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GENERALIZATIONS OF ALZER'S AND KUANG'S INEQUALITY

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Abstract. Let f be a strictly increasing convex (or concave) functions on $(0,1]$, then, for k being a nonnegative integer and n a natural number, the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f(\frac{i}{n+k})$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)dt$. Form this, some new inequalities involving $\sqrt[n]{(n+k)!/k!}$ are deduced. By the Hermie-Hadamard inequality, several inequalities are obtained.

1. Introduction

In [1], H. Alzer, using the mathematical induction and other techniques, proved that for $r > 0$ and $n \in \mathbb{N}$,

$$\frac{n}{n+1} \le \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{1/r} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1}$$

By the Cauchy's mean-value theorem and the mathematical induction, the author in [7] presented that, if n and m are natural numbers, k is a nonnegative integer, $r > 0$, then

$$\frac{n+k}{n+m+k} < \left(\frac{1}{n}\sum_{i=k+1}^{n+k} i^r \Big/ \frac{1}{n+m}\sum_{i=k+1}^{n+m+k} i^r\right)^{1/r}. \tag{2}$$

The lower bound is best possible.

From the Stirling's formula, for all nonnegative integers k and natural numbers n and m, the author in [8] obtained

$$\left(\prod_{i=k+1}^{n+k} i\right)^{1/n} \Big/ \left(\prod_{i=k+1}^{n+m+k}\right)^{1/(n+m)} \le \sqrt{\frac{n+k}{n+m+k}}. \tag{3}$$

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Let f be a strictly increasing convex (or concave) function in $(0,1]$, J.-C. Kuang in [2] verified that

$$\frac{1}{n}\sum_{k=1}^{n} f\Big(\frac{k}{n}\Big) > \frac{1}{n+1}\sum_{k=1}^{n+1} f\Big(\frac{k}{n+1}\Big) > \int_0^1 f(x)dx. \tag{4}$$

The study of Alzer's and Minc-Sathre's inequality has many literature, for examples, [1]-[9].

In this article, motivated by [2, 7], i.e. the inequalities in (2), (3) and (4), considering the convexity of a function, we get

Theorem 1. *Let f be strictly increasing convex (or concave) function in $(0,1]$, then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f(\frac{i}{n+k})$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)dt$, that is,*

$$\frac{1}{n}\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k}\Big) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} f\Big(\frac{i}{n+k+1}\Big) > \int_0^1 f(t)dt, \tag{5}$$

where k is a nonnegative integer, n a natural number.

If let $f(x) = x^r$, $r > 0$, or let $k = 0$ in (5), then the inequalities in (1), (2) and (4) could be deduced. Therefore, inequality (5) generalizes Alzer's and Kuang's inequality in [1, 2] and inequality (2) above.

Corollary 1. *For a nonnegative integer k and a natural number $n > 1$, we have*

$$\frac{n+k}{n+k+1} < \left[\frac{(2n+2k)!}{(n+2k)!}\right]^{1/n} \Big/ \left[\frac{(2n+2k+2)!}{(n+2k+1)!}\right]^{1/(n+1)}$$
$$< \left[\frac{(n+k)!}{k!}\right]^{1/n} \Big/ \left[\frac{(n+k+1)!}{k!}\right]^{1/(n+1)} < \left[\frac{k!(k+2)!}{(k+3)^2}\right]^{1/n(n+1)}. \tag{6}$$

For a larger n, the upper bound in the third inequality of (6) is not better than (3) for $m = 1$. From the Hermite-Hadamard inequality in [3] and [4, pp. 10-12], we get the following

Theorem 2. *Let f be a nonlinear convex function in $(0,1]$, then*

$$\frac{1}{n+k}\sum_{i=k+1}^{n+k}\left[f\Big(\frac{i}{n+k}\Big) - f\Big(\frac{2i-1}{2(n+k)}\Big)\right]$$
$$> \frac{1}{n+k}\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k}\Big) - \int_{k/(n+k)}^1 f(t)dt$$
$$> \frac{1}{2(n+k)}\left[f(1) - f\Big(\frac{k}{n+k}\Big)\right]. \tag{7}$$

Further, if f satisfies the Lipschitz condition

$$|f(x) - f(y)| \le M|x-y|^{\alpha}, \quad 0 < \alpha \le 1, \tag{8}$$

then

$$\frac{n}{n+k}\cdot\frac{M}{[2(n+k)]^{\alpha}} > \frac{1}{n+k}\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k}\Big) - \int_{k/(n+k)}^{1} f(t)dt. \tag{9}$$

If let $k=0$ in Theorem 2, the related result in [2] follows.

2. Proof of Theorems

Proof of Theorem 1. Let us first assume that f be a strictly increasing convex function. Taking $x_1=\frac{i-1}{n+k}$, $x_2=\frac{i}{n+k}$, $\alpha=\frac{i-k-1}{n}$ and using the convexity and monotonicity of f yields

$$\begin{aligned}
&\frac{i-k-1}{n}f\Big(\frac{i-1}{n+k}\Big)+\Big(1-\frac{i-k-1}{n}\Big)f\Big(\frac{i}{n+k}\Big)\\
&\geq f\Big(\frac{i-k-1}{n}\cdot\frac{i-1}{n+k}+\frac{n-i+k+1}{n}\cdot\frac{i}{n+k}\Big)\\
&= f\Big(\frac{ni-i+k+1}{n(n+k)}\Big)\\
&> f\Big(\frac{i}{n+k+1}\Big)
\end{aligned}$$

for $i=k+1,k+2,\dots,n+k$. Summing up leads to

$$\sum_{i=k+1}^{n+k}\left[\frac{i-k-1}{n}f\left(\frac{i-1}{n+k}\right)+\left(1-\frac{i-k-1}{n}\right)f\left(\frac{i}{n+k}\right)\right] > \sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k+1}\right),$$

$$\sum_{i=k+1}^{n+k}\Big[(i-k-1)f\Big(\frac{i-1}{n+k}\Big)+(n+k-i+1)f\Big(\frac{i}{n+k}\Big)\Big] > n\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k+1}\Big),$$

$$n\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k+1}\Big)+nf(1) < (n+1)\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k}\Big),$$

$$n\sum_{i=k+1}^{n+k+1} f\Big(\frac{i}{n+k+1}\Big) < (n+1)\sum_{i=k+1}^{n+k} f\Big(\frac{i}{n+k}\Big).$$

The inequality (5) is proved.

By similar procedure, if f is a strictly increasing concave function in $(0,1]$, then for $k<i\leq n+k$, we have

$$\begin{aligned}
&\frac{i-k}{n+1}f\Big(\frac{i+1}{n+k+1}\Big)+\frac{n+k-i+1}{n+1}f\Big(\frac{i}{n+k+1}\Big)\\
&\leq f\Big(\frac{i-k}{n+1}\cdot\frac{i+1}{n+k+1}+\frac{n+k-i+1}{n+1}\cdot\frac{i}{n+k+1}\Big)\\
&= f\Big(\frac{ni+2i-k}{(n+1)(n+k+1)}\Big) < f\Big(\frac{i}{n+k}\Big),
\end{aligned}$$

$$
\begin{aligned}
&\sum_{i=k+1}^{n+k}\left[\frac{i-k}{n+1}f\Big(\frac{i+1}{n+k+1}\Big)+\frac{n+k-i+1}{n+1}f\Big(\frac{i}{n+k+1}\Big)\right]\\
&=\frac{n}{n+1}\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k+1}\Big)+\frac{n}{n+1}f(1)<\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k}\Big),\\
&n\sum_{i=k+1}^{n+k+1}f\Big(\frac{i}{n+k+1}\Big)<(n+1)\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k}\Big).
\end{aligned}
$$

The proof is complete.

Proof of Corollary 1. Substituting f be $\ln(1+x)$ or by $\ln(x/(1+x))$ in (5) and simplifying yields the first or the second inequality in (6), respectively.

Since

$$
\begin{aligned}
\frac{[(n+k)!/k!]^{n+1}}{[(n+k+1)!/k!]^{n}}&=\sum_{j=3}^{n}\left\{\frac{[(j+k)!/k!]^{j+1}}{[(j+k+1)!/k!]^{j}}-\frac{[(j+k-1)!/k!]^{j}}{[(j+k)!/k!]^{j-1}}\right\}+\frac{[(k+2)!/k!]^{3}}{[(k+3)!/k!]^{2}}\\
&<\frac{k!(k+2)!}{(k+3)^2},
\end{aligned}
$$

the third inequality in (6) is obtained.

Proof of Theorem 2. Using the Hermite-Hadamard inequality in [3] and [4, pp. 10-12], we have

$$
\begin{aligned}
&\sum_{i=k+1}^{n+k}f\Big(\frac{2i-1}{2(n+k)}\Big)\\
&<(n+k)\sum_{i=k+1}^{n+k}\int_{(i-1)/(n+k)}^{i/(n+k)}f(x)dx\\
&<\frac{1}{2}\sum_{i=k+1}^{n+k}\left[f\Big(\frac{i}{n+k}\Big)+f\Big(\frac{i-1}{n+k}\Big)\right]\\
&=\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k}\Big)-\frac{1}{2}\left[f(1)-f\Big(\frac{k}{n+k}\Big)\right],
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{1}{n+k}\sum_{i=k+1}^{n+k}f\Big(\frac{2i-1}{2(n+k)}\Big)<\int_{k/(n+k)}^{1}f(x)dx\\
&<\frac{1}{n+k}\sum_{i=k+1}^{n+k}f\Big(\frac{i}{n+k}\Big)-\frac{1}{2(n+k)}\left[f(1)-f\Big(\frac{k}{n+k}\Big)\right].
\end{aligned}
$$

The inequality (7) is proved. Combining (8) with (7) yields inequality (9). The proof of theorem 2 is complete.

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