TAMKANG JOURNAL OF MATHEMATICS
Volume 54, Number 1, 43-55, 2023
DOI:10.5556/j.tkjm.54.2023.3971

Existence of periodic traveling wave solutions for a K-P-Boussinesq type system

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Abstract. In this paper, via a variational approach, we show the existence of periodic traveling waves for a Kadomtsev-Petviashvili-Boussinesq type system that describes the propagation of long waves in wide channels. We show that those periodic solutions are characterized as critical points of some functional, for which the existence of critical points follows as a consequence of the Mountain Pass Theorem and Arzela-Ascoli Theorem.

Keywords. Boussinesq system, periodic traveling waves, variational methods

1 Introduction

The aim of this work is to use the direct method of the calculus of variations to prove the existence of 2D and 1D periodic traveling waves for a Kadomtsev-Petviashvili-Boussinesq type system,

$$\begin{cases} \eta_t + \partial_x \Phi + \eta \partial_x \eta &= 0, \quad (x,y) \in \mathbb{R}^2, t > 0, \\ \Phi_t + \partial_x \left(\eta \Phi + \eta - \partial_x^2 \eta \right) + \partial_x^{-1} \partial_y^2 \eta &= 0, \end{cases} \tag{1.1}$$

that describes the propagation of long waves in wide channels or open seas, where η is the amplitude and Φ is the horizontal velocity. This system was obtained by an appropriate approximation from the basic equations of hydrodynamics (see [4], [8]).

For models that describe the evolution of nonlinear waves, it is very important to determine the well-posedness for the associated Cauchy problem, and the existence of special solutions as the traveling waves. For instance, traveling wave solutions are important in the study of dynamics of wave propagation in many applied models such as fluid dynamics, acoustic, oceanography, and weather forecasting. An important application is the use of solitons (traveling waves in the energy space) as an efficient means of long-distance communication.

In the work [8], F. Soriano showed the local and global well-posedness in the space Y^s for $s > 2$, where

$$Y^s = X^s \times H^s,$$

Received date: October 16, 2020; Published online: November 13, 2021.
2010 *Mathematics Subject Classification*. 35Q35, 35C07, 35B10, 35A15.
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with $H^s = H^s(\mathbb{R}^2)$, the standard Sobolev space and

$$X^s = \{w \in H^s \ : \ w_x \in H^s \ \text{ and } \ \partial_x^{-1} w_y \in H^s\},$$

where $\partial_x^{-1} w_y$ is defined via the Fourier transformation as

$$\widehat{\partial_x^{-1} w_y} = \frac{\xi_2}{\xi_1} \widehat{w}(\xi_2, \xi_1).$$

In addition, F. Soriano established the existence of 2D-solitons, i.e. the existence of solutions of the form

$$\eta(x,y,t) = u(x-ct,y), \quad \Phi(x,y,t) = v(x-ct,y), \tag{1.2}$$

which propagate with speed of wave c, with $0 < |c| < 1$.

Note that if (η, Φ) is a solution of the system (1.1) of type (1.2), then the traveling wave profile (u,v) should satisfy the system

$$\begin{cases} -cu + v + \frac{1}{2}u^2 &= 0, \quad (x,y) \in \mathbb{R}^2, \\ -cv + u - \partial_x^2 u + \partial_x^{-2}\partial_y^2 u + uv &= 0, \end{cases} \tag{1.3}$$

where

$$\partial_x^{-1} w(x,y) = \int_{-\infty}^{x} w(s,y) ds.$$

F. Soriano showed in [8], using the Concentration-Compactness Theorem, the existence of solutions for the system (1.3) in the space $Z = Z(\mathbb{R}^2) = X(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ for the speed of wave satisfying $0 < |c| < 1$, where the space $X = X(\mathbb{R}^2)$ is equipped with the norm

$$\|u\|_X^2 = \|u\|_{L^2(\mathbb{R}^2)}^2 + \|u_x\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}^2$$

and

$$\|(u,v)\|_Z^2 = \|u\|_{L^2(\mathbb{R}^2)}^2 + \|u_x\|_{L^2(\mathbb{R}^2)}^2 + \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}^2 + \|v\|_{L^2(\mathbb{R}^2)}^2.$$

In this paper we consider instead of (1.3) the following system on the infinite strip

$$Q_k = [-k,k] \times \mathbb{R} \subset \mathbb{R}^2,$$

with $k \in \mathbb{R}^+$,

$$\begin{cases} -cu + v + \frac{1}{2}u^2 &= 0, \quad \text{in } \ Q_k, \\ -cv + u - \partial_x^2 u + \partial_{x,k}^{-2}\partial_y^2 u + uv &= 0, \end{cases} \tag{1.4}$$

where (u,v) is $2k$-periodic in x and

$$\partial_{x,k}^{-1} w(x,y) = \int_{-k}^{x} w(s,y) ds.$$

Using the Mountain Pass Theorem we will show the existence of solutions for the system (1.4) in the space $Z_k = Z_k(\mathbb{R}^2)$ equipped with the norm

$$\|(u,v)\|_{Z_k}^2 = \|u\|_{L_k^2(\mathbb{R}^2)}^2 + \|u_x\|_{L_k^2(\mathbb{R}^2)}^2 + \|\partial_{x,k}^{-1} u_y\|_{L_k^2(\mathbb{R}^2)}^2 + \|v\|_{L_k^2(\mathbb{R}^2)}^2,$$

where

$$L_k^q(\mathbb{R}^2) = \big\{w : \mathbb{R}^2 \to \mathbb{R} \ : \ w \in L^q(Q_k) \text{ and } x\text{-periodic of period } 2k\big\}$$

and

$$\|w\|_{L^q_k(\mathbb{R}^2)} = \|w\|_{L^q(Q_k)}.$$

Also, in this work, we show the existence of 1D periodic traveling waves for the system (1.1). This is, the existence of $2k-$periodic solutions of the form

$$\eta(x,y,t) = \psi(x+y-ct), \quad \Phi(x,y,t) = \phi(x+y-ct),$$

where c denotes the speed of wave. Then, one sees that (ψ,ϕ) must satisfy

$$\begin{cases} -c\psi + \phi + \frac{1}{2}\psi^2 &= 0, \\ -c\phi + 2\psi - \psi'' + \psi\phi &= 0, \end{cases} \tag{1.5}$$

in $[-k,k]$, where (ψ,ϕ) is $2k$-periodic. In this case, using the Arzela-Ascoli, we will show the existence of solutions for the system (1.5) in the space $Z_k = Z_k(\mathbb{R})$ equipped with the norm

$$\|(\psi,\phi)\|^2_{Z_k} = \|\psi\|^2_{L^2_k(\mathbb{R})} + \|\psi'\|^2_{L^2_k(\mathbb{R})} + \|\phi\|^2_{L^2_k(\mathbb{R})},$$

where

$$L^q_k(\mathbb{R}) = \{w : \mathbb{R} \to \mathbb{R} \ : \ w \in L^q[-k,k] \text{ and } x\text{-periodic of period } 2k\}$$

and

$$\|w\|_{L^q_k(\mathbb{R})} = \|w\|_{L^q[-k,k]}.$$

In the 2D case, let us define the profile of the wave at time t to be the graph of the function

$$(x,y) \to (\eta(x,y,t), \Phi(x,y,t)).$$

Then the initial profile (at time $t=0$) is just the graph of (u,v), and at any later time t, the profile at time t is obtained by translating each point $((x,y),(u(x,y),v(x,y)))$ of the initial profile ct units to the right to the point $((x+ct,y),(u(x,y),v(x,y)))$. In other words, the wave profile of a 2D traveling wave just propagates by rigid translation with velocity c. In the 1D case, the traveling waves propagate with wavefront normal to $z=(1,1) \in \mathbb{R}^2$, velocity c, and profile (ψ,ϕ).

We will see that the 2D and 1D periodic traveling waves are characterized as critical points of some action functional. It is straightforward to write the action functional associated to the systems (1.4) and (1.5). For the system (1.4) is relatively standard to establish that this functional has a minimun in a suitable set of periodic functions, we follow a similar approach by A. Pankov and K. Pflüger in [6] for the Kadomtsev-Petviashvili equation,

$$\eta_t - \partial_x^3\eta + \eta\partial_x\eta + \partial_x^{-1}\partial_y^2\eta = 0.$$

For the system (1.5) we follow the approach by H. Brezis and J. Mawhin (see [3]) in a recent work related with the existence of periodic classical solutions for a differential equation

$$\phi(u') - g(x,u) = h(x),$$

where $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism, g is a Charatéodory function k-periodic with respect to x, 2π-periodic with respect to u, of mean value zero on $[0,k]$, and $h \in L_{loc}(\mathbb{R})$ is k-periodic and has mean value zero. A special case of this interesting model is the relativistic forced pendulum differential equation

$$\left(\frac{u'}{\sqrt{1-u^2}}\right) + A\sin(u) = h(x).$$

In a work in preparation (see [5]), we establish some properties of the x-periodic traveling waves. In particular we study the inter-relation between x-periodic traveling wave solutions of period k and traveling wave solutions of finite energy (solitons); in the spirit of [7], we want to show the existence of a sequence of $k-$periodic 2D traveling waves, $k \in \mathbb{N}$, which is uniformly bounded in norm and converges to a soliton in a suitable topology, indicating that the shape of x-periodic 2D traveling waves of period k and solitons are almost the same, as the period k is big enough.

Throughout this paper, if not specified, we denote by C a generic constant varying line by line.

2 Two-Dimensional Periodic Traveling Waves

In this section we will establish the existence of x-periodic solutions with period $2k$ for the system (1.4). To do this we use a variational approach based on the Mountain Pass Theorem without the Palais-Smale condition together with the existence of a compact embedding result.

First, formally we define the space in order to look at x-periodic solutions. Denote by $C_k^\infty(\mathbb{R}^2)$ the space of smooth functions which are x-periodic with period $2k$ and have compact support in y and define

$$Y_k = Y_k(\mathbb{R}^2) = \left\{ D_x\varphi|_{Q_k} \ : \ \varphi \in C_k^\infty(\mathbb{R}^2) \right\}, \quad Q_k = [-k, k] \times \mathbb{R}.$$

Then any function u from Y_k satisfies

$$\int_{-k}^{k} u(x, y)\, dx = 0.$$

We define $X_k = X_k(\mathbb{R}^2)$ as the completion of Y_k with respect to the norm given by

$$\|u\|_{X_k}^2 = \int_{Q_k} \left[u^2 + (u_x)^2 + \left(\partial_{x,k}^{-1} u_y\right)^2 \right] dxdy$$

and the inner product

$$(u, U)_{X_k} = \int_{Q_k} \left[uU + u_x U_x + \left(\partial_{x,k}^{-1} u_y\right)\left(\partial_{x,k}^{-1} U_y\right)\right] dxdy.$$

From the local version of the embedding theorem for anisotropic Sobolev spaces (see [2], p. 187), for any compact $\Omega \subset \mathbb{R}^2$ and $2 \leq q \leq 6$, we have that

$$\|u\|_{L^q(\Omega)}^2 \leq C \int_{\Omega} \left[u^2 + (u_x)^2 + \left(\partial_{x,k}^{-1} u_y\right)^2 \right] dxdy. \tag{2.1}$$

Moreover, A. Pankov and K. Pflüger in the work [6] showed the following embedding result, in the case of $x-$periodic traveling waves for the generalized Kadomtsev-Petviashvili equation.

Lemma 2.1. *The embedding $X_k \hookrightarrow L^q(Q_k)$ is continuous for $2 \leq q \leq 6$, where the embedding constants are uniformly bounded with respect to k, and $X_k \hookrightarrow L^q_{loc}(Q_k)$ is compact for $2 \leq q < 6$.*

The existence of solutions for the system (1.4) in the space $Z_k = X_k \times L^2_k(\mathbb{R}^2)$ is a consequence of a variational approach which apply a minimax type result, since solutions (u, v) of (1.4) are critical points of the functional J_k given by

$$J_k = I_k + G_k,$$

where the functionals I_k and G_k are defined on the space Z_k by

$$I_k(u, v) = \frac{1}{2}\int_{Q_k} \left[u^2 + (u_x)^2 + \left(\partial_{x,k}^{-1}u_y\right)^2 + v^2 - 2cuv\right] dxdy,$$
$$G_k(u, v) = \frac{1}{2}\int_{Q_k} vu^2 dxdy.$$

First we have that $I_k, G_k, J_k \in C^1(Z_k, \mathbb{R})$ and its derivatives in (u, v) in the direction of (U, V) are given by

$$\langle I_k'(u, v), (U, V)\rangle = \int_{Q_k} \left[uU + u_xU_x + \left(\partial_{x,k}^{-1}u_y\right)\left(\partial_{x,k}^{-1}U_y\right) + vV - c\,(uV + vU)\right] dxdy,$$
$$\langle G_k'(u, v), (U, V)\rangle = \int_{Q_k} \left(uvU + \frac{1}{2}u^2V\right) dxdy.$$

As a consequence of this, after integration by parts, we conclude that

$$J_k'(u, v) = \begin{pmatrix} u - \partial_x^2 u + \partial_{x,k}^{-2}\partial_y^2 u - cv + uv \\ v - cu + \frac{1}{2}u^2 \end{pmatrix}, \tag{2.2}$$

meaning that critical points of the functional J_k satisfy the traveling wave system (1.4). Hereafter, we will say that solutions for (1.4) are critical points of the functional J_k. In particular, we have that

$$\begin{aligned} \langle J_k'(u, v), v\rangle &= 2I_k(u, v) + 3G_k(u, v) \\ &= 2J_k(u, v) + G_k(u, v). \end{aligned} \tag{2.3}$$

Thus, on any critical point (u, v) we have that

$$I_k(u, v) = -\frac{3}{2}G_k(u, v), \quad J_k(u, v) = -\frac{1}{2}G_k(u, v), \quad J_k(u, v) = \frac{1}{3}I_k(u, v). \tag{2.4}$$

On the other hand, using that the embedding $X_k \hookrightarrow L^4(Q_k)$ is continuous, one can see easily that the functional G_k is well-defined on Z_k. In particular we have that

$$|G_k(u, v)| \le \frac{1}{2}\|v\|_{L^2_k(\mathbb{R}^2)}\|(u, v)\|^2_{L^4_k(\mathbb{R}^2)} \le C\|v\|^3_{Z_k}. \tag{2.5}$$

Moreover, if $0 < |c| < 1$ then there are some positive constants $C_1(c) < C_2(c)$ such that

$$C_1\|(u, v)\|^2_{Z_k} \le I_k(u, v) \le C_2\|(u, v)\|^2_{Z_k}. \tag{2.6}$$

Our approach to show the existence of a non trivial critical point for J_k is to use the Mountain Pass Theorem without the Palais-Smale condition (see M. Willem [10], A. Ambrosetti *et al.* [1]) to build a Palais-Smale sequence for J_k for a minimax value and use the local embedding result to obtain a critical point for J_k as a weak limit of such Palais-Smale sequence.

Theorem 2.1. *Let X be a Hilbert space, $J \in C^1(X,\mathbb{R})$, $e \in X$ and $r > 0$ such that $\|e\|_X > r$ and*

$$b = \inf_{\|u\|_X = r} J(u) > J(0) \geq J(e).$$

Then, given $n \in \mathbb{N}$, there is $u_n \in X$ such that

$$J(u_n) \to d, \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } X',$$

where

$$d = \inf_{\pi \in \Gamma} \max_{t \in [0,1]} \varphi(\pi(t)), \quad \text{and} \quad \Gamma = \{\pi \in C([0,1], X) : \pi(0) = 0, \quad \pi(1) = e\}.$$

Before we go further, we establish an important result for our analysis, which is related with the characterization of "vanishing" sequences in Z_k. Define for $\zeta \in \mathbb{R}^2$ and $r > 0$ the rectangle

$$R_{r,k}(\zeta) = [-k, k] \times [\zeta - r, \zeta + r].$$

Lemma 2.2. *If $\{(u_n, v_n)\}_n$ is a bounded sequence in Z_k and there is a positive constant $r > 0$ such that*

$$\lim_{n \to \infty} \sup_{\zeta \in \mathbb{R}} \int_{R_{r,k}(\zeta)} v_n^2 \, dxdy = 0, \tag{2.7}$$

then we have that

$$\lim_{n \to \infty} G_k(u_n, v_n) = 0.$$

Proof. From Hölder inequality and the local embedding (2.1) we see that

$$\begin{aligned} |G_k(u_n, v_n)| &\leq C \|v_n\|_{L^2(R_{r,k}(\zeta))} \|u_n\|^2_{L^4(R_{r,k}(\zeta))} \\ &\leq C \|v_n\|_{L^2(R_{r,k}(\zeta))} \int_{R_{r,k}(\zeta)} \left[u_n^2 + (\partial_x u_n)^2 + \left(\partial_{x,k}^{-1} \partial_y u_n \right)^2 \right] dxdy. \end{aligned}$$

Covering Q_k by a countable number of rectangles such that every point in Q_k is contained in at most 3 rectangles $R_{r,k}(\zeta)$, we obtain that

$$|G_k(u_n, v_n)| \leq 3C \sup_{\zeta \in \mathbb{R}} \|v_n\|_{L^2(R_{r,k}(\zeta))} \|u_n\|^2_{X_k}.$$

We conclude using the condition (2.7) and that $\{u_n\}_n$ is a bounded sequence in X_k that

$$\lim_{n \to \infty} |G_k(u_n, v_n)| = 0.$$

□

Now, we want to verify the Mountain Pass Theorem hypotheses given in Theorem 2.1 and to build a Palais-Smale sequence for J_k.

Lemma 2.3. *Let $0 < |c| < 1$. Then*

1. *There exists $\rho > 0$ small enough such that*

$$b := \inf_{\|z\|_{Z_k} = \rho} J_k(z) > 0.$$

2. *There is $e \in Z_k$ with $\|e\|_{Z_k} \geq \rho$ such that $J_k(e) \leq 0$.*

3. *If d is defined as*

$$d = \inf_{\pi \in \Gamma} \max_{t \in [0,1]} J_k(\pi(t)), \quad \Gamma = \{\pi \in C([0,1], Z_k) \ : \ \pi(0) = 0, \ \pi(1) = e\},$$

then $d \geq b$ and there is a sequence $(u_n, v_n)_n \subset Z_k$ such that

$$J_k(u_n, v_n) \to d, \quad J_k'(u_n, v_n) \to 0 \ \ in \ \ (Z_k)'.$$

Proof. From inequalities (2.5)-(2.6), we have for any $(u, v) \in Z_k$ that

$$\begin{aligned} J_k(u,v) &\geq C_1 \|(u,v)\|_{Z_k}^2 - C\|(u,v)\|_{Z_k}^3 \\ &\geq (C_1 - C\|(u,v)\|_{Z_k}) \|(u,v)\|_{Z_k}^2. \end{aligned}$$

Then for $\rho > 0$ small enough such that

$$C_1 - \rho C > 0, \tag{2.8}$$

we conclude for $\rho = \|(u,v)\|_{Z_k}$ that

$$J_k(u,v) \geq (C_1 - \rho C)\, \rho^2 := \delta > 0.$$

In particular, we also have that

$$b = \inf_{\|(u,v)\|_{Z_k} = \rho} J_k(u,v) \geq \delta > 0. \tag{2.9}$$

Now, it is not hard to prove that there exist $(u_0, v_0) \in C_0^\infty(Q_k) \times C_0^\infty(Q_k)$ such that $G_k(u_0, v_0) < 0$. Then for any $t \in \mathbb{R}$ we have that

$$\begin{aligned} J_k(t(u_0,v_0)) &= t^2 I_k(u_0,v_0) + t^3 G_k(u_0,v_0) \\ &= t^2 \left(I_k(u_0,v_0) + tG_k(u_0,v_0)\right). \end{aligned}$$

As a consequence of this, we have that

$$\lim_{t \to \infty} J_k(t(u_0,v_0)) = -\infty,$$

and so, there is $t_0 > 0$ such that $e = t_0(u_0, v_0) \in Z_k$ satisfies that $t_0\|(u_0,v_0)\|_{Z_k} = \|e\|_{Z_k} > \rho$ and that $J_k(e) \leq J_k(0) = 0$. The third part follows by applying Theorem 2.1. □

Theorem 2.2. *For $0 < |c| < 1$, the system (1.4) has a nontrivial solution in Z_k.*

Proof. We will see that d is in fact a critical value of J_k. Let $\{(u_n, v_n)\}_n \subset Z_k$ be the sequence given by previous lemma. First note from (2.9) that $d(c) \geq b(c) \geq \delta$. Using the definition of J_k and (2.3) we have that

$$I_k(u_n,v_n) = 3J_k(u_n,v_n) - \langle J_k'(u_n,v_n), (u_n,v_n)\rangle.$$

But from (2.6) we conclude for n large enough that

$$C_1\|(u_n,v_n)\|_{Z_k}^2 \leq I_k(u_n,v_n) \leq 3(d(c)+1) + \|(u_n,v_n)\|_{Z_k}.$$

Then we have shown that $\{(u_n, v_n)\}_n$ is a bounded sequence in Z_k. We claim that

$$\delta^* = \overline{\lim_{n\to\infty}} \sup_{\zeta \in \mathbb{R}} \int_{R_{1,k}(\zeta)} v_n^2 \, dxdy > 0.$$

If we suppose that

$$\overline{\lim_{n\to\infty}} \sup_{\zeta\in\mathbb{R}} \int_{R_{1,k}(\zeta)} v_n^2\, dxdy = 0.$$

Hence from Lemma 2.2 we conclude that

$$\lim_{n\to\infty} G(u_n, v_n) = 0.$$

Now, using (2.3), (2.5) and (2.9) we have that

$$\begin{aligned} 0 < \delta \le d &= J_k(u_n, v_n) - \frac{1}{2}\langle J_k'(u_n, v_n), (u_n, v_n)\rangle + o(1) \\ &= -\frac{1}{2} G_k(u_n, v_n) + o(1) \\ &\le |G_k(u_n, v_n)| + o(1) \\ &\le o(1). \end{aligned}$$

But this is a contradiction. Thus, there is a subsequence of $\{v_n\}_n$, denoted by the same symbol, and a sequence $\zeta_n \in Q_k$ such that

$$\int_{R_{1,k}(\zeta_n)} v_n^2\, dxdy \ge \frac{\delta^*}{2}. \tag{2.10}$$

We define the sequence $(\tilde{u}_n(x,y), \tilde{v}_n(x,y)) = (u_n(x, y+\zeta_n), v_n(x, y+\zeta_n))$. For this sequence we have that

$$\|(\tilde{u}_n, \tilde{v}_n)\|_{Z_k} = \|(u_n, v_n)\|_{Z_k}, \quad J_k(\tilde{u}_n, \tilde{v}_n) \to d, \quad J_k'(\tilde{u}_n, \tilde{v}_n) \to 0 \ \text{ in } \ {Z_k}'.$$

Then $\{(\tilde{u}_n, \tilde{v}_n)\}_n$ is a bounded sequence in Z_k. Thus, for some subsequence of $\{(\tilde{u}_n, \tilde{v}_n)\}_n$, denoted by the same symbol, and for some $(u, v) \in Z_k$ we have that

$$(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (u, v), \quad \text{as } n \to \infty \quad (\text{weakly in } Z_k).$$

Since the embedding $X_k \hookrightarrow L^q_{loc}(Q_k)$ is compact for $2 \le q < 6$ we see that

$$\tilde{v}_n \to v \ \text{ in } L^q_{loc}(Q_k).$$

Then $v \neq 0$ because using (2.10) we have that

$$\int_{R_{1,k}(0)} v^2\, dxdy = \lim_{n\to\infty} \int_{R_{1,k}(0)} (\tilde{v}_n)^2\, dxdy \ge \frac{\delta^*}{2}.$$

Moreover, if $(U, V) \in C_0^\infty(Q_k) \times C_0^\infty(Q_k)$, then for $K = supp\,(U, V)$ we have that

$$\begin{aligned} &\langle I_k'(u, v), (U, V)\rangle \\ &= \int_K \Big[uU + u_x U_x + \left(\partial_{x,k}^{-1} u_y\right)\left(\partial_{x,k}^{-1} U_y\right) + vV - cuV - cvU\Big] dxdy \\ &= \lim_{n\to\infty} \int_K \Big[\tilde{u}_n U + (\tilde{u}_n)_x U_x + \left(\partial_{x,k}^{-1} (\tilde{u}_n)_y\right)\left(\partial_{x,k}^{-1} U_y\right) + \tilde{v}_n V - c\tilde{u}V - c\tilde{v}U\Big] dxdy \\ &= \lim_{n\to\infty} \langle I_k'(\tilde{u}_n, \tilde{v}_n), (U, V)\rangle. \end{aligned}$$

Now, noting that $(\tilde{u}_n)^2 \rightharpoonup v^2$ and $\tilde{u}_n\tilde{v}_n \rightharpoonup uv$ in $L^2(Q_k)$ then (taking a subsequence, if necessary) we see that

$$\int_K \tilde{u}_n\tilde{v}_n U\,dxdy \to \int_K uvV\,dxdy, \quad \int_K (\tilde{u}_n)^2 V\,dxdy \to \int_K u^2 V\,dxdy.$$

In other words, we have shown that

$$\langle G_k'(u,v),(U,V)\rangle = \lim_{n\to\infty} \langle G_k'(\tilde{u}_n,\tilde{v}_n),(U,V)\rangle\,,$$

and also that

$$\langle J_k'(u,v),(U,V)\rangle = \lim_{n\to\infty} \langle J_k'(\tilde{u}_n,\tilde{v}_n),(U,V)\rangle = 0.$$

Now, let $(U,V)\in Z_k$. By density, there is $(U_n,V_n)\in C_0^\infty(\mathbb{R}^2)\times C_0^\infty(\mathbb{R}^2)$ such that

$$(U_m,V_m)\to(U,V) \ \text{ in } \ Z_k.$$

Then

$$\begin{aligned}|\langle J_k'(u,v),(U,V)\rangle| &\le |\langle J_k'(u,v),(U,V)-(U_n,V_n)\rangle| + |\langle J_k'(u,v),(U_n,V_n)\rangle| \\ &\le \|J_k'(u,v)\|_{Zk'}\,\|(U,V)-(U_n,V_n)\|_{Z_k} + |\langle J_k'(u,v),(U_n,V_n)\rangle| \to 0.\end{aligned}$$

Thus we have already established that $J_k'(u,v)=0$. In other words, (u,v) is a nontrivial solution for equation (1.4). □

3 One-Dimensional Periodic Traveling Waves

In this section we show the existence of 1D periodic traveling waves of period $2k$ for the system (1.1). The result will be a direct consequence of the coerciveness of the associated functional to the system (1.5) and that such functional is (sequentially) weakly lower semi-continuous.

We can see that solutions (ψ,ϕ) of the system (1.5) are critical points of the functional J_k, in this case, given by

$$J_k = I_k + G_k,$$

where the functionals I_k and G_k are defined by

$$\begin{aligned}I_k(\psi,\phi) &= \frac{1}{2}\int_{-k}^{k}\Big[2\psi^2+(\psi')^2+\phi^2-2c\psi\phi\Big]dx,\\ G_k(\psi,\phi) &= \frac{1}{2}\int_{-k}^{k}\phi\psi^2\,dx.\end{aligned}$$

A direct computation shows that

$$\begin{aligned}\langle I_k'(\psi,\phi),(\Psi,\Theta)\rangle &= \int_{-k}^{k}\Big[2\psi\Psi+\psi'\Psi'+\phi\Theta-c\psi\Theta-c\phi\Psi\Big]dx,\\ \langle G_k'(\psi,\phi),(\Psi,\Theta)\rangle &= \int_{-k}^{k}\Big[\tfrac{1}{2}\psi^2\Theta+\psi\phi\Psi\Big]\,dx.\end{aligned}$$

As a consequence of this we conclude that

$$J_k'(\psi,\phi) = \begin{pmatrix}-c\phi+2\psi-\psi''+\psi\phi\\ -c\psi+\phi+\tfrac{1}{2}\psi^2\end{pmatrix},$$

meaning that a critical point ϕ of the functional J_k satisfies the traveling wave equation (1.5). Hereafter, we will say that weak solutions for (1.5) are critical points of the functional J_k. In particular, we have that

$$\langle J_k'(\psi,\phi),(\psi,\phi)\rangle = 2I_k(\psi,\phi)+3G_k(\psi,\phi) = 2J_k(\psi,\phi)+G_k(\psi,\phi). \tag{3.1}$$

Now, we define the space $Z_k = Z_k(\mathbb{R})$ as

$$Z_k = H_k^s(\mathbb{R}) \times L_k^2(\mathbb{R}),$$

where $H_k^s(\mathbb{R}) = H_k^s([-k,k])$ denotes the usual Sobolev space of $2k$-periodic functions. Then, following the same way as in (2.5)-(2.6) we have some properties of I_k and G_k,

Lemma 3.1. *For $0<|c|<1$, we have that $I_k(\psi,\phi)\geq 0$. Moreover, there is a positive constant $C_1 = C_1(c)$ such that*

$$C_1^{-1}\|(\psi,\phi)\|_{Z_k}^2 \leq I_k(\psi,\phi) \leq C_1\|(\psi,\phi)\|_{Z_k}^2. \tag{3.2}$$

Lemma 3.2. *There is $C_2>0$ such that*

$$|G_k(\psi,\phi)| \leq C_2\|(\psi,\phi)\|_{Z_k}^3. \tag{3.3}$$

Next, we show the following result on J_k.

Lemma 3.3. *Assume that the sequence $(\psi_n,\phi_n)_n \subset Z_k$ converges weakly to $(\psi_0,\phi_0)\in Z_k$. If $\{\psi_n\}_n$ converges uniformly to ψ_0 on $[-k,k]$, then we have that*

$$\liminf_{n\to\infty} J_k(\psi_n,\phi_n) \geq J_k(\psi_0,\phi_0). \tag{3.4}$$

Proof. Recall that $J_k = I_k + G_k$. Now, from (3.2) we have that I_k is like a norm in Z_k, so is convex. More exactly, for $\lambda\in(0,1)$ we have that

$$I_k(\psi_n,\phi_n) \geq I_k(\lambda(\psi_0,\phi_0)) + \big\langle I_k'(\lambda(\psi_0,\phi_0)),(\psi_n,\phi_n)-\lambda(\psi_0,\phi_0)\big\rangle. \tag{3.5}$$

Using the formula of I_k' we have that

$$\begin{aligned}&\big\langle I_k'(\lambda(\psi_0,\phi_0)),(\psi_n-\lambda\psi_0,\phi_n-\lambda\phi_0)\big\rangle\\ =&\lambda\int_{-k}^{k}\Big[2\psi_0(\psi_n-\lambda\psi_0)+\psi_0'(\psi_n'-\lambda\psi_0')+\phi_0(\phi_n-\lambda\phi_0)\Big]dx\\ &-\lambda c\int_{-k}^{k}\Big[\psi_0(\phi_n-\lambda\phi_0)+\phi_0(\psi_n-\lambda\psi_0)\Big]dx.\end{aligned}$$

Since the sequence $\{(\psi_n,\phi_n)\}_n$ converges weakly to (ψ_0,ϕ_0) in Z_k we conclude that

$$\lim_{n\to\infty}\big\langle I_k'(\lambda(\psi_0,\phi_0)),(\psi_n-\lambda\psi_0,\phi_n-\lambda\phi_0)\big\rangle = 2\lambda(1-\lambda)I_k(\psi_0,\phi_0).$$

In other words, we have that

$$\liminf_{n\to\infty} I_k(\psi_n,\phi_n) \geq I_k(\lambda(\psi_0,\phi_0)) + 2\lambda(1-\lambda)I_k(\psi_0,\phi_0) = \lambda(2-\lambda)I_k(\psi_0,\phi_0),$$

which implies after taking $\lambda\to 1^-$ that

$$\liminf_{n\to\infty} I_k(\psi_n,\phi_n) \geq I_k(\psi_0,\phi_0). \tag{3.6}$$

Now, we need to observe that

$$\int_k^k \phi_n\psi_n^2\,dx = \int_k^k \phi_n\left((\psi_n)^2-(\psi_0)^2\right)dx + \int_k^k \phi_n(\psi_0)^2dx.$$

Since we know that $(\psi_n)^2, (\psi_0)^2 \in L^2[-k,k]$, we conclude that

$$\lim_{n\to\infty}\int_k^k \phi_n(\psi_0)^2\,dx = \int_k^k \phi_0(\psi_0)^2\,dx.$$

Moreover, using the uniform convergence of $(\psi_n)_n$ to ψ_0 we also have that

$$\begin{aligned}\left|\int_k^k \phi_n\left((\psi_n)^2-(\psi_0)^2\right)dx\right| &\le \int_k^k |\phi_n||\psi_n-\psi_0|\,(|\psi_n|+|\psi_0|)\,dx\\ &\le \sup_{[-k,k]}|\psi_n-\psi_0|\|\phi_n\|_{L^2}\left(\|\psi_n\|_{L^2}+\|\psi_0\|_{L^2}\right)\\ &\le \sup_{[-k,k]}|\psi_n-\psi_0|\|\phi_n\|_{Z_k}\left(\|\psi_n\|_{Z_k}+\|\psi_0\|_{Z_k}\right).\end{aligned}$$

which means, after recalling that the sequence $(\psi_n,\phi_n)_n$ is bounded, that

$$\lim_{n\to\infty}\int_k^k \phi_n(\psi_n)^2\,dx = \int_k^k \phi_0(\psi_0)^2\,dx.$$

Therefore

$$\lim_{n\to\infty} G_k(\psi_n,\phi_n) = G_k(\psi_0,\phi_0).$$

As a consequence of previous remarks, we conclude that

$$\liminf_{n\to\infty} J_k(\psi_n,\phi_n) = \liminf_{n\to\infty}\left(I_k(\psi_n,\phi_n)+G_k(\psi_n,\phi_n)\right) \ge J_k(\psi_0,\phi_0).$$

□

We consider the weakly closed subset of Z_k

$$Z_{\alpha,k} = \{(\psi,\phi)\in Z_k \;:\; |\psi(x)|\le\alpha,\quad \text{a. e. } x\in\mathbb{R}\}.$$

Lemma 3.4. *1. There are positive constants C_1 and C_2 such that for any $(\psi,\phi)\in Z_k$, we have that*

$$J_k(\psi,\phi) \ge C_1\|(\psi,\phi)\|_{Z_k}^2 - C_2\|(\psi,\phi)\|_{Z_k}^3. \tag{3.7}$$

2. There exits $\alpha_0>0$ such that for $0<\alpha<\alpha_0$ the functional J_k is coercive on $Z_{\alpha,k}$. More exactly, there is $C_3>0$ such that for $(\psi,\phi)\in Z_{\alpha,k}$,

$$J_k(\psi,\phi) \ge C_3\|(\psi,\phi)\|_{Z_{\alpha,k}}^2. \tag{3.8}$$

Proof. 1. From inequalities (3.2)-(3.3), there are positive constants C_1 and C_2 such that

$$J_k(\psi,\phi) = I_k(\psi,\phi) + G_k(\psi,\phi) \ge C_1^{-1}\|(\psi,\phi)\|_{Z_k}^2 - C_2\|(\psi,\phi)\|_{Z_k}^3.$$

2. Let $(\psi,\phi)\in Z_{\alpha,k}$. Then $|\psi(x)|\le\alpha$ for a. e. $x\in\mathbb{R}$. Thus,

$$|G_k(\psi,\phi)| \le \frac{\alpha}{2}\int_k^k |\phi\psi|\,dx \le \frac{\alpha}{2}\|(\psi,\phi)\|_{Z_k}^2.$$

Hence, there is $C > 0$ such that

$$|G_k(\psi, \phi)| \leq \alpha C \|(\psi, \phi)\|^2_{Z_k}.$$

So, using inequality (3.2) and previous one, we have that

$$J_k(\psi, \phi) \geq C_1^{-1}\|(\psi, \phi)\|^2_{Z_k} - \alpha C\|(\psi, \phi)\|^2_{Z_k} = \left(C_1^{-1} - \alpha C\right)\|(\psi, \phi)\|^2_{Z_k},$$

as desired. □

Our goal now is to show the existence of a non trivial critical point for J_k. The result will be a direct consequence of the coerciveness of J_k and that J_k is (sequentially) weakly lower semi-continuous on $Z_{k,\alpha}$ for $0 < \alpha < \alpha_0$. We will use the Arzela-Ascoli Theorem and the following result (see Theorem 1.2 in [9]).

Theorem 3.1. *Let X be a Hilbert space and let $M \subset X$ be a weakly closed subset of X. Suppose that $E : M \to \mathbb{R} \cup \{+\infty\}$ is coercive and that is (sequentially) weakly lower semi-continuous on M with respect to X, that is, suppose the following conditions are fulfilled:*

1. *$E(u) \to \infty$ as $\|u\| \to \infty$, with $u \in M$.*
2. *For any $u \in M$, any sequence $(u_n)_n$ in M such that $u_n \rightharpoonup u$ (weakly) in X there holds:*

$$E(u) \leq \liminf_{n\to\infty} E(u_n).$$

Then E is bounded below on M and attains its minimum in M.

Theorem 3.2. *If $0 < |c| < 1$ then for $0 < \alpha < \alpha_0$, J_k has a minimum over $Z_{\alpha,k}$. Therefore, the system (1.5) has a nontrivial solution in Z_k.*

Proof. We will verify that J_k satisfies the hypotheses in Theorem 3.1. It is straightforward to check that $Z_{\alpha,k}$ is weakly closed subset of Z_k. In fact, let $\{(\psi_n, \phi_n)\}_n \subset Z_{\alpha,k}$ be a sequence that converges weakly to (ψ_0, ϕ_0). Then we have that the sequence $\{(\psi_n, \phi_n)\}_n$ is bounded in $Z_{\alpha,k}$. Now, we see that

$$|\psi_n(x) - \psi_n(y)| \leq \int_y^x |\psi_n'(r)|dr \leq |x - y|^{\frac{1}{2}}\|\psi_n\|_{Z_k} \leq M|x - y|^{\frac{1}{2}}.$$

In other words $\{\psi_n\}_n$ is equicontinuous, then by using the Arzela-Ascoli Theorem we have for some subsequence (which we denote by the same symbol) that $(\psi_n)_n$ converges uniformly to ψ_0 on $[-k, k]$, since we have that $|\psi_n(x)| \leq \alpha$ for a. e. $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. From this fact and the uniform convergence of $\{\psi_n\}_n$ we conclude that $|\psi_0(x)| \leq \alpha$ for a. e. $x \in \mathbb{R}$. Then $(\psi_0, \phi_0) \in Z_{\alpha,k}$, meaning that $Z_{\alpha,k}$ is weakly closed subset of Z_k. Now note that the coerciveness property of J_k and condition (1) in Theorem 3.1 are obtained using the inequality (3.8) in previous lemma. We need now to verify condition (2). Let $(\psi_0, \phi_0) \in Z_{\alpha,k}$ and let $\{(\psi_n, \phi_n)\}_n \subset Z_k$ such that $(\psi_n, \phi_n) \rightharpoonup (\psi_0, \phi_0)$ (weakly) in $Z_{\alpha,k}$. This sequence $\{(\psi_n, \phi_n)\}_n$ is bounded in Z_k and the same type of arguments show that$\{\psi_n\}_n$ converges uniformly to ψ_0 on $[-k, k]$ (up to a subsequence), so by Lemma 3.3 we conclude that

$$\liminf_{n\to\infty} J_k(\psi_n, \phi_n) \geq J_k(\psi_0, \phi_0).$$

Then, from Theorem 3.1 we conclude that J_k attains a minimum over $Z_{\alpha,k}$. □

Acknowledgments

A. Montes was supported by Universidad del Cauca under the project ID 5101.

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