TAMKANG JOURNAL OF MATHEMATICS
Volume 27, Number 4, Winter 1996

A NOTE OF PACHPATTE ON HARDY LIKE INTEGRAL INEQUALITIES

DAY-YAN HWANG

Abstract. In [4], B. G. Pachpatte establish two new integral inequalities of the Hardy type with constant $M = \max\{\frac{a+2}{a+1}, \frac{2(p+q)}{a+1}\}$. The aim of the present note is to generalize his inequalities and the constant is better than M.

I. Introduction

In [4] B. G. Pachpatte established the following two new integral inequalities which in the special cases reduces to the inequalities similar to that of the variants of Hardy's inequality [2] given by Adams [1,p.402], Zygmund [5,p.20] and Izumi and Izumi [3,p.279].

Theorem A. *Let $\alpha \geq 0$, $p \geq 0$, $q \geq 1$ be real constants, f be a nonnegative and integrable function on $(0,a)$ for fixed $a > 0$. If $F(x) = \int_0^x f(t)dt$, then*

$$\int_0^a x^\alpha F^{p+q}(x)dx \leq M^q \int_0^a x^{\alpha+q} F^p(x)\{\frac{F(x)}{a} + f(x)\}^q dx \tag{1}$$

where $M = \max\{\frac{\alpha+2}{\alpha+1}, \frac{2(p+q)}{\alpha+1}\}$.

Theorem B. *Let α, p, q, f be as defined in Theorem A. If $G(x) = \int_{x/2}^x f(t)dt$, then*

$$\int_0^\alpha x^\alpha G^{p+q}(x)dx \leq M^q \int_0^\alpha x^{\alpha+q} G^p(x)\{\frac{G(x)}{a} + |f(x) - \frac{1}{2}f(\frac{x}{2})|\}^q dx \tag{2}$$

where M is as defined in Theorem A.

In this paper we shall generalize the inequalities (1) and (2) and the constant is less than or equal to M. Throughout the present note we shall assume that all the integrals exists on the respective domains of their definitions.

Received June 26, 1995.
1991 *Mathematics Subject Classification.* Primary 26D15, Secondary 26D20.
Key words and phrases. Integral inequalities, Hardy type, integrable function, Hölder's inequality.

2. Main Results

Theorem 1. *Let $\alpha \geq 0$, $p \geq 0$, $q \geq 1$ be real constants, f be a nonnegative and integrable function and r be a positive and absolutely continuous on $(0,a)$ for fixed $a > 0$, If $F(x) = \int_0^x \frac{r(t)f(t)}{t}dt$, then*

$$\int_0^a x^\alpha F^{p+q}(x)dx \leq N^q \int_0^\alpha x^{\alpha+q}F^p(x)\{\frac{F(x)}{a} + \frac{r(x)f(x)}{x}\}^q dx \tag{3}$$

where $N = \max\{\frac{\alpha+2}{\alpha+1}, \frac{p+q}{\alpha+1}\}$.

Proof. If f is null, then inequality (3) in Theorem 1 is trivially true. We assume that f is not null. Integrating by parts, we have the following identity:

$$\begin{aligned}&\int_0^a [x^{\alpha+1} - \frac{1}{a}x^{\alpha+2}]F^{p+q-1}(x)[\frac{r(x)f(x)}{x}]dx\\ =&\frac{-1}{p+q}\int_0^a [(\alpha+1)x^\alpha - \frac{1}{a}(\alpha+2)x^{\alpha+1}]F^{p+q}(x)dx.\end{aligned} \tag{4}$$

From (4) we observe that

$$\begin{aligned}&\int_0^a x^\alpha F^{p+q}(x)dx\\ =&\frac{a+2}{(a+1)a}\int_0^a x^{\alpha+1}F^{p+q}(x)dx - \frac{p+q}{\alpha+1}\int_0^a [x^{\alpha+1} - \frac{1}{a}x^{\alpha+2}]F^{p+q-1}(x)[\frac{r(x)f(x)}{x}]dx\\ \leq&\frac{\alpha+2}{(\alpha+1)a}\int_0^a x^{\alpha+1}F^{p+q}(x)dx + \frac{p+q}{\alpha+1}\int_0^a (\frac{x}{a})x^{\alpha+1}F^{p+q-1}(x)[\frac{r(x)f(x)}{x}]dx\\ \leq&\frac{\alpha+2}{(\alpha+1)}\int_0^a x^{\alpha+1}F^{p+q-1}(x)[\frac{F(x)}{a}]dx + \frac{p+q}{\alpha+1}\int_0^a x^{\alpha+1}F^{p+q-1}(x)[\frac{r(x)f(x)}{x}]dx\\ \leq&N\int_0^a x^{\alpha+1}F^{p+q-1}(x)[\frac{F(x)}{a} + \frac{r(x)f(x)}{x}]dx\\ =&N\int_0^a \{x^{\alpha+1-\alpha(q-1/q)}F^{p/q}(x)[\frac{F(x)}{a} + \frac{r(x)f(x)}{x}]\}\cdot\{x^{\alpha(q-1/q)}F^{p+q-1-p/q}(x)\}dx.\end{aligned} \tag{5}$$

By using the Holder's inequality with indices q, $\frac{q}{q-1}$ on the right side of (5) we have

$$\begin{aligned}&\int_0^a x^\alpha F^{p+q}(x)dx\\ \leq&N\{\int_0^a x^{\alpha+q}F^p(x)[\frac{F(x)}{a} + \frac{r(x)f(x)}{x}]^q dx\}^{1/q}\cdot\{\int_0^a x^\alpha F^{p+q}(x)dx\}^{q-1/q}.\end{aligned}$$

Now dividing the above inequality by the last factor on the right and raising the result to the qth power, we obtain the required inequality in (3). This completes the proof of Theorem 1.

Remark 1. Theorem 1 reduces to Theorem A when $r(x) = x$ for every x in $(0, a)$ and $N < M$ when $p + \dot{q} > \alpha + 2$.

Theorem 2. *let* α, p, q, f, r, *be as defined in Theorem 1. If* $G(x) = \int_{\frac{x}{2}}^{x} \frac{r(t)f(t)}{t} dt$, *then*

$$\int_0^a x^\alpha G^{p+q}(x)dx \le N^q \int_0^a x^{\alpha+q} G^p(x) \{ \frac{G(x)}{a} + \left| \frac{r(x)f(x)}{x} - \frac{r(x/2)f(x/2)}{x} \right| \}^q dx \quad (6)$$

where N is as defined in Theorem 1.

Proof. We assume as in the proof of Theorem 1, the function f is not null. Integrating by parts, we get

$$\begin{aligned} &\int_0^a [x^{\alpha+1} - \frac{1}{a} x^{\alpha+2}] G^{p+q-1}(x) [\frac{r(x)f(x)}{x} - \frac{r(x/2)f(x/2)}{x}] dx \\ =&\frac{-1}{p+q} \int_0^a (\alpha+1) x^\alpha - \frac{1}{a} (\alpha+2) x^{\alpha+1}] G^{p+q}(x) dx. \end{aligned} \quad (7)$$

Define $h(x) = [\frac{r(x)f(x)}{x} - \frac{r(x/2)f(x/2)}{x}], h^+(x) = \max\{h(x), 0\}$ and $h^-(x) = -\min\{h(x), 0\}$ for every x in $(0, a)$.

From (7) and the definition of h we observe that

$$\begin{aligned} &\int_0^\alpha x^\alpha G^{p+q}(x)dx \\ =&\frac{\alpha+2}{(\alpha+1)a} \int_0^a x^{\alpha+1} G^{p+q}(x)dx - \frac{p+q}{\alpha+1} \int_0^a [x^{\alpha+1} - (1/a)^{x^{\alpha+2}}] G^{p+q-1}(x) h(x) dx \\ =&\frac{\alpha+2}{(\alpha+1)a} \int_0^a x^{\alpha+1} G^{p+q}(x)dx - \frac{p+q}{\alpha+1} \int_0^a (1 - x/a) x^{\alpha+1} G^{p+q-1}(x) [h^+(x) - h^-(x)] dx \\ \le&\frac{\alpha+2}{(\alpha+1)a} \int_0^a x^{\alpha+1} G^{p+q}(x)dx + \frac{p+q}{\alpha+1} \int_0^a x^{\alpha+1} G^{p+q-1}(x) h^-(x) dx \\ \le&N \int_0^a x^{\alpha+1} G^{p+q-1}(x) [\frac{G(x)}{a} + h^-(x)] dx \\ \le&N \int_0^a x^{\alpha+1} G^{p+q-1}(x) [\frac{G(x)}{a} + |h(x)|] dx \\ =&N \int_0^a \{x^{\alpha+1-\alpha(q-1/q)} G^{p/q}(x) [\frac{G(x)}{a} + |h(x)|]\} \cdot \{x^{\alpha(q-1/q)} G^{p+q-1-p/q}(x)\} dx. \end{aligned}$$

Now by following exactly the same arguments as in the proof of Theorem 1 given below the identity (5) with suitable modifications, we get the desired inequality in (6). The proof of Theorem 2 is complete.

Remark 2. Theorem 2 reduces to Theorem b when $r(x) = x$ for every x in $(0, a)$ and $N < M$ when $p + q > \alpha + 2$.

References

[1] R. A. Adams, "Some integral inequalities with applications to the imbedding of Sobolev spaces defined over irregular domains," *Trans. Amer. Math., Soc.* 178 (1973), 401-429.

[2] G. H. Hardy, "Note on a theorem of Hilbert," *Math Z.* 6 (1920), 314-317.

[3] M. Izumi and S. Izumi, "On some inequalities for Fourier Series," *J. Analyes Math.* 21 (1968), 277-291.

[4] B. G. Pachpatte, "A note on Hardy like integral inequalities," *Tamkang. J. Math.* 25, 3, (1994), 221-224.

[5] A. Zygmund, "Trigonometrical series, Vol. I. 2nd edition," *Cambridge Univ., Press, New York,* 1959.

Kuang Wu Institute of Technology and Commerce Peito, Taipei, Taiwan, R.O.C.