda\|f\|_n \le \delta$. Then we have $\|\lambda f\|_n \le \delta$ and so $\|L(\lambda f)| \le \varepsilon$. Thus $|L(f)| \le \frac{\varepsilon}{\lambda}$. If $f \in H(\Omega_w \times \Omega_z)$ is such that $\|f\|_n = 0$, we can take λ arbitrarily large and so $|L(f)| = 0$. If $\|f\|_n \neq 0$, we can take $\lambda = \frac{\delta}{\|f\|_n}$ and so, in any case, we have $|L(f)| \le M\|f\|_n$ with $M = \frac{\varepsilon}{\delta}$.

For this n, $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ is a Banach space and $H(\Omega_w \times \Omega_z) \subset B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$. Since L is a linear functional on $H(\Omega_w \times \Omega_z)$ and $|L(f)| \le M\|f\|_n$ for all $f \in H(\Omega_w \times \Omega_z)$, L can be extended to all of $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ with the same bound by Hahn-Banach theorem.

Consequently L is defined for all functions $\frac{1}{w-\eta}\frac{1}{z-\zeta}$, $\eta \in \overline{\Omega}_{wn}^C$ and $\zeta \in \overline{\Omega}_{zn}^C$, which, considered as functions of (w,z), are elements of $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$. Define a function ℓ of two variables as follows:

$$
\begin{aligned}
\ell(\eta,\zeta) &= L(\frac{1}{w-\eta}\frac{1}{z-\zeta}) \\
\ell(\infty,\zeta) &= 0 \\
\ell(\eta,\infty) &= 0 \\
\ell(\infty,\infty) &= 0
\end{aligned} \tag{2}
$$

Theorem 3. *The function ℓ is holomorphic on $\overline{\Omega}_{wn}^C \times \overline{\Omega}_{zn}^C$.*

Proof. If $(\eta_0,\zeta_0) \in \overline{\Omega}_{wn}^C \times \overline{\Omega}_{zn}^C$ is such that $\eta_0 \neq \infty$ and $\zeta_0 \neq \infty$, then for $(\eta,\zeta),(\eta_0,\zeta_0)$ in $\overline{\Omega}_{wn}^C \times \overline{\Omega}_{zn}^C$ we have $\frac{\ell(\eta,\zeta)-\ell(\eta_0,\zeta)}{\eta-\eta_0} = \frac{1}{\eta-\eta_0}\cdot L(\frac{1}{w-\eta}\frac{1}{z-\zeta} - \frac{1}{w-\eta_0}\frac{1}{z-\zeta}) = L(\frac{1}{\eta-\eta_0}[\frac{1}{w-\eta}\frac{1}{z-\zeta} - \frac{1}{w-\eta_0}\frac{1}{z-\zeta}]) = L(\frac{1}{\eta-\eta_0}[\frac{1}{w-\eta} - \frac{1}{w-\eta_0}]\frac{1}{z-\zeta})$. As $\eta \to \eta_0$, $\frac{1}{\eta-\eta_0}[\frac{1}{w-\eta} - \frac{1}{w-\eta_0}]\frac{1}{z-\zeta} \to \frac{1}{(w-\eta_0)^2}\frac{1}{z-\zeta}$ with respect to the norm of $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$. Since $\frac{1}{(w-\eta_0)^2}\frac{1}{z-\zeta}$ is in $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ and L is continuous on $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$, we have $\lim_{\eta\to\eta_0} \frac{\ell(\eta,\zeta)-\ell(\eta_0,\zeta)}{\eta-\eta_0} = L(\frac{1}{(w-\eta_0)^2}\frac{1}{z-\zeta})$ and ℓ is differentiable with respect to η at η_0. Similarly $\lim_{\zeta\to\zeta_0} \frac{\ell(\eta,\zeta)-\ell(\eta,\zeta_0)}{\zeta-\zeta_0} = L(\frac{1}{w-\eta}\frac{1}{(z-\zeta_0)^2})$ and ℓ is differentiable with respect to ζ at ζ_0. It then follows from Hartogs' theorem that ℓ is holomorphic at (η_0,ζ_0).

If $\eta_0 = \infty$ and $\zeta_0 \neq \infty$, then $\ell(\eta,z)$ is holomorphic on a neighborhood of (∞,ζ_0). Since $\frac{1}{w-\eta}\frac{1}{z-\zeta}$ converges to 0 with respect to the norm of $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ as $(\eta,\zeta) \to (\infty,\zeta_0)$, $\ell(\eta,\zeta) \to 0$ as $(\eta,\zeta) \to (\infty,\zeta_0)$. Hence ℓ is also holomorphc at (∞,ζ_0) and vanishes there. Similarly if $\eta_0 \neq \infty$ and $\zeta_0 = \infty$, then $\ell(\infty,\zeta)$ is holomorphic at (η_0,∞) and 0 there. If $\eta_0 = \infty$ and $\zeta_0 = \infty$, then $\ell(\infty,\zeta)$ is holomorphic for all ζ such that $\zeta \neq \infty$. Since $\zeta \to \infty$ implies $\ell(\infty,\zeta) \to 0$, ℓ is holomorphic at (∞,∞). Hence for each $\eta \in \overline{\Omega}_{wn}^C$, ℓ is analytic on $\overline{\Omega}_{zn}^C$. Similarly for each $\zeta \in \overline{\Omega}_{zn}^C$, ℓ is analytic on $\overline{\Omega}_{wn}^C$ and we conclude that ℓ is holomorphic on $\overline{\Omega}_{wn}^C \times \overline{\Omega}_{zn}^C$.

Theorem 4. *Let $f(w,z)$ be in $LH(\overline{\Omega}_{wn}\times\overline{\Omega}_{zn})$. Let L be a continuous linear functional on $B(\overline{\Omega}_{wn}\times\overline{\Omega}_{zn})$ and ℓ be defined as in (2). Then*

$$L(f)=\frac{1}{(2\pi i)^2}\int_{\Gamma_\zeta}\int_{\Gamma_\eta} f(\eta,\zeta)\ell(\eta.\zeta)d\eta d\zeta$$

where Γ_ζ is the boundary of some domain $G_z\supset\overline{\Omega}_{zn}$ and Γ_η is the boundary of another domain $G_w\supset\overline{\Omega}_{wn}$.

Proof. Since $f(w,z)\in LH(\overline{\Omega}_{wn}\times\Omega_{zn})$, there exist domains G_w and G_z with $G_w\supset\overline{\Omega}_{wn}$ and $G_z\supset\overline{\Omega}_{zn}$ such that for all $(w,z)\in G_w\times G_z$, $f(w,z)=\frac{1}{(2\pi i)^2}\int_{\Gamma_\zeta}\int_{\Gamma_\eta}\frac{f(\eta,\zeta)}{(\eta-w)(\zeta-z)}$ $d\eta d\zeta$ where Γ_ζ and Γ_η are the boundaries of G_z and G_w respectively. Let $\{f_k\}$ be a sequence of Riemann sums defined by

$$f_k(w,z)=\frac{1}{(2\pi i)^2}\sum_{1,m}\frac{f(\eta_1^{(k)},\zeta_m^{(k)})\Delta\eta_1^{(k)}\cdot\Delta\zeta_m^{(k)}}{(\eta_1^{(k)}-w)(\zeta_m^{(k)}-z)}.$$

Then $\lim_{k\to\infty}f_k(w,z)=f(w,z)$ for all $(w,z)\in G_w\times G_z$. Let $(w,z)\in\overline{\Omega}_{wn}\times\overline{\Omega}_{zn}$. Let $|\Gamma_\zeta|$ and $|\Gamma_\eta|$ be the lengths of Γ_ζ and Γ_η respectively. Let δ_η be the distance between $\overline{\Omega}_{wn}$ and Γ_η and δ_ζ the distance between $\overline{\Omega}_{zn}$ and Γ_ζ. Let $\|f\|_G=\sup_{(w,z)\in\overline{G}_w\times\overline{G}_z}|f(w,z)|$. Then $|f_k(w,z)|\le\frac{1}{4\pi^2}\frac{\|f\|_G|\Gamma_\eta||\Gamma_\zeta|}{\delta_\eta\cdot\delta_\zeta}=M$ for all $k=1,2,\dots$ and for all $(w,z)\in\overline{\Omega}_{wn}\times\overline{\Omega}_{zn}$. Replacing $\overline{\Omega}_{wn}\times\overline{\Omega}_{zn}$ by any compact subset of $G_w\times G_z$ and repeating the above reasoning, we have $\{f_k\}$ is uniformly bounded on every compact subset of $G_w\times G_z$. Since f_k converges to f on $G_w\times G_z$, f_k converges to f uniformly on every compact subset of $G_w\times G_z$ by Vitali's theorem. In particular, f_k converges to f uniformly on $\overline{\Omega}_{wn}\times\overline{\Omega}_{zn}$ and $\|f_k-f\|_n\to 0$ as $k\to\infty$. Since L is a continuous linear functional on $B(\overline{\Omega}_{wn}\times\overline{\Omega}_{zn})$, $L(f)=\lim_{k\to\infty}L(f_k)=\lim_{k\to\infty}\frac{1}{(2\pi i)^2}\sum_{1,m}f(\eta_1^{(k)},\zeta_m^{(k)})\,L(\frac{1}{\eta_1^{(k)}-w}\frac{1}{\zeta_m^{(k)}-z})\Delta\eta_1^{(k)}\cdot\Delta\zeta_m^{(k)}=\frac{1}{(2\pi i)^2}\int_{\Gamma_\zeta}\int_{\Gamma_\eta}f(\eta,\zeta)L(\frac{1}{(\eta-w)}\frac{1}{(\zeta-z)})d\eta d\zeta=\frac{1}{(2\pi i)^2}\int_{\Gamma_\zeta}\int_{\Gamma_\eta}f(\eta,\zeta)\ell(\eta,\zeta)d\eta d\zeta$

Theorem 5. *Let ℓ be holomorphic on $\overline{\Omega}_{wn}^C\times\overline{\Omega}_{zn}^C$. Let Γ_η and Γ_ζ be the boundaries of G_w and G_z respectively and $\overline{\Omega}_{wn}\times\overline{\Omega}_{zn}\subset G_w\times G_z\subset\Omega_w\times\Omega_z$. Then L defined on $B(\overline{G}_w\times\overline{G}_z)$ by $L(f)=\frac{1}{(2\pi i)^2}\int_{\Gamma_\zeta}\int_{\Gamma_\eta}f(\eta,\zeta)\ell(\eta,\zeta)d\eta d\zeta$ is a continuous linear functional on $B(\overline{G}_w\times\overline{G}_z)$.*

Proof. Since the double integral exists and satisfies linearity condition, L is a linear functional on $B(\overline{G}_w\times\overline{G}_z)$. Let $\|f\|_G=\sup_{(w,z)\in\overline{G}_w\times\overline{G}_z}|f(w,z)|$. Let $|\Gamma_\zeta|$ and $|\Gamma_\eta|$ be the lengths of Γ_ζ and Γ_η respectively. Let $\|\ell\|=\sup_{(\eta,\zeta)\in\Gamma_\eta\times\Gamma_\zeta}|\ell(\eta,\zeta)|$. Then $|L(f)|\le\frac{1}{4\pi^2}\|f\|_G\|\ell\|\,|\Gamma_\zeta|\,|\Gamma_\eta|\equiv M\|f\|_G$ for all $f\in B(\overline{G}_w\times\overline{G}_z)$ where $M=\frac{1}{4\pi^2}\|\ell\|\,|\Gamma_\zeta|\,|\Gamma_\eta|$. Hence L is continuous.

We are now in the position to use Theorem 4 and 5 to describe the dual space of $H(\Omega_w\times\Omega_z)$. Let L be a continuous linear functional on $H(\Omega_w\times\Omega_z)$. We have shown that L can be extended to a Banach space $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$. By Theorem 4, there exist G_w and G_z properly contained in Ω_w and Ω_z respectively with $\overline{\Omega}_{wn} \times \overline{\Omega}_{zn} \subset G_w \times G_z$ and ℓ holomorphic on $\overline{\Omega}^C_{wn} \times \overline{\Omega}^C_{zn}$ such that

$$L(f) = \frac{1}{(2\pi i)^2} \int_{\Gamma_\zeta} \int_{\Gamma_\eta} f(\eta, \zeta)\ell(\eta, \zeta) d\eta d\zeta \tag{6}$$

for all $f \in H(\Omega_w \times \Omega_z)$ where Γ_ζ and Γ_η are the boundaries of G_z and G_w respectively.

By Theorem 5, if ℓ is holomorphic on $\overline{\Omega}^C_{wn} \times \overline{\Omega}^C_{zn} \supset G^C_w \times G^C_z \supset \Omega^C_w \times \Omega^C_z$, then (6) defines a continuous linear functional on $B(\overline{G}_w \times \overline{G}_z)$. This means that given $\varepsilon > 0$ there exists $\delta > 0$ such that $|L(f)| < \varepsilon$ for all f with $\|f\|_G < \delta$ where $\| \ \|_G$ is the norm of $B(\overline{G}_w \times \overline{G}_z)$. Since $\overline{G}_w \times \overline{G}_z \subset \overline{\Omega}_{wm} \times \overline{\Omega}_{zm}$ for some m, $\|f\|_G \leq \|f\|_m$ and thus given $\varepsilon > 0$ there exist δ and $\| \ \|_m$ such that $\|f\|_m < \delta$ implies $|L(f)| < \varepsilon$. But this means that (6) also defines a continuous linear functional on $H(\Omega_w \times \Omega_z)$.

We observe that $\overline{\Omega}^C_{wn} \times \overline{\Omega}^C_{zn}$ is a domain containing $\Omega^C_w \times \Omega^C_z$. A domain such as this will be called an open neighborhood of $\Omega^C_w \times \Omega^C_z$. Let G be an open neighborhood of $\Omega^C_w \times \Omega^C_z$. Then the value of (6) is independent of Γ_ζ and Γ_η, provided that Γ_ζ and Γ_η lie in G; this follows from Cauchy's theorem. For the same reason, two holomorphic functions ℓ_1 and ℓ_2, defined on neighborhoods G_1 and G_2 of $\Omega^C_w \times \Omega^C_z$ respectively, define the same linear functional if $\ell_1 = \ell_2$ on a neighborhood contained $G_1 \cap G_2$. We say that two such functions are equivalent and call a class of equivalent functions a locally holomorphic function on $\Omega^C_w \times \Omega^C_z$. For convenience we use ℓ to denote the equivalence class to which ℓ belongs. We note here that we have defined locally holomorphic for functions of two variables analogously to the way it was defined for functions of one variable. Using $H(\Omega^C_w \times \Omega^C_z)$ to denote all the locally holomorphic functions on $\Omega^C_w \times \Omega^C_z$, we can state our result in the following theorem.

Theorem 7. *Let Ω_w and Ω_z be simply connected domains in the complex plane. Then the dual space of $H(\Omega_w \times \Omega_z)$ is $H(\Omega^C_w \times \Omega^C_z)$, i.e. if L is a continuous linear functional on $H(\Omega_w \times \Omega_z)$, then there exists ℓ locally holomorphic on $\Omega^C_w \times \Omega^C_z$ such that L has representation* (6) *and conversely, if ℓ is a locally holomorphic function on $\Omega^C_w \times \Omega^C_z$, then* (6) *defines a continuous linear functional on $H(\Omega_w \times \Omega_z)$.*

Theorem 8. *The extension of a continuous linear functional on $H(\Omega_w \times \Omega_z)$ to $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ is unique, i.e., $H(\Omega_w \times \Omega_z)$ is dense in $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$.*

Proof. Let L be a continuous linear functional on $H(\Omega_w \times \Omega_z)$ and ℓ its corresponding locally holomorphic function on $\Omega^C_w \times \Omega^C_z$ with representation (6). Assume L has two extensions L_1 and L_2. If $f \in LH(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ and $f \notin H(\Omega_w \times \Omega_z)$, then it follows from representation (6) that $L_1(f) = L_2(f)$. If $f \in B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ and $f \notin LH(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$, then it follows from Theorem 1 that $L_1(f) = \lim_{k\to\infty} L_1(f_k) = \lim_{k\to\infty} L_2(f_k) = L_2(f)$ where $\{f_k\}$ is a sequence in $LH(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ converging to f. Hence $L_1 \equiv L_2$.

Assume $H(\Omega_w \times \Omega_z)$ were not dense in $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ and let g be in the complemnet of the closure of $H(\Omega_w \times \Omega_z)$. Then it follows from Hahn-Banach theorem that there exists a continuous linear functional L^* on $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ such that $L^*(f) = 0$ for all f in the closure of $H(\Omega_w \times \Omega_z)$ but $L^*(g) \neq 0$. Since the null functional on $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$ is an extension of the null functional on $H(\Omega_w \times \Omega_z)$, we have obtained two extensions from $H(\Omega_w \times \Omega_z)$ to $B(\overline{\Omega}_{wn} \times \overline{\Omega}_{zn})$, contrary to the first part of the theorem.

Continuous linear operators on $H(\Omega_w \times \Omega_z)$

Let $\Omega_w, \Omega_z, \Omega_\eta$, and Ω_ζ be simply connected domains in the complex plane. We are mainly interested in finding a way to represent continuous linear operators from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$. In view of Köthe's integral representations of continuous linear operators from $H(\Omega_w)$ to $H(\Omega_z)$, we shall first consider continuous linear operators from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_\eta \times \Omega_\zeta)$ and obtain a result analogous to Köthe's.

Theorem 9. *Let $\Omega_w, \Omega_z, \Omega_\eta$ and Ω_ζ be simply connected domains in the complex plane. To each continuous linear operator T from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_\eta \times \Omega_\zeta)$ there corresponds a function $M(w, z, \eta, \zeta)$ locally holomorphic on $\Omega_w^C \times \Omega_z^C \times \Omega_\eta \times \Omega_\zeta$ such that T has the integral representation*

$$T(f)(\eta, \zeta) = \frac{1}{(2\pi i)^2} \int_{\Gamma_z} \int_{\Gamma_w} f(w, z) M(w, z, \eta, \zeta) dw dz$$

where $\Gamma_w \subset \Omega_w$ and $\Gamma_z \subset \Omega_z$ are simple closed contours such that there is a bounded open subset A of $\Omega_\eta \times \Omega_\zeta$ with the property that for (η, ζ) in $\overline{A}$, M is holomorphic in (w, z) for w outside and on Γ_w and z outside and on Γ_z.

Proof. We note that we have constructed the space $H(\Omega_w \times \Omega_z) H(\Omega_\eta \times \Omega_\zeta)$ respectively) at the beginning of this paper and we only recall that the topology of $H(\Omega_w \times \Omega_z)$ is defined by a sequence of norms $||f||_n = \sup_{(w,z) \in \overline{D}_n} |f(w, z)|$ where $D_n = \Omega_{wn} \times \Omega_{zn} (||f||_n = \sup_{(\eta,\zeta) \in \overline{D}_n} |f(\eta, \zeta)|$ where $D_n = \Omega_{\eta n} \times \Omega_{\zeta n}$ respectively).

Since T is continuous, given a neighborhood $N_{\eta\zeta} = \{g \in H(\Omega_\eta \times \Omega_\zeta) \mid ||g||_n \leq 1\}$ of 0, for each n, there exist a positive integer $N(n)$ and a positive number $\delta(n)$ such that $N_{wz} = \{f \in H(\Omega_w \times \Omega_z) \mid ||f||_{N(n)} \leq \delta(n)\}$ and $T(N_{wz}) \subset N_{\eta\zeta}$ Hence T is a continuous linear operator defined on $H(\Omega_w \times \Omega_z) \subset B(\overline{\Omega}_{wN(n)} \times \overline{\Omega}_{zN(n)})$ to $B(\overline{\Omega}_{\eta n} \times \overline{\Omega}_{\zeta n})$. Since $B(\overline{\Omega}_{wN(n)} \times \overline{\Omega}_{zN(n)})$ is complete and $H(\Omega_w \times \Omega_z)$ is dense in $B(\overline{\Omega}_{wN(n)} \times \overline{\Omega}_{zN(n)})$ by Theorem 8, T can be uniquely extended to T_n:

$B(\overline{\Omega}_{wN(n)} \times \overline{\Omega}_{zN(n)}) \to B(\overline{\Omega}_{\eta n} \times \overline{\Omega}_{\zeta n})$ and $||T_n(f)||_n \leq K_n ||f||_{N(n)}$ for all $f \in B(\overline{\Omega}_{wN(n)} \times \overline{\Omega}_{zN(n)})$. For $(s, t) \in \overline{\Omega}_{wN(n)}^C \times \overline{\Omega}_{zN(n)}^C$, $\frac{1}{w-s} \cdot \frac{1}{z-t}$, considered as a function of (w, z), is in $B(\overline{\Omega}_{wN(n)} \times \overline{\Omega}_{zN(n)})$. Hence we can define a function M_n as follows:

$$\begin{aligned} M_n(s,t,\eta,\zeta) &= T_n(\frac{1}{w-s}\frac{1}{z-t}) \text{ for finite } (s,t) \\ M_n(s,\infty,\eta,\zeta) &= 0 \\ M_n(\infty,t,\eta,\zeta) &= 0 \\ M_n(\infty,\infty,\eta,\zeta) &= 0 \end{aligned}$$

It follows from the definition that for fixed (s,t), M_n, considered as a function of (η,ζ), is holomorphic on $\Omega_{\eta n} \times \Omega_{\zeta n}$ and continuous on $\overline{\Omega}_{\eta n} \times \overline{\Omega}_{\zeta n}$. Conversely for fixed (η,ζ) we can show as in the proof of Theorem 3 that M_n, as a function of (s,t), is holomorphic on $\overline{\Omega}^C_{wN(n)} \times \overline{\Omega}^C_{zN(n)}$. Therefore $M_n(s,t,\eta,\zeta)$ is holomorphic on $\overline{\Omega}^C_{wN(n)} \times \overline{\Omega}^C_{zN(n)} \times \Omega_{\eta n} \times \Omega_{\zeta n}$. Since $\|T_n(f)\|_n \leq K_n \|f\|_{N(n)}$ for all $f \in B(\overline{\Omega}_{wN(n)\times\overline{\Omega}_{zN(n)}})$, we have $\|M_n(s,t,\eta,\zeta)\|_\eta \leq K_n \frac{1}{\delta_{N(n)}(s,t)}$ where $\delta_{N(n)}(s,t)$ is the distance between (s,t) and $\overline{\Omega}_{wN(n)} \times \overline{\Omega}_{zN(n)}$. Thus if $s \to \infty$ or $t \to \infty$ or both, we obtain $M_n \to 0$. The functions M_n defined as in the above have the following property: If $n < m$, then M_n and M_m are identical on $[(\overline{\Omega}^C_{wN(n)} \times \overline{\Omega}^C_{zN(n)}) \cap (\overline{\Omega}^C_{wN(m)} \times \overline{\Omega}^C_{zN(m)})] \times [(\Omega_{\eta n} \times \Omega_{\zeta n}) \cap (\Omega_{\eta m} \times \Omega_{\zeta m})] = \overline{\Omega}^C_{wN(m)} \times \overline{\Omega}^C_{zN(m)} \times \Omega_{\eta n} \times \Omega_{\zeta n}$. Because of this property, we can define a function $M(s,t,\eta,\zeta)$ as the collection of $\{M_n(s,t,\eta,\zeta)\}_{n=1}^\infty$ and thus obtain a well-defined locally holomorphic function on $\Omega^C_w \times \Omega^C_z \times \Omega_\eta \times \Omega_\zeta = \bigcup_{n=1}^\infty (\overline{\Omega}^C_{wN(n)} \times \overline{\Omega}^C_{zM(n)} \times \Omega_{\eta n} \times \Omega_{\zeta n})$. Let $f \in H(\Omega_w \times \Omega_z)$. For any $\Omega_{wN(n)} \times \Omega_{zN(n)}$ with boundary $\Gamma_{sN(n)} \times \Gamma_{tN(n)}$, we have $f(w,z) = \frac{1}{(2\pi i)^2} \int_{\Gamma_{tN(n)+1}} \int_{\Gamma_{sN(n)+1}} \frac{f(s,t)}{(s-w)(t-z)} dsdt$ for all $(w,z) \in \Omega_{wN(n)} \times \Omega_{zN(n)}$ by Cauchy integral formula for product domain. Using Riemann sums and the continuity of T as in the proof of Theorem 4, we have

$$\begin{aligned} T(f)(\eta,\zeta) =& \frac{1}{(2\pi i)^2} \int_{\Gamma_{tN(n)+1}} \int_{\Gamma_{sN(n)+1}} f(s,t) T(\frac{1}{(s-w)(t-z)} dsdt \\ =& \frac{1}{(2\pi i)^2} \int_{\Gamma_{tN(n)+1}} \int_{\Gamma_{sN(n)+1}} f(s,t) M(s,t,\eta,\zeta) dsdt \end{aligned}$$

for all $(\eta,\zeta) \in \overline{\Omega}_{\eta n} \times \overline{\Omega}_{\zeta n}$. We note that (η,ζ) can be arbitrary in $\Omega_\eta \times \Omega_\zeta$ since then there exists n such that $(\eta,\zeta) \in \Omega_{\eta n} \times \Omega_{\zeta n}$ and to this n there corresponds $N(n)$ with which we started to use Cauchy theorem to derive the integral representation for T.

Theorem 10. *Let Ω_w, Ω_z and Ω_ζ be simply connected domains in the complex plane. To each continuous linear operator T from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ there corresponds a function $M(z,w,\zeta)$ locally holomorphic on $\Omega^C_z \times \Omega_w \times \Omega_\zeta$ such that T has the integral representation*

$$T(f)(w,\zeta) = \frac{1}{2\pi i} \int_{\Gamma_z} f(w,z) M(z,w,\zeta) dz$$

where $\Gamma_z \subset \Omega_z$ is a simple closed contour such that there is a bounded open subset A of $\Omega_w \times \Omega_\zeta$ with the property that for (w,ζ) in $\overline{A}$, M is holomorphic outside and on Γ_z.

Proof. For fixed $w_0 \in \Omega_w$, T maps functions $f(w_0,z)$ in $H(\Omega_z)$ to functions $T(f)(w_0,\zeta)$ in $H(\Omega_\zeta)$. By Köthe's representation theorem, there exists a function $M_{w_0}(z,\zeta)$ locally holomorphic on $\Omega_z^C \times \Omega_\zeta$ such that

$$T(f)(w_0,\zeta) = \frac{1}{2\pi i}\int_{\Gamma_z} f(w_0,z)M_{w_0}(z,\zeta)dz \tag{11}$$

where $\Gamma_z \subset \Omega_z$ is a simple closed contour such that for all ζ in the closure of a bounded open subset of Ω_ζ, M_{w_0} is holomorphic outside and on Γ_z.

By Theorem 9, there exists a function $M_1(s,t,w,\zeta)$ locally holomorphic on $\Omega_w^C \times \Omega_z^C \times \Omega_w \times \Omega_\zeta$ such that T has the integral representation

$$T(f)(w,\zeta) = \frac{1}{(2\pi i)^2}\int_{\Gamma_z}\int_{\Gamma_s} f(s,z)M_1(s,z,w,\zeta)dsdz \tag{12}$$

where $\Gamma_s \subset \Omega_w$ and $\Gamma_z \subset \Omega_z$ are simple closed contours such that for (w,ζ) in the closure of a bounded open subset of $\Omega_w \times \Omega_\zeta$, M_1 is holomorphic in (s,z) for s ouside and on Γ_s and z outside and on Γ_z.

For fixed w_0, (12) becomes

$$T(f)(w_0,\zeta) = \frac{1}{(2\pi i)^2}\int_{\Gamma_z}\int_{\Gamma_z} f(s,z)M_1(s,z,w_0,\zeta)dsdz \tag{13}$$

Compare (11) with (13). If we can show that $f(w_0,z)M_{w_0}(z,\zeta) = \frac{1}{2\pi i}\int_{\Gamma_s} f(s,z) M_1(s,z,w_0,\zeta)ds$, then for those functions $f(s,z) \in H(\Omega_w \times \Omega_z)$ which are the product of $g(s) \in H(\Omega_w)$ and $h(z) \in H(\Omega_z)$, we have $g(w_0)h(z)M_{w_0}(z,\zeta) = \frac{1}{2\pi i}\int_{\Gamma_s} g(s)h(z)M_1 (s,z,w_0,\zeta)ds$. In particular; if $g \equiv 1$, then we have $M_{w_0}(z,\zeta) = \frac{1}{2\pi i}\int_{\Gamma_s} M_1(s,z,w_0,\zeta)ds$. Rewrite $M_{w_0}(z,\zeta)$ as $M(z,w_0,\zeta)$ and let w_0 vary throughout Ω_w. Then the right hand side of the last equation represents a function locally holomorphic on $\Omega_z^C \times \Omega_w \times \Omega_\zeta$. Hence $M(z,w,\zeta)$ is locally holomorphic on $\Omega_z^C \times \Omega_w \times \Omega_\zeta$. Therefore it remais to show that $f(w_0,z)M_{w_0}(z,\zeta) = \frac{1}{2\pi i}\int_{\Gamma_s} f(s,z)M_1(s,z,w_0,\zeta)ds$ for those functions $f(s,z)$ which are product of $g(s) \in H(\Omega_w)$ and $h(z) \in H(\Omega_z)$. From (11) and (13), we have $\frac{1}{2\pi i}\int_{\Gamma_z}[\frac{1}{2\pi i}\int_{\Gamma_s} f(s,z)M_1(s,z,w_0,\zeta)ds - f(w_0,z)M_{w_0}(z,\zeta)dz = 0$ for all $f \in H(\Omega_w \times \Omega_z)$. If $f(s,z) = g(s)h(z)$, then $\frac{1}{2\pi i}\int_{\Gamma_z}[\frac{1}{2\pi i}\int_{\Gamma_s} g(s)h(z)M_1(s,z,w_0,\zeta)ds - g(w_0)h(z)M_{w_0}(z,\zeta)]dz = \frac{1}{2\pi i}\int_{\Gamma_z}[\frac{1}{2\pi i}\int_{\Gamma_s} g(s)M_1(s,z,w_0,\zeta)ds - g(w_0)M_{w_0}(z,\zeta)]h(z)dz = 0$. Let $G(z,\zeta) = \frac{1}{2\pi i}\int_{\Gamma_s} g(s)M_1(s,z,w_0,\zeta)ds - g(w_0)M_{w_0}(z,\zeta)$. Then for each $\zeta \in \Omega_\zeta$, G, as a function of z, is in $BK[\Omega_z]$. Hence for each fixed $\zeta_0 \in \Omega_\zeta$, $\frac{1}{2\pi i}\int_{\Gamma_z} G(z,\zeta_0)h(z)dz = 0$ for all h in $H(\Omega_z)$. By Köthe's duality theorem, $G(z,\zeta_0)$ is a null linear functional on $H(\Omega_z)$. We conclude that $G(z,\zeta_0) = 0$ for all $\zeta_0 \in \Omega_\zeta$ and the proof is complete.

Pólya property preserving operators

We start with the following definition:

Definition 14. Let Ω_w, Ω_z and Ω_ζ be simply connected domains in the complex plane. Let T be a continuous linear operator from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ and $f(w,z) \in H(\Omega_w \times \Omega_z)$. Then T is said to be a *Pólya property preserving operator* if $T(f)(w,\zeta)$ has the Pólya property on Ω_ζ whenever $f(w,z)$ has the Pólya property on Ω_z.

Theorem 15. *Let Ω_w, Ω_z and Ω_ζ be simply connected domains in the complex plane. Let T be a continuous linear operator from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ and $M(z,w,\zeta)$ its corresponding kernel function. If $M(z,w,\zeta)$ is of the form $[g(z)+h(w)]K(w,\zeta)$ where $g \not\equiv 0$ and $K(w,\zeta)$ has the Pólya property on Ω_ζ, then T is a Pólya property preserving operator.*

Proof. Let $f(w,z) \in H(\Omega_w \times \Omega_z)$ have the Pólya property on Ω_z. We must show that $T(f)(w,\zeta)$ has the Pólya property on Ω_ζ. let $\Gamma_\zeta \subseteq \Omega_\zeta$ be a simple closed contour and F a function analytic outside and on Γ_ζ with $F(\infty)=0$ such that $\int_{\Gamma_\zeta} T(f)(w,\zeta)F(\zeta)d\zeta = 0$ for all w in Ω_w. Since $T(f)(w,\zeta) = \frac{1}{2\pi i}\int_{\Gamma_z} f(w,z)M(z,w,\zeta)dz$ by Theorem 10, we have $\int_{\Gamma_\zeta}(\frac{1}{2\pi i}\int_{\Gamma_z} f(w,z)M(z,w,\zeta)dz)F(\zeta)d\zeta = 0$ for all w in Ω_w. Since $f(w,z)M(z,w,\zeta)F(\zeta)$ is continuous on the compact set $\Gamma_\zeta \times \Gamma_z$, we can interchange the order of integration and obtain $\int_{\Gamma_z} f(w,z)[\int_{\Gamma_\zeta} M(z,w,\zeta)F(\zeta)d\zeta]dz = 0$ for all w in Ω_w. Substituting $[g(z)+h(w)]K(w,\zeta)$ for $M(z,w,\zeta)$, we have $\int_{\Gamma_z} f(w,z)[\int_{\Gamma_\zeta}(g(z)+h(w))K(w,\zeta)F(\zeta)d\zeta]dz = \int_{\Gamma_z} f(w,z)[\int_{\Gamma_\zeta} g(z)K(w,\zeta)F(\zeta)d\zeta]dz + \int_{\Gamma_z} f(w,z)[\int_{\Gamma_\zeta} h(w)K(w,\zeta)F(\zeta)d\zeta]dz = \int_{\Gamma_z} f(w,z)$ $g(z)[\int_{\Gamma_\zeta} K(w,\zeta)F(\zeta)d\zeta]dz + h(w)\int_{\Gamma_z} f(w,z)dz \int_{\Gamma_\zeta} K(w,\zeta)F(\zeta)d\zeta = 0$ for all w in Ω_w. Since $\int_{\Gamma_z} f(w,z)dz \equiv 0$, we obtain $\int_{\Gamma_z} f(w,z)g(z)dz \int_{\Gamma_\zeta} K(w,\zeta)\,F(\zeta)d\zeta = 0$ for all w in Ω_w. This implies that either $\int_{\Gamma_z} f(w,z)g(z)dz = 0$ for all w in Ω_w or $\int_{\Gamma_\zeta} K(w,\zeta)F(\zeta)d\zeta = 0$ for all w in Ω_w. If $\int_{\Gamma_z} f(w,z)g(z)dz = 0$ for all w in Ω_w, then since g is analytic outside and on Γ_z with $g(\infty) = 0$ and $f(w,z)$ has the Pólya property on Ω_z, we would have $g \equiv 0$ which contradicts our assumption on $M(z,w,\zeta)$. Hence $\int_{\Gamma_z} f(w,z)g(z) \not\equiv 0$ and $\int_{\Gamma_\zeta} K(w,\zeta)F(\zeta)d\zeta = 0$ for all w in Ω_w. Since F is analytic outside and on Γ_ζ and $K(w,\zeta)$ has the Pólya property on Ω_ζ, we conclude that $F \equiv 0$ and thus $T(f)(w,\zeta)$ has the Pólya property on Ω_ζ.

Theorem 16. *Let Ω_w, Ω_z and Ω_ζ be simply connected domains in the complex plane. Let T be a continuous linear operator from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ and $M(z,w,\zeta)$ its corresponding kernel function. If $M(z,w,\zeta)$ is of the form $h(w)K(z,\zeta)$ where $h \not\equiv 0$ and $K(z,\zeta)$ has the Pólya property on Ω_ζ, then T is a Pólya property preserving operator.*

Proof. Let $f(w,z) \in H(\Omega_w \times \Omega_z)$ have the Pólya property on Ω_z. We must show that $T(f)(w,\zeta)$ has the Pólya property on Ω_ζ. Let $\Gamma_\zeta \subset \Omega_\zeta$ be a simple closed contour and F a function analytic outside and on Γ_ζ with $F(\infty)=0$ such that $\int_{\Gamma_\zeta} T(f)(w,\zeta)F(\zeta)d\zeta = 0$ for all w in Ω_w. Since $T(f)(w,\zeta)=\frac{1}{2\pi i}\int_{\Gamma_z} f(w,z)M(z,w,\zeta)dz$ by Theorem 10, we have $\int_{\Gamma_\zeta}(\frac{1}{2\pi i}\int_{\Gamma_z} f(w,z)M(z,w,\zeta)dz)F(\zeta)d\zeta = 0$ for all w in Ω_w. Since $f(w,z)M(z,w,\zeta)F(\zeta)$ is continuous on $\Gamma_\zeta \times \Gamma_z$, we can interchange the order of integration an obtain $\int_{\Gamma_z} f(w,z)$ $[\int_{\Gamma_\zeta} M(z,w,\zeta)F(\zeta)d\zeta]dz = 0$ for all w in Ω_w. Substituting $h(w)K(z,\zeta)$ for $M(z,w,\zeta)$, we have $\int_{\Gamma_z} f(w,z)[\int_{\Gamma_\zeta} h(w)K(z,\zeta)F(\zeta)d\zeta]dz = h(w)\int_{\Gamma_z} f(w,z)[\int_{\Gamma_z} K(z,\zeta)F(\zeta)d\zeta]dz = 0$ for all w in Ω_w. Since $h(w)\not\equiv 0$, $\int_{\Gamma_z} f(w,z)[\int_{\Gamma_\zeta} K(z,\zeta)F(\zeta)d\zeta]dz = 0$ for all w in Ω_w. Since $f(w,z)$ has the Pólya property on Ω_z and $\int_{\Gamma_\zeta} K(z,\zeta)F(\zeta)d\zeta$ is in $K[\Omega_z]$, the last equation implies that $\int_{\Gamma_\zeta} K(z,\zeta)F(\zeta)d\zeta \equiv 0$. but this in turn implies $F\equiv 0$ since $K(z,\zeta)$ is assumed to have the Pólya property on Ω_ζ.

Before we can show that the sufficient conditions in Theorems 15 and 16 are not necessary conditions, we must prove the following interesting result.

Theorem 17. *Let Ω_w, Ω_z, and Ω_ζ be simply connected domains in the complex plane. Let T be a continuous linear operator from $H(\Omega_w\times\Omega_z)$ to $H(\Omega_w\times\Omega_\zeta)$. Then T either preserves the Pólya property for all functions having that property or does not preserve the Pólya property for any function.*

Proof. Let $f(w,z)\in H(\Omega_w\times\Omega_z)$ have the Pólya property on Ω_z. Then given any simple closed contour $\Gamma_z \subseteq \Omega_z$ and any function F analytic outside and on Γ_z with $F(\infty)=0$ such that $\int_{\Gamma_z} f(w,z)F(z)dz = 0$ for all w in Ω_w we have $F\equiv 0$. In fact, the condition that $\int_{\Gamma_z} f(w,z)F(z)dz=0$ for all w in Ω_w may well be replaced by $\int_{\Gamma_z} f(w,z)F(z)dz = 0$ for a sequence $\{w_n\}$ in Ω_w with a limit point in Ω_w. This follows from the uniqueness theorem for analytic functions. Let $\{w_n\}$ be such a sequence and denote $f(w_n,z)$ by $f_n(z)$ for all $n=0,1,2,\ldots$. Then given any simple closed contour $\Gamma_z \subseteq \Omega_z$ and any function F analytic outside and on Γ_z with $F(\infty)=0$ such that $\int_{\Gamma_z} f_n(z)F(z)dz = 0$ for all $n=0,1,2,\ldots$, we have $F\equiv 0$. Each $f_n(z)$ is in $H(\Omega_z)$ and each function in $H(\Omega_z)$ can also be considered as a function in $H(\Omega_w\times\Omega_z)$. With T restricted to $H(\Omega_z)$, T maps $H(\Omega_z)$ to $H(\Omega_\zeta)$ and its conjugate operator T' maps $BK[\Omega_\zeta]$ to $BK[\Omega_z]$. We first show that if T' is one-to-one, then T is Pólya property preserving. Let $f(w,z),\{w_n\},\{f_n\},\Gamma_z$ and F be as in the above with $\int_{\Gamma_z} f_n(z)F(z)dz = 0$ for all $n=0,1,2,\ldots$ implying $F\equiv 0$. We now let $\Gamma_s \subseteq \Omega_\zeta$ be any simple closed contour and G a function analytic outside and on Γ_ζ with $G(\infty)=0$ such that $\int_{\Gamma_\zeta} T(f_n)(\zeta)G(\zeta)d\zeta = 0$ for all $n=0,1,2,\ldots$. Since $\int_{\Gamma_\zeta} T(f_n)(\zeta)G(\zeta)d\zeta = \int_{\Gamma_\zeta} f_n(z)T'(G)(z)dz = 0$ for all $n=0,1,2,\ldots$ and $T'(G)\in BK[\Omega_z]$, we have $T'(G)\equiv 0$. Since T' is one-to-one, $G\equiv 0$ and this shows that T is Pólya property preserving. We now show that if T' is not one-to-one, then T does not preserve the Pólya property for any function. Again we let $f(w,z)$, $\{w_n\}$, $\{f_n\}$, Γ_z and F be as above satisfying the condition that if $\int_{\Gamma_z} f_n(z)F(z)dz = 0$ for all $n=0,1,2,\ldots$ then $F\equiv 0$. Since T' is not one-to-one, there exists $G_0\in BK[\Omega_\zeta]$ with $G\not\equiv 0$ such that $T'(G_0)\equiv 0$. Hence $0=\int_{\Gamma_z} f_n(z)T'(G_0)(z)dz = \int_{\Gamma_\zeta} T(f_n)(\zeta)G_0(\zeta)d\zeta$

for all $n = 0, 1, 2, \ldots$ where $\Gamma_\zeta \subseteq \Omega_\zeta$ is a simple closed contour such that G_0 is analytic outside and on Γ_ζ. But $G_0 \not\equiv 0$. Hence $T(f)$ does not have the Pólya property. Since T' is either one-to-one or not one-to-one, T either preserves the Pólya property for all functions having that property or does not preserve the Pólya property for any function.

Remark. By Theorem 10, to each continuous linear operator T from $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ there corresponds a kernel function $M(z, w, \zeta)$. If $M(z, w, \zeta)$ is either of the form $[g(z) + h(w)]K(w, \zeta)$ or of the form $h(w)K(z, \zeta)$, then its corresponding T is a Pólya property preserving operator by Theorems 15 and 16. If $M(z, w, \zeta)$ is neither of the form $[g(z) + h(w)]K(w, \zeta)$ nor of the form $h(w)K(z, \zeta)$, then we appeal to Theorem 17. To test whether its corresponding operator T preserves the Pólya property. it suffices to check just one function which is known to have the Pólya property. Both e^{wz} and $\frac{1}{w-z}$ are good choices to consider as we demonstrate in the following example which also shows the sufficient conditions in Theorem 15 and 16 are not necessary conditions.

Example 18. Let Ω_w, Ω_z and Ω_ζ be simply connected domains in the complex plane such that they are mutually disjoint. Let Ω_z contain the origin and $M(z, w, \zeta) = \frac{e^{\frac{w}{z}}}{\zeta - w}$. Then $M(z, w, \zeta)$ is locally holomorphic on $\Omega_z^C \times \Omega_w \times \Omega_\zeta$. Moreover, $M(z, w, \zeta)$ has the Pólya property on Ω_ζ and $M(z, w, \zeta)$ is neither of the form $[g(z)+h(w)]K(w, \zeta)$ nor of the form $h(w)K(z, \zeta)$. To show that the operator T with kernel $M(z, w, \zeta)$ preserves the Pólya property, we let $f(w, z) = \frac{1}{z-w}$ which is in $H(\Omega_w \times \Omega_z)$ and note that $f(w, z)$ has the Pólya property on Ω_z. Then $T(f)(w, \zeta) = \frac{1}{2\pi i} \int_\Gamma f(w, z) M(z, w, \zeta) dz$ where $\Gamma \subseteq \Omega_z$ is a simple closed contour such that for (w, ζ) in the closure of a bounded open subset of $\Omega_w \times \Omega_\zeta$, M is holomorphic outside and on Γ. Thus $T(f)(w, \zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z-w} \cdot \frac{e^{\frac{w}{z}}}{\zeta - w} dz = \frac{1}{\zeta - w} \frac{1}{2\pi i} \int_\Gamma \frac{e^{\frac{w}{z}}}{z-w} dz = \frac{-1}{\zeta - w} \frac{1}{2\pi i} \int_{-\Gamma} \frac{e^{\frac{w}{z}}}{z-w} dz = \frac{-e}{\zeta - w}$ which has the Pólya property on Ω_ζ. Since T either preserves the Pólya property for all functions having that property or does not preserve the Pólya property for any function by Theorem 17, we conclude that this T with kernel $\frac{e^{\frac{w}{z}}}{\zeta - w}$ is a Pólya property preserving operator.

Application. It is now clear how to use functions which have the Pólya property to generate new functions which also have the Pólya property. All we have to do is to pick a function which is known to have the Pólya property and let the function be the kernel of an operator. Then with domains suitably chosen, the operator maps functions which have the Pólya property to new functions which have the Pólya property. We shall give a simple example where the kernel is of the form $h(w)K(z, \zeta)$ as in Theorem 16 with $h(w) \equiv 1$ and $K(z, \zeta)$ has the Pólya property on Ω_ζ.

Example 19. Let Ω_ζ be the unit disk and $m(w, \zeta) = A(\zeta)e^{w\zeta} + B(\zeta)e^{-w\zeta}$ where A and B are analytic on Ω_ζ. Let $A(\zeta)A(-\zeta) - B(\zeta)B(-\zeta) \neq 0$ for all ζ in Ω_ζ. Then $m(w, \zeta)$ has the Pólya property on Ω_ζ [5]. The Borel transform of $m(w, \zeta)$ with respect to w is $M(z, \zeta) = \frac{A(\zeta)}{z-\zeta} + \frac{B(\zeta)}{z+\zeta}$ which has the Pólya property on Ω_ζ. Let Ω_w be a simply connected domain and $\Omega_z = \Omega_\zeta$. Since $M(z, \zeta)$ has the Pólya property on Ω_ζ, the operator T defined on $H(\Omega_w \times \Omega_z)$ to $H(\Omega_w \times \Omega_\zeta)$ with $M(z,\zeta)$ as kernel preserves the Pólya property. Let $f(w,z) = e^{wg(z)}$ where g is analytic and univalent on Ω_z. Then $f(w,z)$ has the Pólya property on Ω_z and so $T\ (f)(w,\zeta)\frac{1}{2\pi i}\int_\Gamma f(w,z)M(z,\zeta)dz = \frac{1}{2\pi i}\int_\Gamma e^{wg(z)}(\frac{A(\zeta)}{z-\zeta} + \frac{B(\zeta)}{z+\zeta})dz = A(\zeta)e^{wg(\zeta)} + B(\zeta)e^{wg(-\zeta)}$ has Pólya property on Ω_ζ.

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