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A CLASS OF P-VALENTLY ANALYTIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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Abstract. Certain subclass of p-valently analytic functions with positive coefficients is introduced, and some interesting properties belonging to the this class is obtained.

1. Introduction

Let $T_p(n)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \ (a_k \geq 0;\ n \in N;\ p \in N = \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the unit disk

$$U = \{z : z \in C \text{ and } |z| < 1\}.$$

A function $f(z) \in T_p(n)$ is said to be in the class $T_p(n, i, \lambda, \beta)$ if it satisfies the inequality:

$$\Re\{z^{i-p}(1 + \lambda(p-i))f^{(i)}(z) - \lambda z f^{(1+i)}(z))\} > \beta \tag{1.2}$$

for some $\lambda(\lambda > \frac{1}{n})$, $\beta(0 \leq \beta < \frac{p!}{(p-i)!})$, $n \in N$, $p \in N, i < p$, $i \in N \cup \{0\}$, and for all $z \in U$. Here, and throughout this paper, $f^{(i)}$ defined by for a function $f(z)$ fiven by (1.1):

$$f^{(i)}(z) = \frac{p!}{(p-i)!} z^{p-i} + \sum_{k=n+p}^{\infty} \frac{k!}{(k-i)!} a_k z^{k-i} \tag{1.3}$$

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for all $z \in U,\ n \in N,\ p \in N,\ i \in N \cup \{0\}$, and $i \leq p$.

Other subclasses of the class $T_p(n)$ were studied recently by (for example) Saitoh et al. [1], Yaguchi [2], and Nunokowa ([3], [4]), and (see also) Chen et al. [5].

2. A Theorem on Coefficient Bounds

Theorem 1. *Let a function $f(z) \in T_p(n)$ is in the class $T_p(n, i, \lambda, \beta)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{k![\lambda(k-p)-1]}{(k-i)!} a_k$$
$$\leq \frac{p!}{(p-i)!} - \beta \Big[\lambda > \frac{1}{n}; 0 \leq \beta < \frac{p!}{(p-i)!};\ n \in N;\ p \in N;\ i \in N \cup \{0\};\ i < p\Big] \quad (2.1)$$

The result is sharp.

Proof. Suppose that $f(z) \in T_p(n, i, \lambda, \beta)$. Then we find from (1.1), (1.2) and (1.3) that

$$\Re\Big\{\frac{p!}{(p-i)!} - \sum_{k=n+p}^{\infty} \frac{k![\lambda(k-p)-1]}{(k-i)!} a_k z^{k-p}\Big\}$$
$$> \beta \Big[\lambda > \frac{1}{n};\ 0 \leq \beta < \frac{p!}{(p-i)!};\ n \in N;\ p \in N;\ i \in N \cup \{0\};\ i < p;\ z \in U\Big].$$

If we choose z to be real and $z \to 1^-$, we get

$$\frac{p!}{(p-i)!} - \sum_{k=n+p}^{\infty} \frac{k![\lambda(k-p)-1]}{(k-i)!} a_k$$
$$\geq \beta \Big[\lambda > \frac{1}{n};\ 0 \leq \beta < \frac{p!}{(p-i)!};\ n \in N;\ p \in N;\ i \in N \cup \{0\};\ i < p\Big],$$

which is precisely the assertion (2.1) of Theorem 1.

Conversely, suppose that the inequality (2.1) holds true and let

$$U = \{z : z \in C \text{ and } |z| = 1\}.$$

Then we have

$$|z^{i-p}[(1 + \lambda(p-i)) f^{(i)}(z) - \lambda z f^{(1+i)}(z)] - \frac{p!}{(p-i)!}|$$
$$\leq | - \sum_{k=n+p}^{\infty} \frac{k![\lambda(k-p)-1]}{(k-i)!} a_k z^{k-p}| \leq \sum_{k=n+p}^{\infty} \frac{k![\lambda(k-p)-1]}{(k-i)!} a_k |z|^{k-p}$$
$$\leq \frac{p!}{(p-i)!} - \beta \Big[\lambda > \frac{1}{n};\ 0 \leq \beta < \frac{p!}{(p-i)!};\ n \in N;\ p \in N;\ i \in N \cup \{0\};\ i < p;\ z \in \partial(U)\Big],$$

provided the inequality (2.1) is satisfied. Hence, by the maximum modulus theorem we have $f(z) \in T_p(n, i, \lambda, \beta)$.

Finally, we note that the assertion (2.1) of theorem 1 is sharp, the extremal function being given by

$$f(z) = z^p + \frac{(n+p-i)![p! - \beta(p-i)!]}{(n\lambda - 1)(n+p)!(p-i)!} z^{n+p} \quad (n \in N;\ p \in N;\ i \in N \cup \{0\}). \tag{2.2}$$

Corollary 1. *If $f(z) \in T_p\ (n, i, \lambda, \beta)$, then*

$$a_{n+p} \leq \frac{(n+p-i)![p! - \beta(p-i)!]}{(n\lambda - 1)(n+p)!(p-i)!} \ (n \in N;\ p \in N;\ i \in N \cup \{0\}). \tag{2.3}$$

Corollary 2. *Let a function $f(z) \in T_p(n)$ is in the class $T_p(n, 0, \lambda, \beta)$ if and only if*

$$\sum_{k=n+p}^{\infty} [\lambda(k-p) - 1]a_k \leq 1 - \beta \ (n \in N;\ p \in N; \lambda > \frac{1}{n}). \tag{2.4}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p + \frac{1-\beta}{n\lambda - 1} z^{n+p} \ (n \in N;\ p \in N). \tag{2.5}$$

Corollary 3. *Let a function $f(z) \in T_1(n)$ is in the class $T_1(n, 0, \lambda, \beta)$ if and only if*

$$\sum_{k=n+1}^{\infty} [\lambda(k-1) - 1]a_k \leq 1 - \beta \ (n \in N;\ \lambda > \frac{1}{n}). \tag{2.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z + \frac{1-\beta}{n\lambda - 1} z^{n+1} \ (n \in N). \tag{2.7}$$

Theorem 2. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by*

$$g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \ (b_k \geq 0;\ n \in N; p \in N) \tag{2.8}$$

be in the same class $T_p(n, i, \lambda, \beta)$. Then the function $h(z)$ defined by

$$h(z) = (1-\mu)f(z) + \mu g(z) = z^p + \sum_{k=n+p}^{\infty} c_k z^k \tag{2.9}$$

$$(c_k = (1-\mu)a_k + \mu b_k \geq 0;\ 0 \leq \mu \leq 1;\ n \in N;\ p \in N)$$

is also in the class $T_p(n, i, \lambda, \beta)$.

Proof. Suppose that each of functions $f(z)$ and $g(z)$ are in the class $T_p(n, i, \lambda, \beta)$. Then, making use of the inequality (2.1), we see that

$$\begin{aligned} &\sum_{k=n+p}^{\infty} \frac{k![\lambda(k-p)-1]}{(k-i)!} c_k \\ =&(1-\mu) \sum_{k=n+p}^{\infty} \frac{k![\lambda(k-p)-1]}{(k-i)!} a_k + \mu \sum_{k=n+p}^{\infty} \frac{k![\lambda(k-p)-1]}{(k-i)!} b_k \\ \leq&(1-\mu)\Big[\frac{p!}{(p-i)!} - \beta\Big] + \mu\Big[\frac{p!}{(p-i)!} - \beta\Big] = \frac{p!}{(p-i)!} - \beta \\ &\Big[\lambda > \frac{1}{n};\ 0 \leq \beta < \frac{p!}{(p-i)!};\ 0 \leq \mu \leq 1;\ n \in n;\ p \in N;\ i \in N \cup \{0\};\ i < p\Big], \end{aligned}$$

which completes the proof of Theorem 2.

Next we define the modified Hadamard product of the functions $f(z)$ and $g(z)$, which are defined by (1.1) and (2.8), respectively, by

$$f \times g(z) = z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k \quad (a_k \geq 0;\ b_k \geq 0;\ n \in N;\ p \in N) \tag{2.10}$$

Theorem 3. *If each of the functions* $f(z)$ *and* $g(z)$ *are in the class* $T_p(n, i, \lambda, \beta)$, *then* $f \times g(z) \in T_p(n, i, \lambda, \delta)$, *where*

$$\delta \leq \frac{p!}{(p-i)!} - \Big[\frac{p! - \beta(p-i)!}{(p-i)!}\Big]^2 \frac{(n+p-i)!}{(n\lambda - 1)(n+p)!} \quad (n \in N; p \in N; i \in N \cup \{0\}). \tag{2.11}$$

The result is sharp for the functions $f(z)$ *and* $g(z)$ *given by*

$$f(z) = g(z) = z^p + \frac{(n+p-i)![p! - \beta(p-i)!]}{(n\lambda - 1)(n+p)!(p-i)!} z^{n+p} \tag{2.12}$$

Proof. From the inequality (2.1), we have

$$\sum_{k=n+p}^{\infty} \frac{k!(p-i)![\lambda(k-p)-1]}{(k-i)![p! - \beta(p-i)!]} a_k \leq 1 \quad (n \in N;\ p \in N;\ i \in N \cup \{0\}) \tag{2.13}$$

and

$$\sum_{k=n+p}^{\infty} \frac{k!(p-i)![\lambda(k-p)-1]}{(k-i)![p! - \beta(p-i)!]} b_k \leq 1 \quad (n \in N;\ p \in N;\ i \in N \cup \{0\}. \tag{2.14}$$

we have to find the largest δ such that

$$\sum_{k=n+p}^{\infty} \frac{k!(p-i)![\lambda(k-p)-1]}{(k-i)![p!-\delta(p-i)!]} a_k b_k \leq 1 \quad (n \in N;\ p \in N; i \in N \cup \{0\}). \tag{2.15}$$

From (2.13) and (2.14) we find, by means of Cauchy-Schwarz inequality, that

$$\sum_{k=n+p}^{\infty} \frac{k!(p-i)![\lambda(k-p)-1]}{(k-i)![p!-\beta(p-i)!]} \sqrt{a_k b_k} \leq 1 \quad (n \in N;\ p \in N;\ i \in N \cup \{0\}). \tag{2.16}$$

Therefore, (2.15) holds true if

$$\sqrt{a_k b_k} \leq \frac{p!-\delta(p-i)!}{p!-\beta(p-i)!} \quad (k \geq n+p;\ n \in N;\ p \in N;\ i \in N \cup \{0\}), \tag{2.17}$$

that is, if

$$\frac{(k-i)![p!-\beta(p-i)!]}{k!(p-i)![\lambda(k-p)-1]} \leq \frac{p!-\delta(p-i)!}{p!-\beta(p-i)!} \quad (k \geq n+p;\ n \in N;\ i \in N \cup \{0\}), \tag{2.18}$$

which readily yields

$$\delta \leq \frac{p!}{(p-i)!} - \left[\frac{p!-\beta(p-i)!}{(p-i)!}\right]^2 \frac{(k-i)!}{k![\lambda(k-p)-1]} \tag{2.19}$$

$$(k \geq n+p;\ n \in N;\ p \in N;\ i \in N \cup \{0\}).$$

Finally, letting

$$\gamma(k) = \frac{p!}{(p-i)!} - \left[\frac{p!-\beta(p-i)!}{(p-i)!}\right]^2 \frac{(k-i)!}{k![\lambda(k-p)-1]} \tag{2.20}$$

$$(k \geq n+p;\ n \in N;\ p \in N;\ i \in N \cup \{0\}),$$

we see that the function $\gamma(k)$ is increasing in k. This shows that

$$\delta \leq \gamma(n+p) = \frac{p!}{(p-i)!} - \left[\frac{p!-\beta(p-i)!}{(p-i)!}\right]^2 \frac{(n+p-i)!}{(n\lambda-1)(n+p)!} \tag{2.21}$$

$$n(n \in N;\ p \in N;\ i \in N \cup \{0\}),$$

which completes the proof of Theorem 3.

Setting $i = 0$ in Theorem 3, we obtain

Corollary 4. *If each of the functions $f(z)$ and $g(z)$ are in the class $T_p(n, 0, \lambda, \beta)$, then $f \times g(z) \in T_p(n, 0, \lambda, \delta)$, where*

$$\delta \leq 1 - \frac{(1-\beta)^2}{n\lambda - 1} \qquad (n \in N;\ p \in N; \quad \lambda > \frac{1}{n}). \tag{2.22}$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by (2.5).

Setting $p = 1$ in Corollary 4, we obtain

Corollary 5. *If each of the functions $f(z)$ and $g(z)$ are in the class $T_1(n, 0, \lambda, \beta)$, then $f \times g(z) \in T_1(n, 0, \lambda, \delta)$, where*

$$\delta \leq 1 - \frac{(1-\beta)^2}{n\lambda - 1} \qquad (n \in N;\ \lambda > \frac{1}{n}). \tag{2.23}$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by (2.7).

3. Distortion Theorems

Theorem 4. *If $f(z) \in T_p(n, i, \lambda, \beta)$, then*

$$|f^{(j)}(z)| \leq \Big[\frac{p!}{(p-j)!} + \frac{(n+p-i)![p! - \beta(p-i)!]}{(p-i)!(n+p-j)!(n\lambda - 1)}|z|^n\Big]|z|^{p-j} \tag{3.1}$$

and

$$|f^{(j)}(z)| \geq \Big[\frac{p!}{(p-j)!} - \frac{(n+p-i)![p! - \beta(p-i)!]}{(p-i)!(n+p-j)!(n\lambda - 1)}|z|^n\Big]|z|^{p-j} \tag{3.2}$$

for all $z \in U$; $n \in N$; $p \in N$; $i \in N \cup \{0\}$; $j \in N \cup \{0\}$; $i < p$; $j < p$.

The result is sharp for the function $f(z)$ given by (2.2).

Proof. Suppose that $f(z) \in T_p(n, p, i, \alpha)$. We then find from (2.1), that

$$\sum_{k=n+p}^{\infty} k! a_k \leq \frac{(n+p-i)![p! - \beta(p-i)]}{(p-i)!(n\lambda - 1)} \quad (i \in N \cup \{0\};\ n \in N;\ p \in N;\ i < p). \tag{3.3}$$

and, we have from (1.1) and (1.3) that

$$\begin{aligned} f^{(j)}(z) &= \frac{p!}{(p-j)!} z^{p-j} + \sum_{k=n+p}^{\infty} \frac{k!}{(k-j)!} a_k z^{k-j} \\ &= \frac{p!}{(p-j)!} z^{p-j} + \sum_{k=n+p}^{\infty} k! \phi(k) a_k z^{k-j} \ (p \in N;\ n \in N;\ j \in N \cup \{0\};\ j<p), \end{aligned} \tag{3.4}$$

where, for convenience,

$$\phi(k) = \frac{1}{(k-j)!} \qquad (p \in N;\ n \in N;\ j \in N \cup \{0\};\ k \geq n+p;\ j < p). \tag{3.5}$$

Clearly, the function $\phi(k)$ is decreasing in k, and we have

$$0 < \phi(k) \leq \phi(n+p) = \frac{1}{(n+p-j)!}$$
$$(p \in N;\ n \in N;\ j \in N \cup \{0\};\ j < p). \tag{3.6}$$

Making use of (3.3), (3.4) and (3.6), that

$$\begin{aligned} |f^{(j)}(z)| &\leq \left[\frac{p!}{(p-j)!} + |z|^n \phi(n+p) \sum_{k=n+p}^{\infty} k! a_k\right] |z|^{p-j} \\ &\leq \left[\frac{p!}{(p-j)!} + \frac{(n+p-i)![p! - \beta(p-i)!]}{(p-i)!(n+p-j)!(n\lambda - 1)} |z|^n\right] |z|^{p-j} \\ &\quad (n \in N;\ p \in N;\ i \in N \cup \{0\};\ j \in N \cup \{0\};\ i < p;\ j < p;\ z \in U), \end{aligned}$$

which is precisely the (3.1), and that

$$\begin{aligned} |f^{(j)}(z)| &\geq \left[\frac{p!}{(p-j)!} - |z|^n \phi(n+p) \sum_{k=n+p}^{\infty} k! a_k\right] |z|^{p-j} \\ &\geq \left[\frac{p!}{(p-j)!} - \frac{(n+p-i)![p! - \beta(p-i)!]}{(p-i)!(n+p-j)!(n\lambda - 1)} |z|^n\right] |z|^{p-j} \\ &\quad (n \in N;\ p \in N;\ i \in N \cup \{0\};\ j \in N \cup \{0\};\ i < p;\ j < p;\ z \in u), \end{aligned}$$

which is the same as the assertion (3.2).

In order to complete the proof of Theorem 4, it is easily observed the equalities in (3.1) and (3.2) are satisfied by the function $f(z)$ given by (2.2).

One of the interesting properties of this work is that we can easily determine the lower and upper bounds for the functions $f(z)$ in this classes and the derivatives of $f(z)$ of arbitrary order whenever we choose the index i (or j) $= 0, 1, 2, \ldots$ appropriately. For this purpose on mind, we give the following corrollaries:

Setting $j = 0$ in Theorem 4, we obtain

Corollary 6. *If $f(z) \in T_p(n, i, \lambda, \beta)$, then*

$$|f(z)| \leq |z|^p \left[1 + \frac{(n+p-i)![p! - \beta(p-i)!]}{(n\lambda - 1)(p-i)!(n+p)!} |z|^n\right] \tag{3.7}$$

and

$$|f(z)| \geq |z|^p \left[1 - \frac{(n+p-i)![p! - \beta(p-i)!]}{(n\lambda - 1)(p-i)!(n+p)!} |z|^n\right] \tag{3.8}$$

for all $z \in U$; $n \in N$; $p \in N$; $i \in N \cup \{0\}$; $i < p$.
The result is sharp for the function $f(z)$ given by (2.2).
Setting $i = 0$ in Corollary 6, we obtain

Corollary 7. *If $f(z) \in T_p(n, 0, \lambda, \beta)$, then*

$$|z|^p \left[1 - \frac{1-\beta}{n\lambda - 1} |z|^n\right] \leq |f(z)| \leq |z|^p \left[1 + \frac{1-\beta}{n\lambda - 1} |z|^n\right] \tag{3.9}$$

for all $z \in U$; $n \in N$; *and* $p \in N$.

The result is sharp for the function $f(z)$ *given by* (2.5).

Setting $j = 1$ *in Theorem 4, we obtain*

Corollary 8. *If* $f(z) \in T_p(n, i, \lambda, \beta)$, *then*

$$|f'(z)| \leq |z|^{p-1}\left[p + \frac{(n+p-i)![p! - \beta(p-i)!]}{(n\lambda - 1)(p-i)!(n+p-1)!}|z|^n\right] \tag{3.10}$$

and

$$|f'(z)| \geq |z|^{p-1}\left[p - \frac{(n+p-i)![p! - \beta(p-i)!]}{(n\lambda - 1)(p-i)!(n+p-1)!}|z|^n\right] \tag{3.11}$$

for all $z \in N$; $n \in N$; $p \in N$; $i \in N \cup \{0\}$; $i < p$.

The result is sharp for the function $f(z)$ *given by* (2.2).

Setting $i = 1$ *in Corollary 8, we obtain*

Corollary 9. *If* $f(z) \in T_p(n, 1, \lambda, \beta)$, *then*

$$|z|^p\left[p - \frac{p-\beta}{n\lambda - 1}|z|^n\right] \leq |f'(z)| \leq |z|^p\left[p + \frac{p-\beta}{n\lambda - 1}|z|^n\right] \tag{3.12}$$

for all $z \in U$; $n \in N$; *and* $p \in N$.

The result is sharp for the function $f(z)$ *given by*

$$f(z) = z^p + \frac{p-\beta}{(n+p)(n\lambda - 1)}z^{n+p} \quad (n \in N;\ p \in N). \tag{3.13}$$

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