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A REFINEMENT OF HADAMARD'S INEQUALITY FOR ISOTONIC LINEAR FUNCTIONALS

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Abstract. A general refinement of Hadamard's inequality for isotonic linear functionals and some applications to norm and discrete inequalities are given.

I. Introduction

Let $f : I \subseteq R \to R$ be a convex function. The following double inequality

$$f(\frac{x+y}{2}) \le \frac{1}{y-x}\int_x^y f(t)dt \le \frac{f(x)+f(y)}{2} \tag{1.1}$$

where $x, y \in I$ is known in literature as Hadamard's inequality (see [7][9][5] or [6]). For some recent results in connection with this famous integral inequality we refer to [2-5] and [9-11] where further applications are given.

In this note we will give a refinement of Hadamard's inequality for isotonic linear functionals (compare with [8]), and some natural applications. As in [1], let E be a nonempty set and let L be a linear class of real-value functions $g : E \to R$ having the properties:

$$L_1 : f, g \in L \text{ imply } (af + bg) \in L \text{ for all } a, b \in R;$$
$$L_2 : 1 \in L, \text{ that is, if } f(t) = 1\ (t \in E) \text{ then } f \in L.$$

We also consider isotonic linear functionals $A : L \to R$. That is

$$A1 : A(af + bg) = aA(f) + bA(g) \text{ for all } f, g \in L;$$
$$A2 : f \in L,\ f(t) \ge 0 \text{ on } E \text{ implies } A(f) \ge 0$$

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We note that some examples of such isotonic linear functionals A are given by $A(g) = \int_E g du$ or $A(g) = \sum_{K\in E} p_k g_k$ where u is a positive measure on E in the first case and E is a subset of natural number N in the second case with $p_k \geq 0$ for $k \in E$.

Jessen's Inequality: Let L satisfy properties $L1$, $L2$ on a nonempty set E and suppose Φ is a convex function on an interval $I \subseteq R$. If A is any isotonic linear functional with $A(1) = 1$, then for all $g \in L$ so that $\Phi(g) \in L$, we have $A(g) \in I$ and $\Phi(A(g)) \leq A(\Phi(g))$.

2. Main Result

We will start with the following simple lemma.

Lemma 2.1. *Let X be a real linear space and C be a convex subset of X. If $f : C \to R$ is convex on C, then for all x, y in C the mapping $g_{xy} : [0,1] \to R$ given by*

$$g_{xy}(t) = \alpha f\Big[\frac{t}{2\alpha}x + \Big(1 - \frac{t}{2\alpha}\Big)y\Big] + \beta f\Big[\Big(1 - \frac{t}{2\beta}\Big)x + \frac{t}{2\beta}y\Big]$$

where $\beta \geq \alpha > 0$, $\alpha + \beta = 1$ is also convex on $[0, 2\alpha]$.

In addition, we have the inequality:

$$f(\alpha y + \beta x) \leq g_{xy}(t) \leq \alpha f(y) + \beta f(x) \tag{2.1}$$

for all x, y in C and $0 \leq t \leq 2\alpha$.

Proof. Suppose $x, y \in C$ and let $t_1, t_2 \in [0, 2\alpha]$ and $a, b \geq 0$ with $a + b = 1$. Then

$$\begin{aligned}
&g_{x,y}(at_1 + bt_2)\\
=&\alpha f\Big[\frac{at_1 + bt_2}{2\alpha}x + \Big(1 - \frac{at_1 + bt_2}{2\alpha}\Big)y\Big] + \beta f\Big[\Big(1 - \frac{at_1 + bt_2}{2\beta}\Big)x + \frac{at_1 + bt_2}{2\beta}y\Big]\\
=&\alpha f\Big[a\Big(\frac{t_1}{2a}x + \Big(1 - \frac{t_1}{2a}\Big)y\Big) + b\Big(\frac{t_2}{2a}x + \Big(1 - \frac{t_2}{2a}\Big)y\Big)\Big]\\
&+ \beta f\Big[a\Big(\Big(1 - \frac{t_1}{2\beta}\Big)x + \frac{t_1}{2\beta}y\Big) + b\Big(\Big(1 - \frac{t_2}{2\beta}\Big)x + \frac{t_2}{2\beta}y\Big)\Big]\\
\leq&a\alpha f\Big[\frac{t_1}{2\alpha}x + \Big(1 - \frac{t_1}{2\alpha}\Big)y\Big] + b\alpha f\Big[\frac{t_2}{2\alpha}x + \Big(1 - \frac{t_2}{2\alpha}\Big)y\Big]\\
&+ a\beta f\Big[(1 - \frac{t_1}{2\beta}\Big)x + \frac{t_1}{2\beta}y\Big] + b\beta f\Big[\Big(1 - \frac{t_2}{2\beta}\Big)x + \frac{t_2}{2\beta}y\Big]\\
=&ag_{x,y}(t_1) + bg_{x,y}(t_2)
\end{aligned}$$

which shows that $g_{x,y}$ is convex in $[0, 2\alpha]$.

By the convexity of f we have

$$g_{x,y}(t) \geq f\Big[\alpha\frac{t}{2\alpha}x + \alpha\Big(1-\frac{t}{2\alpha}\Big)y + \beta\Big((1-\frac{t}{2\beta}\Big)x + \beta\frac{t}{2\beta}y\Big]$$
$$= f(\alpha y + \beta x)$$
$$g_{x,y}(t) \leq \alpha\Big[\frac{t}{2\alpha}f(x) + \Big(1-\frac{t}{2\alpha}\Big)f(y)\Big] + \beta\Big[\Big(1-\frac{t}{2\beta}\Big)x + \frac{t}{2\beta}f(y)\Big]$$
$$= \alpha f(y) + \beta f(x)$$

for all t in $[0, 2\alpha]$. This completes the proof.

Remark. By the inequality (2.1) we have:
$\sup_{t\in[0,1]} g_{x,y}(t) = \alpha f(y) + \beta f(x)$ and $\inf_{t\in[0,1]} g_{x,y}(t) = f(\alpha y + \beta x)$
for all $x, y \in C$.

Now we given our main result.

Theorem 2.1. *Let $f : C \subseteq X \to R$ be a convex function on a convex set C. Let L and A satisfy conditions $L1$, $L2$ and $A1$, $A2$ and let $h : E \to R$, $0 \leq h(t) \leq 2\alpha (t \in E), h \in L$ is so that $f(hx + (1-h)y)$, $f((1-h)x + hy)$ belong to L for x, y fixed in C. If $A(1) = 1$, then we have the inequality:*

$$f(\alpha y + \beta x) \leq \alpha f\Big[\frac{A(h)}{2\alpha}x + \Big(1-\frac{A(h)}{2\alpha}\Big)y\Big] + \beta f\Big[\Big(1-\frac{A(h)}{2\beta}\Big)x + \frac{A(h)}{2\beta}y\Big]$$
$$\leq \alpha A f\Big[\frac{h}{2\alpha}x + \Big(1-\frac{h}{2\alpha}\Big)y\Big) + \beta A f\Big[\Big(1-\frac{h}{2\beta}\Big)x + \frac{h}{2\beta}y\Big]$$
$$\leq \alpha f(y) + \beta f(x) \tag{2.2}$$

where $\beta \geq \alpha > 0$ with $\alpha + \beta = 1$.

Proof. Consider the mapping $g_{x,y} : [0,1] \to R$ given above. Then by lemma 2.1 we know that $g_{x,y}$ is convex $[0,1]$. Applying Jessen's inequality for the mapping $g_{x,y}$ we get

$$g_{x,y}(A(h)) \leq A(g_{x,y}(h))$$

where $g_{x,y}(A(h)) = \alpha f\Big[\frac{A(h)}{2\alpha}x + \Big(1-\frac{A(h)}{2\alpha}\Big)y\Big] + \beta f\Big[\Big(1-\frac{A(h)}{2\beta}\Big)x + \frac{A(h)}{2\beta}y\Big]$ and
$A(g_{x,y}(h)) = \alpha A f\Big[\frac{h}{2\alpha}x + \Big(1-\frac{h}{2\alpha}\Big)y\Big] + \beta A f\Big[\Big(1-\frac{h}{2\beta}\Big)x + \frac{h}{2\beta}y\Big]$. This proves the second inequality in (2.2).

To prove the first inequality in (2.2) we observe that, by (2.1), we can write $f(\alpha y + \beta x) \leq g_{x,y}(A(h))$.

Finally , we have

$$\alpha f\Big[\frac{h}{2\alpha}x + \Big(1-\frac{h}{2\alpha}\Big)y\Big] + \beta f\Big[\Big(1-\frac{h}{2\beta}\Big)x + \frac{h}{2\beta}y]$$
$$\leq \frac{h}{2}f(x) + \alpha f(y) - \frac{h}{2}f(y) + \beta f(x) - \frac{h}{2}f(x) + \frac{h}{2}f(y)$$
$$= \alpha f(y) + \beta f(x)$$

. Hence, the last part of (2.2) follows from

$$A\Big[\frac{\alpha f\big[\frac{h}{2\alpha}x+(1-\frac{h}{2\alpha})y\big]+\beta f\big[(1-\frac{h}{2\beta})x+\frac{h}{2\beta}y\big]}{\alpha f(y)+\beta f(x)}\Big]\le A(1)=1$$

Remark. If we choose: $A=\int_0^{2\alpha}$, $E=[0,1], h(t)=t$, $C=[x,y]\subset R$, then a simple calculation shows that

$$\int_0^{2\alpha} f\Big[\frac{t}{2\alpha}x+\Big(1-\frac{t}{2\alpha}\Big)y\Big]dt=\frac{2\alpha}{y-x}\int_x^y f(t)dt$$

$$\int_0^{2\alpha} f\Big[\Big(1-\frac{t}{2\beta}\Big)x+\frac{t}{2\beta}y\Big]=\frac{2\beta}{y-x}\int_x^{(1-\frac{\alpha}{\beta})x+\frac{\alpha}{\beta}y} f(t)dt$$

it follows from (2.2) that

$$f(\alpha x+\beta y)\le\frac{2\alpha^2}{y-x}\int_x^y f(t)dt+\frac{2\beta^2}{y-x}\int_x^{(1-\frac{\alpha}{\beta})x\frac{\alpha}{\beta}y} f(t)dt\le\alpha f(x)+\beta f(y) \tag{2.3}$$

By letting $\alpha=\beta=\frac{1}{2}$ in (2.3), we recapture the Hadamard's inequality

$$f\Big(\frac{x+y}{2}\Big)\le\frac{1}{y-x}\int_x^y f(t)dt\le\frac{f(x)+f(y)}{2}.$$

Lemma 2.2 *Let X be a real linear space and C be a convex subset of X. If $f:C\to R$ is convex, then*

$$g_{x,y,z}(t)=\frac{1}{3}[f((1-t)x+ty)+f((1-t)y+tz)+f((1-t)z+tx)]$$

is also convex on $[0,1]$. In addition, we have the inequality:

$$f\Big(\frac{x+y+z}{3}\Big)\le g_{x,y,z}(t)\le\frac{f(x)+f(y)+f(z)}{3} \tag{2.4}$$

for all x,y,z in C and $t\in[0,1]$.

Proof. Suppose $x,y,z\in C$ and let $t_1,t_2\in[0,1]$ and $a,b\ge 0$ with $a+b=1$. Then

$$\begin{aligned}
&g_{x,y,z}(at_1+bt_2)\\
=&\frac{1}{3}f((1-at_1bt_2)x+(at_1+bt_2)y)+f((1-at_1-bt_2)y+(at_1+bt_2)z)\\
&+f((1-at_1-bt_2)z+(at_1+bt_2)x))]\\
=&\frac{1}{3}f\{a[1-t_1)x+t_1y]+b[(1-t_2)x+t_2y]\}+\frac{1}{3}f\{a[(1-t_1)y+t_1z]+b[(1-t_2)y+t_2z]\}\\
&+\frac{1}{3}f\{a[(1-t_1)z+t_1x]+b[(1-t_2)z+t_2x]\}\\
\le&\frac{a}{3}f[(1-t_1)x+t_1y]+\frac{b}{3}f[1-t_2)x+t_2y]+\frac{a}{3}f[(1-t_1)y+t_1z]+\frac{b}{3}f[(1-t_2)y+t_2z]\\
&+\frac{a}{3}f[(1-t_1)z+t_1x]+\frac{b}{3}f[(1-t_2)z+t_2x]\\
=&ag_{x,y,z}(t_1)+bg_{x,y,z}(t_2)
\end{aligned}$$

so that $g_{x,y,z}(t)$ is convex on $[0,1]$. By the convexity of f, we have

$$g_{x,y,z}(t) \geq f\Big[\frac{(1-t)x+ty}{3}+\frac{(1-t)y+tz}{3}+\frac{(1-t)z+tx}{3}\Big] = f\Big(\frac{x+y+z}{3}\Big)$$

and $g_{x,y,z}(t) \leq \frac{(1-t)}{3}f(x)+\frac{t}{3}f(y)+\frac{(1-t)}{3}f(y)+\frac{t}{3}f(z)+\frac{(1-t)}{3}f(z)+\frac{t}{3}f(x) = \frac{f(x)+f(y)+f(z)}{3}$ for all t in $[0,1]$.

Remark. It follows from the inequality (2.4) that $\sup_{t\in[0,1]} g_{x,y,z}(t) = \frac{f(x)+f(y)+f(z)}{3}$ and $\inf_{t\in[0,1,]} g_{x,y,z}(t) = f\Big(\frac{x+y+z}{3}\Big)$ for all x, y, z in C.

Theorem 2.2. *Let $f : C \subseteq X \to R$ be a convex function on a convex set C, L and A satisfy conditions L1, L2 and A1, A2 and $h : E \to R$, $0 \leq h(t) \leq 1$ $(t \in E)$, $h \in L$ is so that $f((1-h)x+hy)$, $f((1-h)y+hz)$, $f((1-h)z+hx\}$ belong to L for x, y, z fixed in C. If $A(1) = 1$, then we have the inequalities:*

$$\begin{aligned} f\Big(\frac{x+y+z}{3}\Big) &\leq \frac{f[(1-A(h))x+A(h)y]+f[(1-A(h))y+A(h)z]+f[(1-A(h))z+A(h)x]}{3} \\ &\leq \frac{Af[(1-h)x+hy]+Af[(1-h)y+hz]+Af[(1-h)z+hx]}{3} \\ &\leq \frac{f(x)+f(y)+f(z)}{3} \end{aligned} \tag{2.5}$$

Proof. Consider the mapping $g_{x,y,z}[0,1] \to R$ given as in lemma 2.2 above. Applying Jessen's inequality for the mapping $g_{x,y,z}$ we get

$$g_{x,y,z}(A(h)) \leq A(g_{x,y,z}(h))$$

where

$$g_{x,y,z}(A(h)) = \frac{f[(1-A(h))x+A(h)y]+f[(1-A(h))Y+A(h)z]+f[(1-A(h))z+A(h)x]}{3}$$

and

$$A(g_{x,y,z}(h)) = \frac{Af[(1-h)x+hy]+Af(1-h)y+hz]+Af[(1-h)z+hx]}{3}$$

so the second inequality in (2.5) is proved.

To prove the first inequality in (2.5) we observe that by (2.4), we have $f\Big(\frac{x+y+z}{3}\Big) \leq g_{x,y,z}(A(h))$ which is exactly the desired statement.

Finally, we have

$$\begin{aligned} g_{x,y,z}(h) &= \frac{f[(1-h)x+hy]+f[(1-h)y+hz]+f[(1-h)z+hx]}{3} \\ &\le \frac{(1-h)}{3}f(x)+\frac{h}{3}f(y)+\frac{(1-h)}{3}f(y)+\frac{h}{3}f(z)+\frac{(1-h)}{3}f(z)+\frac{h}{3}f(x) \\ &= \frac{f(x)+f(y)+f(z)}{3}. \end{aligned}$$

Hence the last inequality of (2.5) follows from

$$A\Big[g_{x,y,z}(h)\frac{1}{\frac{f(x)+f(y)+f(z)}{3}}\Big] \le A(1) = 1.$$

Remark. if we choose $A=\int_0^1$, $E=[0,1]$, $h(t)=t$, $C=[x,y]\cup[y,z]\subset R$, then, a simple calculation shows that

$$\begin{aligned} \int_0^1 f[(1-t)x+ty]dt &= \frac{1}{y-x}\int_x^y f(t)dt \\ \int_0^1 f[(1-t)y+tz]dt &= \frac{1}{z-y}\int_y^z f(t)dt \\ \int_0^1 f[(1-t)z+tx]dt &= \frac{1}{x-z}\int_z^x f(t)dt \end{aligned}$$

and the inequality (2.5) reduces to

$$\begin{aligned} f\Big(\frac{x+y+z}{3}\Big) &\le \frac{1}{3(y-x)}\int_x^y f(t)dt+\frac{1}{3(z-y)}\int_y^z f(t)dt+\frac{1}{3(x-z)}\int_z^x f(t)dt \\ &\le \frac{f(x)+f(y)+f(z)}{3}. \end{aligned} \tag{2.6}$$

If we put $z=\frac{x+y}{2}$ in (2.6), we recapture the Hadamard's inequality.

3. Applications

1. Let $h:[0,1]\to[0,2\alpha]$ be a Riemann integrable function on $[0,1]$ and $p\ge 1$. Then for all x,y in a norm space $(X;\|\ \|)$ with $\beta\ge\alpha>0$, $\alpha+\beta=1$, we have

$$\begin{aligned} \|\alpha y+\beta x\|^p \le& \alpha\Big\|\Big(\frac{1}{2\alpha}\int_0^1 h(t)dt\Big)x+\Big(1-\frac{1}{2\alpha}\int_0^1 h(t)dt\Big)y\Big\|^p \\ &+\beta\Big\|\Big(1-\frac{1}{2\beta}\int_0^1 h(t)dt\Big)x+\Big(\frac{1}{2\beta}\int_0^1 h(t)dt\Big)dt\Big\|^p \\ \le& \alpha\int_0^1\Big\|\frac{h(t)}{2\alpha}x+\Big(1-\frac{h(t)}{2\alpha}\Big)y\Big\|^p dt+\beta\int_0^1\Big\|\Big(1-\frac{h(t)}{2\beta}\Big)x+\frac{h(t)}{2\beta}y\Big\|^p dt \\ \le& \alpha\|y\|^p+\beta\|x\|^p. \end{aligned} \tag{3.1}$$

If we choose $h(t) = t$, we obtain

$$\begin{aligned}\|\alpha y+\beta x\|^p &\le \alpha\int_0^1\left\|\frac{t}{2\alpha}x+\left(1-\frac{t}{2\alpha}\right)y\right\|^p dt+\beta\int_0^1\left\|\left(1-\frac{t}{2\beta}\right)x+\frac{t}{2\beta}y\right\|^p dt\\ &\le \alpha\|y\|^p+\beta\|x\|^p \qquad (3.2)\end{aligned}$$

for all x, y in X.

The inequality (3.1) follows by theorem 2.1 for the functional $A = \int_0^1$.

2. Let $f : C \subseteq X \to R$ be a convex function on a convex subset C of a linear space X, and $t_i \in [0, 2\alpha], i = 1, 2, \ldots, n$. If $\beta \ge \alpha > 0, \alpha + \beta = 1$, then we have the inequality:

$$\begin{aligned}f(\alpha y+\beta x) &\le \alpha f\left[\frac{1}{n}\sum_{i=1}^n\frac{t_i}{2\alpha}x+\frac{1}{n}\sum_{i=1}^n\left(1-\frac{t_i}{2\alpha}\right)y\right]+\beta f\left[\frac{1}{n}\sum_{i=1}^n\left(1-\frac{t_i}{2\beta}\right)x+\frac{1}{n}\sum_{i=1}^n\frac{t_i}{2\beta}y\right]\\ &\le \frac{\alpha}{n}\sum_{i=1}^n f\left[\frac{t_i}{2\alpha}x+\left(1-\frac{t_i}{2\alpha}\right)y\right]+\frac{\beta}{n}\sum_{i=1}^n f\left[\left(1-\frac{t_i}{2\beta}\right)x+\frac{t_i}{2\beta}y\right]\\ &\le \alpha f(y)+\beta f(x). \qquad (3.3)\end{aligned}$$

If we put $t_i = \sin^2\alpha_i$, $\alpha_i \in R$, $i = 1, \ldots, n$, then we have

$$\begin{aligned}&f(\alpha y+\beta x)\\ &\le\alpha f\left[\frac{1}{n}\sum_{i=1}^n\frac{\sin^2\alpha_i}{2\alpha}x+\frac{1}{n}\sum_{i=1}^n\left(1-\frac{\sin^2\alpha_i}{2\alpha}\right)y\right]+\beta f\left[\frac{1}{n}\sum_{i=1}^n\left(1-\frac{\sin^2\alpha_i}{2\beta}\right)x+\frac{1}{n}\sum_{i=1}^n\frac{\sin^2\alpha_i}{2\beta}y\right]\\ &\le\frac{\alpha}{n}\sum_{i=1}^n f\left[\frac{\sin^2\alpha_i}{2\alpha}x+\left(1-\frac{\sin^2\alpha_i}{2\alpha}\right)y\right]+\frac{\beta}{n}\sum_{i=1}^n f\left[\left(1-\frac{\sin^2\alpha_i}{2\beta}\right)x+\frac{\sin^2\alpha_i}{2\beta}\right)y\right]\\ &\le\alpha f(y)+\beta f(x).\end{aligned}$$

The inequality (3.3) follows by (2.2) for

$$A=\frac{1}{n}\sum_{n=1}^n, h(t)=\begin{cases} t_1, & 0\le t\le\frac{1}{n}\\ t_2, & \frac{1}{n}<t\le\frac{2}{n}\\ \vdots, & \vdots\\ t_n, & \frac{n-1}{n}<t\le 1\end{cases}$$

By the use of the inequality (3.3), we have the following arithmetic mean-geometric mean inequality

$$\alpha y+\beta x\ge y^\alpha x^\beta$$

where $x, y \ge 0$. Indeed, chosing $f(x) = -\ln x, x > 0$, we obtain

$$\begin{aligned}\alpha y+\beta x &\ge \left[\frac{1}{n}\sum_{i=1}^n\frac{t_i}{2\alpha}x+\frac{1}{n}\sum_{i=1}^n\left(1-\frac{t_i}{2\alpha}\right)y\right]^\alpha\left[\frac{1}{n}\sum_{i=1}^n\left(1-\frac{t_i}{2\beta}\right)x+\frac{1}{n}\sum_{i=1}^n\frac{t_i}{2\beta}y\right]^\beta\\ &\ge\prod_{i=1}^n\left(\left[\frac{t_i}{2\alpha}x+\left(1-\frac{t_i}{2\alpha}\right)y\right]^\alpha\left[\left(1-\frac{t_i}{2\beta}\right)x+\frac{t_i}{2\beta}y\right]^\beta\right)^{\frac{1}{n}}\ge y^\alpha x^\beta\end{aligned}$$

where inequality holds if and only if $x = y$.

3. Let $h : [0,1] \to [0,1]$ be a Reimann integrable function on $[0,1]$ and $p \geq 1$. Then for all x, y, z vectors in normed space $(X; \|\ \|)$ we have the inequality :

$$\begin{aligned}\Big\|\frac{x+y+z}{3}\Big\|^p \leq & \frac{1}{3}\Big\|\Big(1-\int_0^1 h(t)dt\Big)x+\Big(\int_0^1 h(t)dt\Big)y\Big\|^p \\ & +\frac{1}{3}\Big\|\Big(1-\int_0^1 h(t)dt\Big)y+\Big(\int_0^1 h(t)dt\Big)z\Big\|^p+\frac{1}{3}\Big\|\Big(1-\int_0^1 h(t)dt\Big)x\Big\|^p \\ \leq & \frac{1}{3}\int_0^1 \|(1-h(t)x+h(t)y\|^p dt+\frac{1}{3}\int_0^1 \|(1-h(t))y+h(t)z\|^p dt \\ & +\frac{1}{3}\int_0^1 \|(1-h(t)z+h(t)x\|^p dt \\ \leq & \frac{\|x\|^p+\|y\|^p+\|z\|^p}{3} \end{aligned} \tag{3.4}$$

If we choose $h(t) = t$, we obtain

$$\begin{aligned}\Big\|\frac{x+y+z}{3}\Big\|^p \leq & \frac{1}{3}\int_0^1 \|(1-t)x+ty\|^p dt+\frac{1}{3}\int_0^1 \|(1-t)y+tz\|^p dt+\frac{1}{3}\int_0^1 \|(1-t)z+tx\|^p dt \\ \leq & \frac{\|x\|^p+\|y\|^p+\|z\|^p}{3}\end{aligned} \tag{3.5}$$

for all x, y, z in X.
The inequality (3.4) follows by theorem 2.2 for the functional $A = \int_0^1$.

4. Let $f : C \subseteq X \to R$ be a convex function on a convex subset C of a linear space X, $t_i \in [0,1]$, $i = 1, \ldots, n$. Then we have the inequality:

$$\begin{aligned} & f\Big(\frac{x+y+z}{3}\Big) \\ \leq & \frac{1}{3}f\Big[\sum_{i=1}^n(1-t_i)x+\Big(\frac{1}{n}\sum_{i=1}^n t_i\Big)y\Big]+\frac{1}{3}f\Big[\frac{1}{n}\sum_{i=1}^n(1-t_i)y+\Big(\frac{1}{n}\sum_{i=1}^n t_i\Big)z\Big] \\ & +\frac{1}{3}f\Big[\frac{1}{n}\sum_{i=1}^n(1-t_i)z+\Big(\frac{1}{n}\sum_{i=1}^n t_i\Big)x\Big] \\ \leq & \frac{1}{3n}\sum_{i=1}^n f[(1-t_i)x+t_iy]+\frac{1}{3n}\sum_{i=1}^n f[(1-t_i)y+t_iz]+\frac{1}{3n}\sum_{i=1}^n f[(1-t_i)z+t_ix] \\ \leq & \frac{f(x)+f(y)+f(z)}{3}\end{aligned} \tag{3.6}$$

If we put $t_i = \sin^2 \alpha_i$, $\alpha_i \in R$, $i = 1, \ldots, n$, then

$$f\Big(\frac{x+y+z}{3}\Big)$$

$$\leq \frac{1}{3} f\Big[\Big(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\Big)x + \Big(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\Big)y\Big] + \frac{1}{3} f\Big[\Big(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\Big)y + \Big(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\Big)z\Big]$$
$$+ \frac{1}{3} f\Big[\Big(\frac{1}{n}\sum_{i=1}^{n}\cos^2\alpha_i\Big)z + \Big(\frac{1}{n}\sum_{i=1}^{n}\sin^2\alpha_i\Big)x\Big]$$
$$\leq \frac{f(x)+f(y)+f(z)}{3}.$$

The inequality (3.6) follows by (2.5) for $A = \frac{1}{n}\sum_{i=1}^{n}$, $h(t) = t_i \in [0,1]$.
By use of the inequality (3.6), we obtain arithmetic mean-geometric mean inequality

$$\frac{x+y+z}{3} \geq (xyz)^{\frac{1}{3}} \quad \text{where} \quad x,y,z \geq 0.$$

Indeed, choosing $f(x) = -\ln x, x > 0$, we have

$$\frac{x+y+z}{3} \geq \Big[\frac{1}{n}\sum_{i=1}^{n}(1-t_i)x + \Big(\frac{1}{n}\sum_{i=1}^{n}t_i\Big)y\Big]^{\frac{1}{3}}\Big[\frac{1}{n}\sum_{i=1}^{n}(1-t_i)y + \Big(\frac{1}{n}\sum_{i=1}^{n}t_i\Big)z\Big]^{\frac{1}{3}}$$
$$\Big[\frac{1}{n}\sum_{i=1}^{n}(1-t_i)z + \Big(\frac{1}{n}\sum_{i=1}^{n}t_i\Big)x\Big]^{\frac{1}{3}}$$
$$\geq \prod_{i=1}^{n}\Big\{[(1-t_i)x + t_i y]^{\frac{1}{3}}[(1-t_i)y + t_i z]^{\frac{1}{3}}][(1-t_i)z + t_i x]^{\frac{1}{3}}\Big\}^{\frac{1}{n}}$$
$$\geq (xyz)^{\frac{1}{3}}$$

where equality holds if, and only if $x = y = z$.

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