TAMKANG JOURNAL OF MATHEMATICS
Volume 27, Number 4, Winter 1996

HOLONOMIC FILTERED MODULES IN THE CATEGORY OF MICRO-STRUCTURE SHEAVES

ABD EL AZIZ A. RADWAN AND SALAH EL DIN S. HUSSEIN

0. Introduction

Since the late sixties, Various Auslander regularity conditions have been widely investigated in both commutative and non-commutative cases, [6]. J. E. Bjork studied the Auslander regularity on graded rings and positively filtered Noetherian Noetherian rings, [7]. In [7] the notion of a holonomic module over positively filtered rings has been introduced. Recently, Huishi, in his Ph. D. Thesis [12], investigate Auslander regularity condition and holonomity of graded and filtered modules over Zariski filtered rings.

In this work, using the micro-structure sheaf techniques we characterize a generalized Holonomic sheaf theory. We introduce a general study of Auslander regularity on the micro-structure sheaves. We calculate the global dimension of modules over the micro-structure sheaves $\underline{O}_X^{\mu}$. The main results are contained in Theorem (2.4), Theorem (3.6) and Theorem (3.7).

1. Preliminaries

In this section we collect and recall some basic notions. For full detail we have to refer to the references. Auslander regularity conditions in both commutative and non-commutative cases and holonomic modules may be found in [9], [6] and [7]. In [12] Huishi-Li has studied these subjects but for filtered and graded levels. Basic facts concerning filtered and graded modules, Zariski filtered ring theory may be found in [8], [11]. For micro-localizations of filtered modules and micro-structure sheaves, we have to refer to [5], [14]. For ringed spaces, general notion of a coherent sheaf and the category of $\underline{O}_X$-modules one can use [10], [13]. Micro-structure coherent sheaf, Zariski filtered sheaves and formal quantum sheaves have been studied in [2], [3] and [1]. Finally the coherent filtered modules $HOM(,)$ and $EXT(,)$ over the micro-structure sheaf may be

Received August 23, 1995.

1991 *Mathematics Subject Classification.* 14, 16A60 , 16A03

Key words and phrases. Zariski filtered sheaf - Regular sheaves - Holonomic sheaves.

found in [4].

Remark 1.1. 1- If R is one of the standard rings of differential operators with positive filtration then the class of holonomic R-modules is just the Bernstein class defined in [7].

2- Let R be the n-th weyl algebra over a field K of characteristic 0. If $n \geq 2$, then there are simple R-modules which are not holonomic.

Throughout, unless otherwise specified, we have the Zariski situations. We work over ringed spaces $(X, \underline{O}_X^\mu)$ where $X = spec^g(G(R))$ is the graded prime spectrum of the commutative domain $G(R)$ such that R is Zariski filtered ring. X has a base β of open sets $X(f) = \{p \in X, f \neq p\}$; $f \in G(R)$ homogeneous element. $\underline{O}_X^\mu$ is the micro-structure coherently Zariski filtered sheaf associated to R.

Associating to $X(f) \in \beta$ the micro-localizations $\tilde{Q}_{\tilde{f}}^\mu(\tilde{R})$, resp. $Q_f^\mu(R)$ we obtain the micro-structure coherent sheaves $\tilde{\underline{O}}_x^\mu$, resp. $\underline{O}_x^\mu$ defined on X having the completed stalks at $x \in X$ the Noetherian rings $\tilde{Q}_{\tilde{f}}^\mu(\tilde{R})$,resp.$Q_f^\mu(R)$. Replacing R by a good filtered module $M \in R$-filt in the foregoing leads to the construction of the coherent graded $\tilde{\underline{O}}_x^\mu$-Modules $\tilde{\underline{M}}_X^\mu$ and teh coherently filtered $\underline{O}_x^\mu$-Modules $\underline{M}_X^\mu$ having as the completed stalks at $x \in X$ the finitely generated modules $\tilde{Q}_{\tilde{f}}^\mu(\tilde{M})$, resp. $Q_f^\mu(M)$, [2]. The quantization $F_0\underline{O}_x^\mu$, $F_0\underline{M}_x^\mu$ of the micro-level for $G(R)$, $G(M)$ are obtained by looking at the parts of filtration degree zero, see [2], [1]. The formal level $\underline{O}_{\hat{x}}^\mu$, $\underline{M}_{\hat{x}}^{\mu,\triangle}$ of the micro-level has been studied in [1].

2. Regularity and Global Dimension of the Micro-Structure Sheaves

The projective dimension of a coherently filtered module $\underline{m}_X \in \underline{O}_X^\mu$-Filt is given by

$$P.\dim(\underline{m}_x) = \sum_{x \in X} \{P.\dim \underline{m}_{X,x}\}$$

In this section the work is restricted to the open affine Noethenian base for X. The following results are due to Huishi-Li in the case of filtered modules over Zariski filtered ring, [12]. One can easily modify the proofs in [12] to the case of coherent sheaves over the micro-structure sheaves. Thus we have the following results.

Lemma 2.1. *For $x \in X = spec^g(G(R))$*
a-$p.\dim \underline{m}_{X,x} \leq p.\dim G(\underline{O}_{X,x}^\mu)G(\underline{m}_{X,x})$.
b-$gl.\dim \underline{O}_{X,x}^\mu \leq gr.gl, \dim G(\underline{O}_{X,x}^\mu)$.
where $p.\dim$ stands for projective dimension, $gl.\dim$ stands Global dimension.

Theorem 2.2. *Under the same versoin*
a-$Gl.\dim \underline{O}_X^\mu \leq Gr.Gl.\dim G(\underline{O}_X^\mu)$.
b-$P.\dim \underline{m}_X \leq Gr.P.\dim G(\underline{O}_X^\mu)G(\underline{m}_X)$.

c-Moreover

$$\begin{aligned} Gl.\dim \underline{O}^{\mu}_{\hat{X}} &= Gr.\dim (\underset{\overline{n}}{\underline{\lim}}\, R/I^n)^{\mu}_{X} \\ &= Gl.\dim \underline{\hat{R}^{I}}^{\mu}_{X} \leq gl.\dim \hat{R}^{I} \leq gl.\dim R. \end{aligned}$$

Theorem 2.3. *Let R a Zariski filtered ring with associated graded (then Noetherian) commutative domain $G(R)$.*

$X = spec^g(G(R))$ and $\underline{m}_X \in \underline{O}^{\mu}_X$-Filt coherently filtered module. Then locally $GEXT^{j-1}_{\underline{O}^{\mu}_X}(\underline{m}_X, \underline{O}^{\mu}_X)$ is isomorphic to a subfactor sheaf of $EXT^{j-1}_{\underline{O}^{g}_X}(G(\underline{m}_X), \underline{O}^{g}_X)$; $j = 1, 2, \ldots$

Let $\underline{m}_X \in \underline{O}^{\mu}_X$-Filt be coherently filtered module over $\underline{O}^{\mu}_x$. Then there is a $k \in N$ such that $EXT^{k}_{\underline{O}^{\mu}_X}(\underline{m}_X, \underline{O}^{\mu}_X) \neq 0$; i.e. locally it is non zero. *The grade number* $J = J(\underline{m}_X)$ of $\underline{m}_X$ is given by the unique smallest positive integer such that $EXT^{J}_{\underline{O}^{\mu}_X}(\underline{m}_X, \underline{O}^{\mu}_X) \neq 0$

$\underline{m}_X$ is said to satisfy the Auslander condition if locally for any $0 \leq k \leq gr.\dim \underline{O}^{\mu}_X$ and any subsheaf $\underline{n}_X$ of modules of $EXT^{k}_{\underline{O}^{\mu}_X}(\underline{m}_X, \underline{O}^{\mu}_X)$ we have $J(\underline{n}_X) \geq k$. $\underline{O}^{\mu}_k$ itself is said to be *regular Noetherian* sheaf if every $\underline{m}_X \in \underline{O}^{\mu}_X$-Filt coherently filtered satisfies the Auslander condition.

Theorem 2.4. *With notations and considerations as before $\underline{O}^{\mu}_x$ is regular Noetherian sheaf over $X = spec^g(G(R))$.*

Proof. Consider an affine Noetherian open $X(f) \in \beta$ =base(X) and $x \in X$ and reduce to a local problem. $G(\underline{O}^{\mu}_X)(X(f)) = Q^g_x(G(R))$ is graded Noetherian and $G(\underline{O}^{\mu}_X)_x = Q^g_x(G(R))$ is local regular graded ring. $\underline{O}^{\mu}_X(X(f))$ and $\underline{O}^{\mu}_{X,x}$ are Zariski filtered rings. Hence $\underline{O}^{\mu}_X(X(f))$ and $\underline{O}^{\mu}_{X,x}$ are regular Noetherian. Therefore $\underline{O}^{\mu}_x$ is regular Anoetherian sheaf over $X = spec^g(G(R))$.

Now we are ready to define and construct the theory of holonomic sheaves.

3. Holonomic Sheaves

Let R be a Zariski filtered regular ring with finite global dimension and $G(R)$ be commutative domain. Then, let $Gl.\dim \underline{O}^{\mu}_X = k$. A coherently filtered module $\underline{m}_x \in O^{\mu}_X$-Filt is said to be holonomic if $J(\underline{m}_X) = k$.

Before passing on to the theory of holonomic sheaves we state and prove the following important results.

Theorem 3.1. *The function*

$$spec^g(G(R)) \to N$$
$$x \to k_x$$

is continuous.

Proof. It follows from the definition of the grade number $J(-)$ and theorem (1.1) in [4] that the given function is continuous. Moreover since $G(R)$ is a domain then $x \to k_x$ is a constant function.

Lemma 3.2. *Let $\underline{m}_X \in \underline{O}^{\mu}_X$-Filt be coherently filtered then $\underline{m}_X$ is holonomic if and only if $\underline{m}_X(X(f))$; $X(f) \in \beta$, is holonomic if and only if $\underline{m}_{X,x}; x \in X$, is holonomic.*

Theorem 3.4. *The function $\underline{m}_X \to \underline{m}^E_X = EXT^j_{\underline{O}^{\mu}_X}(\underline{m}_X, \underline{O}^{\mu}_X)$ is exact and moreover $\underline{m}^{EE}_X \cong \underline{m}_X$.*

Proof. Consider an exact sequence in $\underline{O}^{\mu}_X$-Filt of holonomic coherently filtered sheaves

$$0 \to {}^1\underline{m}_X \to {}^2\underline{m}_X \to {}^3\underline{m}_X \to 0.$$

Hence we have an exact sequence, for every $x \in X$,

$$0 \to {}^1\underline{m}_{X,x} \to {}^2\underline{m}_{X,x} \to {}^3\underline{m}_{X,x} \to 0,$$

of holonomic good filtered modules over the Zariski filtered ring $\underline{O}^{\mu}_{X,x}$.

Then

$$0 \to {}^1\underline{m}^E_{X,x} \to {}^2\underline{m}^E_{X,x} \to {}^3\underline{m}^E_{X,x} \to 0,$$

is the only part non-trivial part in the long exact EXT-cohomology sequence. Therefore

$$0 \to {}^1\underline{m}^E_X \to {}^2\underline{m}^E_X, \to {}^3\underline{m}^E_X \to 0,$$

is exact. Now we may choose a resolution of $\underline{m}_X$

$$0 \to \underline{P_j}_X \to \underline{P_{j-1}}_X \to \dots \underline{P_0}_X \to \underline{m}_X \to 0,$$

of coherently filtered projective sheaves. Hence we have a resolution of $\underline{m}_{X,x}$; $x \in X$,

$$0 \to \underline{P_j}_{X,x} \to \underline{P_{j-1}}_{X,x} \to \dots \underline{P_0}_{X,x} \to \underline{m}_{X,x} \to 0,$$

of good filtered projective modules. We can obtain for every $x \in X$ a resolution of $\underline{m}^{EE}_{X,x}$:

$$0 \to \underline{P_j}_{X,x} \to \dots \underline{P_0}_{X,x} \to \underline{m}^{EE}_{X,x} \to 0,$$

Therefore we have

$$0 \to \underline{P_j}_{X,x} \to \dots \underline{P_0}_{X,x} \to \underline{m}^{EE}_{X,x} \to 0,$$

and $\underline{m}_X \cong \underline{m}^{EE}_X$ as well.

Remark 3.5.

1- $\underline{m}_X^E = EXT^j_{\underline{O}^\mu_X}(\underline{m}_X, \underline{O}^u_X)$ of holonomic coherently filtered module $\underline{m}_X$ over $\underline{O}^\mu_x$ is holonomic coherently filtered.

2- The class of all holonomic sheaves over $\underline{O}^\mu_x$ is closed under subsheaves and factor sheaves.

However it is also possible to continue the holonomic sheaf theory and establish the micro-holonomic version of J. P. Serrre's results on global sections of coherent algebra sheaves.

Theorem 3.6. *With notation and conventions as before: Let $M \in R$-filt be a good filtered holonomic module over the Zariski filtered regular ring R, then $\underline{M}^\mu_X$ is holonomic coherently filtered sheaf. Conversely if $\underline{m}_X \in \underline{O}^\mu_X$-Filt is coherently filtered over $\underline{O}^\mu_x$ then there is a holonomic good filtered module M over R such that $\underline{m}_X \approx \underline{M}^\mu_X$.*

Theorem 3.7. *With conventions as above, but assume that R is filtered complete. Let $\{\underline{m_i}_X, \varphi_{ij}\}_{i,j\in\Lambda}$ be an inverse system holonomic coherently filtered sheaf over $\underline{O}^\mu_x$ defined on $X = spec^g(G(R))$ with subjective morphism $\varphi_{ij}; i,j \in$ Lambda. Then the inverse limit is holonomic coherently filtered over $\underline{O}^\mu_x$.*

Proof. Easy.

Corollary 3.8. *With conventions as above but assume that R is a strongly filtered ring. Let $\underline{m}_X \in \underline{O}^\mu_X$-Filt coherently filtered module over $\underline{O}^\mu_X$. Then*

1- $F_0\underline{m}_X \cong \underline{(\tilde{m}_X)_0}_X$ is holonomic coherently filtered sheaf.

2- $F_0\underline{M}^{\mu,\triangle}_{\hat{X}}$ and $\underline{M}^{\mu,\triangle}_{\hat{X}}$ are holonomic coherently filtered sheaves over $\underline{O}^\mu_x$.

References

[1] Abdel-Aziz E. Radwan, *Micro-structure sheaves and coherent sheaves of complete Modules*, Ph. D. Thesis, Department of Math., Faculty of Science, Ain-Shams Uni., Cairo, Egypt. (1991).

[2] Abdel-Aziz E. Radwan and Van oystaeyen F., "Coherent sheaves over micro-structure sheaves," *Bull. Soc. Math. Belguim, V.* 45, N1, S. A. (1993) 261-280.

[3] Abdel-Aziz E. Radwan, "Zariski Filtered sheaves," *Bull. of the calcutta Math. Soc. Manus. no.* 133/93 in press (1995) India.

[4] Abdel-Aziz E. Radwan, *Coherent Modules in the Category $\underline{O}^\mu_x$-Filt, submitted.*

[5] Asensio M., Van den Bergh M. and Van Oystaeyen F., "A new algebraic approach to micro-localization of filtered rings," *Trans. Amer. Math. Soc. V.* 316, N2, (1989) 537-555.

[6] Bass H., Algebraic K-theory, W. Benjamin, (1968) New York.

[7] Bjork J., '*Rings of differential operators, Math. Library 21, North Holland Amsterdam* (1979).

[8] Canstantin N. and Van Oystaeyen F., "Graded ring theory, Math. Library 28," *North Holland Amsterdam* (1982).

[9] Canstantin N. and Van Oystaeyen F., "Dimensions of ring theory," *D. Reidel publ. Co.*, (1987).

[10] Harshorne R., it Algebraic Geometry, G. T. M. 52 Springer Verlag, New York (1977).

[11] Huishi L. and Van Oystaeyen F., "Zariskian filtrations." *Comm. in algebra,* 17(1989) 12, 2945-2970.

[12] Huishi L., (Ph. D. Thesis), "Zariskian filtrations, Department of Mathematics," *Antwerp Uni., Belguim* (1991).

[13] Van Oystaeyen F. and Verschoren A., "Non-Commutative algebraic Geometry," *LNM887, Springer Verlag, Berlin* (1981).

[14] Van Oystaeyen F. and Sallam R., "A micro-structure sheaf and quantum sections over a projective scheme," *J. of algebra,* 158(1993) 201-225.

Department of Mathematics, Faculty of Science, Ain Shams University, Cairo-Egypt.