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THE SOLUTIONS OF A LINEAR HOMOGENEOUS RECURRENCE RELATION WITH UNIT COEFFICIENTS

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Abstract. In this paper, three expressions of solutions of linear homogeneous recurrence relations with unit coefficients

$$x_n = x_{n-1} + x_{n-2} + \cdots + x_{n-(k-1)} + x_{n-k}, \quad \forall n \geq k, \ k \geq 3$$

are derived.

1. Introduction

The Fibonacci sequence, $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots\}$, is one of the oldest recurrence relations, which satisfies

$$\begin{aligned} x_n &= x_{n-1} + x_{n-2}, \\ x_0 &= x_1 = 1, \end{aligned} \tag{1}$$

where x_n denotes the number of pairs of rabbits at the end of the n-th month. After this counting problem about the growth of rabbit population has been posed, the properties of the Fibonacci sequence, Fibonacci matrices and related sequences have been studied broadly and applied in many fields. For example, Turner [5] discussed the expression of the Fibonacci word patterns; Yu and Zhao [6] analyzed and proved the properties of Fibonacci languages; Filipponi [2] showed how the m-by-m symmetric tridiagonal Toeplitz matrices are Fibonacci matrices; Lin [4] discussed the properties of the Fibonacci matrices and the relations between Fibonacci matrices and the units of the field $2(\theta)$; Zhang [7] established some properties of the generalized Fibonacci sequences. In this paper, we focus our attention on the solutions of the recursive functions from three terms to k-terms which generalize the well-known formula of the solution of (1). We pose a question concerning a counting problem first, then discover the recurrence relation of the question, finally, we try to find three explicit counting formulas of the recurrence relation.

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The general method of solving linear homogeneous recurrence relations with constant coefficients is to find an explicit basis for the recurrence relation. There are many methods to solve equation (1), e.g., method of characterisitc polynomials, general counting method for arrangements and selections, eigensystem of matrix and generating functions, etc. But, there is no general rule which will enable one to solve all recurrence relations. Here are three expressions of the solution of (1), (see [3])

1. $$x_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \tag{2}$$

2. $$x_n = \sum_{i=0}^{[\frac{n}{2}]} \frac{(n-i)!}{i!(n-2i)!}, \tag{3}$$

or written as (see [1])

$$x_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-j}{j}, \qquad j = \left[\frac{n}{2}\right], \tag{4}$$

where $[x]$ is the greatest integer less than x.

3. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $A^{n-1}B = \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix}$.

In this paper, three expressions of solutions of linear homogeneous recurrence relations with unit coefficients $x_n = x_{n-1} + x_{n-2} + \cdots + x_{n-(k-1)} + x_{n-k}$, $\forall n \geq k$, $k \geq 3$, are derived. The initial conditions are assumed to be $x_0 = 1$, $x_1 = 1$, $x_2 = 2, \ldots$, and $x_{k-1} = 2^{k-2}$.

2. The Solution of The Recurrence Relation $x_n = x_{n-1} + x_{n-2} + x_{n-3}$

Let $F(n)$ denote the $(n+1)$-th Fibonacci number.

Lemma. $F(i)F(j+1) + F(i-1)F(j) = F(i+j+1)$.

Proof. The proof will be completed by fixed i and induction on j.

$$\begin{aligned}
\text{When } j = 0, \text{ Left } &= F(i)F(1) + F(i-1)F(0) \\
&= F(i) + F(i-1) = F(i+1) = \text{Right}. \\
\text{When } j = 1, \text{ Left } &= F(i)F(2) + F(i-1)F(1) \\
&= F(i)(F(1) + F(0)) + F(i-1)F(1) \\
&= F(i+1)F(1) + F(i)F(0) \\
&= F(i+1)F(1) + F(i)F(1) \\
&= F(i+2)F(1) \\
&= F(i+2) = \text{Right}.
\end{aligned}$$

Assume that when $j = k$, $F(i)F(k+1) + F(i-1)F(k) = F(i+k+1)$ is true. Then when $j = k+1$,

$$\begin{aligned}\text{Left} &= F(i)F(k+2) + F(i-1)F(k+1)\\ &= F(i)(F(k+1) + F(k)) + F(i-1)(F(k) + F(k-1))\\ &= F(i)F(k+1) + F(i-1)F(k) + F(i)F(k) + F(i-1)F(k-1)\\ &= F(i+k+1) + F(i+k)\\ &= F(i+k+2)\\ &= \text{Right}.\end{aligned}$$

Thus, the proof is completed.

Question. Assume there is a staircase of n stairs to climb. Each stepping can cover one stair, two stairs or three stairs. Find a recurrence relation for x_n, the number of different ways to ascend the n-stair staircase.

Suppose the first step is taken, it may climb one stair, two stairs or three stairs, then there are $n-1$, $n-2$ or $n-3$ stairs remained to climb, thus $x_0 = 1$, $x_1 = 1$, $x_2 = 2$, $x_3 = 4$ and

$$x_n = x_{n-1} + x_{n-2} + x_{n-3}, \quad \forall n \geq 3. \tag{5}$$

In fact, $x_0 = F(0)$, $x_1 = F(1)$ and $x_2 = F(2)$. Let $a(n,i)$ denote the number of the ways to climb a stair case of n stairs by stepping 3 stairs i times. It is clear that $a(n,0) = F(n)$. We want to prove $x_n = \sum_{i=0}^{[\frac{n}{3}]} a(n,i)$ is the solution of (5).

In the process of stair climbing, suppose that the stepping of 3 stairs occurs only once, which crosses over the $(i+1)th$, $(i+2)th$ and $(i+3)th$ stairs, $i = 0, 1, 2, \ldots, n-3$. Since the climbing stairs of each step must be 1 or 2 during the first i stairs and the last $n-i-3$ stairs, we have $a(n,1) = \sum_{i=0}^{n-3} a(i,0)a(n-i-3,0) = \sum_{i=0}^{n-3} F(i)F(n-i-3)$. Notice that $a(1,1) = a(2,1) = 0$ corresponding to empty sum.

Proposition 1. *$a(n,1)$ satisfies $a(n,1) = a(n-1,1) + a(n-2,1) + a(n-3,0)$, $\forall n \geq 3$.*

Proof. (1) If $n = 2m$, then

$$\begin{aligned}&a(n,1) - a(n-1,1) - a(n-2,1) - a(n-3,0)\\ &= 2\sum_{i=0}^{m-2} F(i)F(2m-3-i) - F^2(m-2) - 2\sum_{i=0}^{m-3} F(i)F(2m-4-i)\\ &\quad -2\sum_{i=0}^{m-3} F(i)F(2m-5-i) - F(2m-3)\\ &= 2[F(0)F(2m-3) + F(1)F(2m-4) + \cdots + F(m-2)F(m-1)\\ &\quad -(F(0)F(2m-4) + F(1)F(2m-5) + \cdots + F(m-3)F(m-1))\end{aligned}$$

$$
\begin{aligned}
&\quad -(F(0)F(2m-5)+F(1)F(2m-6)+\cdots+F(m-3)F(m-2))] \\
&\quad -F^2(m-2)-F(2m-3) \\
&= 2F(m-2)F(m-1)-F^2(m-2)-F(2m-3) \\
&= F(m-2)F(m-1)-F(m-2)F(m-2)+F(m-2)F(m-1)-F(2m-3) \\
&= F(m-2)F(m-3)+F(m-2)F(m-1)-F(2m-3) \\
&= F(2m-3)-F(2m-3) \\
&= 0.
\end{aligned}
$$

(2) If $n = 2m+1$, then

$$
\begin{aligned}
&\quad a(n,1)-a(n-1,1)-a(n-2,1)-(n-3,0) \\
&= F^2(m-1)+2\sum_{i=0}^{m-2}F(i)F(2m-2-i)-2\sum_{i=0}^{m-2}F(i)F(2m-3-i) \\
&\quad -F^2(m-2)-2\sum_{i=0}^{m-3}F(i)F(2m-4-i)-F(2m-2) \\
&= 2F(m-2)F(m-2)+F^2(m-1)-F^2(m-2)-F(2m-2) \\
&= 2F^2(m-2)+F^2(m-1)-F^2(m-2)-(F^2(m-1)+F^2(m-2)) \\
&= 2F^2(m-2)-2F^2(m-2) \\
&= 0.
\end{aligned}
$$

In the process of stair climbing, suppose that the stepping of 3 stairs happens twice. The first one crosses over the $(i+1)th$ to $(i+3)th$ stairs, where $i = 0, 1, 2, \ldots, n-6$. The other stepping of 3 stairs occurs in the last $(n-i-3)$ stairs after the first one. Since each step in the first i stairs take 1 or 2 stairs, we have $a(n,2) = \sum_{i=0}^{n-6} a(i,0)a(n-i-3,1) = \sum_{i=0}^{n-6} F(i)a(n-i-3,1)$. Notice that $a(i,0) = 0$ for $i = 1, 2, \ldots, 5$, which corresponds to empty sum.

Proposition 2. $a(n,2)$ *satisfies*

$$a(n,2) = a(n-1,2)+a(n-2,2)+a(n-3,1), \quad \forall n \geq 6.$$

Proof.

$$
\begin{aligned}
&\quad a(n,2)-a(n-1,2)-a(n-2,2)-a(n-3,1) \\
&= \sum_{i=0}^{n-6}F(i)a(n-i-3,1)-\sum_{i=0}^{n-7}F(i)a(n-i-4,1)-\sum_{i=0}^{n-8}F(i)a(n-i-5,1)-a(n-3,1) \\
&= F(0)a(n-3,1)+F(1)a(n-4,1)+\sum_{i=2}^{n-6}F(i)a(n-i-3,1)-F(0)a(n-4,1)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{i=1}^{n-7} F(i)a(n-i-4,1) - \sum_{i=0}^{n-8} F(i)a(n-i-5,1) - a(n-3,1) \\
&= \sum_{i=0}^{n-8} F(i+2)a(n-i-5,1) - \sum_{i=0}^{n-8} F(i+1)a(n-i-5,1) - \sum_{i=0}^{n-8} F(i)a(n-i-5,1) \\
&= \sum_{i=0}^{n-8} (F(i+2) - F(i+1) - F(i))a(n-i-5,1) \\
&= 0.
\end{aligned}
$$

In the same way, a general expression of $a(n,k)$ can be obtained, too.

Proposition 3. *$a(n,k)$ satisfies $a(n,k) = \sum_{i=0}^{n-3k} F(i)a(n-i-3,k-1)$ and $a(n,k) = a(n-1,k) + a(n-2,k) + a(n-3,k-1)$, $\forall n \geq 3k$.*

Proof. The method is similar to Proposition 2, so we omit it.

Summarizing above three Propositions, we have the following theorem.

Theorem 1.

(1) $a(n,0) = F(n) = F(n-1) + F(n-2)$;
(2) $a(n,1) = \sum_{i=0}^{n-3} F(i)F(n-3-i)$;
(3) $a(n,k) = \sum_{i=0}^{n-3k} F(i)a(n-i-3,k-1)$, $\forall k \geq 2$;
(4) *$x_n = \sum_{i=0}^{\left[\frac{n}{3}\right]} a(n,i)$ satisfies the recurrence relation $x_n = x_{n-1} + x_{n-2} + x_{n-3}$ with $x_0 = x_1 = 1$ and $x_2 = 2$.*

We denote the sequence x_n by $x_n(3)$ and its values are shown in Table 1 in Appendix.

Theorem 2. *The explicit formula of $a(n,i)$ is $a(n,i) = \sum_{j=0}^{\left[\frac{n-3i}{2}\right]} \frac{(n-2i-j)!}{i!j!(n-3i-2j)!}$. So $\sum_{i=0}^{\left[\frac{n}{3}\right]} a(n,i)$ give another form of solution of $x_n = x_{n-1} + x_{n-2} + x_{n-3}$ with $x_0 = x_1 = 1$ and $x_2 = 2$.*

Proof. Since the empty sum is regarded as 0, the initial conditions match $a(n,i) = 0$ for $n = 0, 1, 2, \ldots, 3i-1$. It suffices to prove $a(n,i) = a(n-1,i) + a(n-2,i) + a(n-3,i-1)$. If $i+n$ is odd, then

$$
\begin{aligned}
&\left[\frac{n-3i}{2}\right] = \left[\frac{n-3i-1}{2}\right], \left[\frac{n-3i-2}{2}\right] = \left[\frac{n-3i}{2}\right] - 1, \\
&a(n,i) - a(n-1,i) - a(n-3,i-1) \\
&= \sum_{j=0}^{\left[\frac{n-3i}{2}\right]} \frac{(n-2i-j-1)!}{i!j!(n-3i-2j-1)!} \left(\frac{n-2i-j}{n-3i-2j} - 1 - \frac{i}{n-3i-2j} \right)
\end{aligned}
$$

$$= \sum_{j=0}^{[\frac{n-3i}{2}]} \frac{(n-2i-j-1)!}{i!j!(n-3i-2j-1)!} \cdot \frac{j}{n-3i-2j}$$
$$= \sum_{j=1}^{[\frac{n-3i}{2}]} \frac{(n-2i-j-1)!}{i!(j-1)!(n-3i-2j)!}$$
$$= \sum_{j=0}^{[\frac{n-3i}{2}]-1} \frac{(n-2i-j-2)!}{i!j!(n-3i-2j-2)!}$$
$$= a(n-2,i).$$

The other case can be proved similarly.

Theorem 3. *Let* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, *then* $A^n B = \begin{bmatrix} x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$, $\forall n \geq 2$, *gives the solution of* (5).

Proof. Because the first row of A is [0 1 0], so the first row of $A^n = A \cdot A^{n-1}$ equal to the second row of A^{n-1}; the second row of A is [0 0 1], so the second row of A^n equal to the third row of A^{n-1}; the third row of A is [1 1 1], so the third row of A^n equal to the sum of the first, second and third row of A^{n-1}.

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & x_0 \\ 1 & 1 & x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, then $A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{bmatrix}$. Let x_k equal to the (3,3) element of A^k, then the third column of A^n is $\begin{bmatrix} x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$, i.e. $A^n B = \begin{bmatrix} x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix}$. Notice that $x_2 = 2$ and $x_n = x_{n-1} + x_{n-2} + x_{n-3}$, $\forall n \geq 3$, give the exact solution of (5).

3. The Solution of The Recurrence Relation $x_n = x_{n-1} + x_{n-2} + \cdots + x_{n-(k-1)} + x_{n-k}$

Question. Assume there is a staircase of n stairs to climb. Each stepping can cover one stair, two stairs, three stairs,. . ., to at most k stairs. Find a recurrence relation for x_n, the number of different ways to ascent the n-stair staircase.

This sequence x_n depends strongly on k, so we use the precise notation $x_n(k)$ in the sequel. Suppose the first step is taken, its climbing stairs may be $1, 2, 3, \ldots, k$, then there are $n-1, n-2, n-3, \ldots, n-k$ stairs remained to climb, thus

$$x_n(k) = x_{n-1}(k) + x_{n-2}(k) + \cdots + x_{n-(k-1)}(k) + x_{n-k}(k), \quad \forall n \geq k. \tag{6}$$

The initial conditions are

$$x_0(k) = 1 \text{ and } x_i(k) = x_i(k-1) = 2^{i-1}, \quad \forall i = 1, \ldots, k-1. \tag{7}$$

Let $a(n,k,i)$ denote the number of the ways to climb a stair case of n stairs by stepping k stairs i times. We generalize theorems in the previous section as follows. Their proofs are similar and omitted.

Theorem 4. *Let $k \geq 3$, then*

(1) $x_0(k) = x_1(k) = 1$, $\forall k$ and $a(n,k,i) = 0$, $\forall k > n$;
(2) $a(n,k,0) = x_n(k-1)$;
(3) $a(n,k,1) = \sum_{i=0}^{n-k} x_i(k-1)x_{n-k-i}(k-1)$;
(4) $a(n,k,j) = \sum_{i=0}^{n-jk} x_i(k-1)a(n-i-k,k,j-1)$, $\forall n \geq k$, *and*
(5) $x_n(k) = \sum_{i=0}^{\left[\frac{n}{k}\right]} a(n,k,i)$ *satisfies the recurrence relation* (6) *and* (7).

The values of $x_n(4)$ are shown in Table 2 in Appendix.

Theorem 5. *Let $k \geq 3$, then*

$$x_n = \sum_{i_1=0}^{\left[\frac{n}{k}\right]} \sum_{i_2=0}^{\left[\frac{n-ki_1}{k-1}\right]} \sum_{i_3=0}^{\left[\frac{n-ki_1-(k-1)i_2}{k-2}\right]} \cdots \sum_{i_{k-1}=0}^{\left[\frac{n-ki_1-(k-1)i_2-\cdots-3i_{k-2}}{2}\right]} \frac{\left(n-\sum_{t=1}^{k-1}(k-t)i_t\right)!}{\left(\prod_{t=1}^{k-1} i_t!\right)\left(n-\sum_{t=1}^{k-1}(k+1-t)i_t\right)!}$$

is another form of the solution of (6) *and* (7).

Theorem 6. *Let*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{k\times k} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{k\times 1}, \text{ then } A^n B = \begin{bmatrix} x_{n-(k-1)} \\ x_{n-(k-2)} \\ x_{n-(k-3)} \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \ \forall n \geq k,$$

gives the solution of (6) *and* (7).

4. Conclusion

Taking $x_n = x_{n-1} + x_{n-2} + x_{n-3}$ for instance, we obtain four ways to evaluate the value of x_n. They are: (i) $x_n = x_{n-1} + x_{n-2} + x_{n-3}$; (ii) Theorem 1; (iii) Theorem 2 and (iv) Theorem 3. By method (i) and (ii), we can evaluate x_n recursively according to the order of n. By method (iii), we found an explicit formula of (5). By method (iv), according to the theorems of the eigenvalues and eigenvectors, an explicit formula similar to (2) can be found. Although the processes of these four ways are very tedious, method (ii) seems to be the best of three. Because it evaluates the detail informations like $a(n,i)$ besides the value of x_n. These details may be applied to count the ways of stops made by a train at different stations.

Four our further research, we are going to discuss the method of generating function for $x_n = x_{n-1} + x_{n-2} + \cdots + x_{n-k}$ and the solutions of the recurrence relation $x_n = c_1x_{n-1} + c_2x_{n-2} + \cdots + c_kx_{n-k}$. However, in this paper, we proposed a new approach to solve the recurrence relation (6), which can be generalized to some general cases.

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Appendix

Table 1. $x_n(3)$ value table

n	$F(n)$	$a(n,1)$	$a(n,2)$	$a(n,3)$	$a(n,4)$	$a(n,5)$	$\cdots$	$x_n(3)$
0	1	0	0	0	0	0		1
1	1	0	0	0	0	0		1
2	2	0	0	0	0	0		2
3	3	1	0	0	0	0		4
4	5	2	0	0	0	0		7
5	8	5	0	0	0	0		13
6	13	10	1	0	0	0		24
7	21	20	3	0	0	0		44
8	34	38	9	0	0	0		81
9	55	71	22	1	0	0		149
10	89	130	51	4	0	0		274
11	144	235	111	14	0	0		504
12	233	420	233	40	1	0		927
13	377	744	474	105	5	0		1705
14	610	1308	942	256	20	0		3136
15	987	2285	1836	594	65	1		5768
16	1597	3970	3522	1324	190	6		10609

Table 2. $x_n(4)$ value table

n	$x_n(3)$	$a(n,4,1)$	$a(n,4,2)$	$a(n,4,3)$	$a(n,4,4)$	$(a,4,5)$	$\cdots$	$x_n(4)$
0	1	0	0	0	0	0		1
1	1	0	0	0	0	0		1
2	2	0	0	0	0	0		2
3	4	0	0	0	0	0		4
4	7	1	0	0	0	0		8
5	13	2	0	0	0	0		15
6	24	5	0	0	0	0		29
7	44	12	0	0	0	0		56
8	81	26	1	0	0	0		108
9	149	56	3	0	0	0		208
10	274	118	9	0	0	0		401
11	504	244	25	0	0	0		773
12	927	499	63	1	0	0		1490
13	1705	1010	153	4	0	0		2872
14	3136	2027	359	14	0	0		5536
15	5768	4040	819	44	0	0		10671
16	10609	8004	1830	125	1	0		20569
17	19513	15776	4018	336	5	0		39648
18	35890	30956	8694	864	20	0		76424
19	66012	60504	18582	2144	70	0		147312
20	121415	117845	39298	5174	220	1		283953

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