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On the Diophantine equation $F_n = x^a \pm x^b \pm 1$ in Mersenne and Fermat Numbers

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Abstract. In this article we investigate on the representation of Fibonacci numbers in the form $x^a \pm x^b \pm 1$, for x in the sequence of Mersenne and Fermat numbers.

Keywords. Fibonacci numbers, Mersenne and Fermat numbers, exponential Diophantine equations, lower bounds for linear forms in logarithms

1 Introduction

The Fibonacci sequence $(F_n)_{n\geq 0}$ is given by $F_0 = 0$, $F_1 = 1$ and

$$F_{n+2} = F_{n+1} + F_n \qquad \text{for all} \qquad n \geq 0.$$

Luca and Szalay [8] showed that the number of quadruples (n, p, a, b) satisfies the Diophantine equatin $F_n = p^a \pm p^b + 1$ is finite. Hernndez [5] has studied the particular cases $p = 2,\ 3,\ 5,\ 7,\ 11,\ 13$ showing that all solutions of the above Diophantine equation, are

$$\begin{aligned}
p &= 2: & & 3 = F_4 = 2^2 - 2^1 + 1, \quad F_5 = 2^3 - 2^2 + 1,\\
& & & 13 = F_7 = 2^3 + 2^2 + 1 = 2^4 - 2^2 + 1,\\
& & & 21 = F_8 = 2^4 + 2^2 + 1;\\
p &= 3: & & 13 = F_7 = 3^2 + 3^1 + 1, \quad F_{10} = 3^4 - 3^3 + 1;\\
p &= 5: & & 21 = F_8 = 5^2 - 5^1 + 1;
\end{aligned}$$

while for $p = 7, 11$ and 13 there are not solutions, hinting that for all prime $p \geq 7$ doesn't have any solution. Laishram and Luca [7] studied a more general Diophantine equation $F_n = x^a \pm x^b \pm 1$ with x composed of two prime divisors, showing that it has only finitely many positive integer solutions (n, x, a, b) with $\max\{a, b\} \geq 2$. Recently, Kafle, Rihane and Togb [6] have investigated about Pell and Pell–Lucas numbers (instead of Fibonacci numbers) of the form $x^a \pm x^b + 1$, completely solving this equation for each $x \in [2, 20]$.

In this paper, we study the Diophantine equation

$$F_n = x^a \pm x^b \pm 1, \qquad \text{for positive integers} \quad a > b \geq 1 \tag{1.1}$$

and x a Mesenne or Fermat number.

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2 The main result

Since $F_0 = 0$ and $F_1 = F_2 = 1$, we will assume $n \geq 3$ to avoid the trivial situations. We put $x := x_\ell = 2^\ell \pm 1$ with $\ell \geq 1$, to denote Mersenne and Fermat numbers. Note that the first Mersenne number is 1, while the second Mersenne number and the first Fermat number are 3, so we can assume also that $\ell \geq 2$.

We prove the following theorem.

Theorem 2.1. *The solutions of Diophantine equation $F_n = x_\ell^a + \epsilon_1 x_\ell^b + \epsilon_2$, with $\ell \geq 2$, $a > b \geq 1$, $n \geq 3$ and $\epsilon_i \in \{\pm 1\}$ (for $i = 1, 2$), are the quadruples (n, x_ℓ, a, b):*

$$
\begin{aligned}
(\epsilon_1, \epsilon_2) &= (1, -1): \quad (10, 7, 2, 1),\ (11, 3, 4, 2),\ (11, 9, 2, 1);\\
(\epsilon_1, \epsilon_2) &= (-1, 1): \quad (8, 5, 2, 1),\ (10, 3, 4, 3);\\
(\epsilon_1, \epsilon_2) &= (-1, -1): \quad (5, 3, 2, 1),\ (13, 3, 5, 2);\\
(\epsilon_1, \epsilon_2) &= (1, 1): \quad (7, 3, 2, 1).
\end{aligned}
$$

3 Some Lemmas

To start, we present an analytic argument which is Lemma 7 from [4].

Lemma 3.1. *If $m \geq 1$ is an integer and y, T are real numbers such that $T > (4m^2)^m$ and*

$$\frac{y}{(\log y)^m} < T, \quad \text{then} \quad y < 2^m T (\log T)^m.$$

3.1 Lower bounds for linear forms in logarithms

In order to find upper bounds for the integer unknowns of exponential Diophantine equation (1.1), we use a Baker–type lower bounds for nonzero linear forms in logarithms of real algebraic numbers. We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers $f(z) := a_0 \prod_{i=1}^{d} (z - \eta^{(i)}) \in \mathbb{Z}[z]$, where the leading coefficient a_0 is positive. The logarithmic height of η is given by

$$h(\eta) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\} \right).$$

In particular, if $\eta = p/q \in \mathbb{Q}$ with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in this paper: $h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2$, $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$ and $h(\eta^s) = |s| h(\eta)$ $(s \in \mathbb{Z})$.

Many Diophantine problems can be solved by reducing them to an instance in which one can apply lower bounds for linear forms in logarithms of algebraic numbers. We will use the following theorem, which is a variation of a result of Matveev [9], proved by Bugeaud, Mignotte and Siksek [1, Theorem 9.1].

Lemma 3.2. *Let $\mathbb{L} \subseteq \mathbb{R}$ be a real algebraic number field of degree $d_{\mathbb{L}}$ over $\mathbb{Q}$, $\eta_1, \ldots, \eta_l$ non–zero elements of $\mathbb{L}$, and $d_1, \ldots, d_l$ rational integers. Put $\Lambda := \eta_1^{d_1} \cdots \eta_l^{d_l} - 1$ and $D \geq$*

$\max\{|d_1|, \dots, |d_l|\}$. *Let* $A_i \geq \max\{d_{\mathbb{L}} h(\eta_i), |\log \eta_i|, 0.16\}$ *be real numbers, for* $i = 1, \dots, l$. *Then, assuming that* $\Lambda \neq 0$, *we have*

$$|\Lambda| > \exp(-1.4 \cdot 30^{l+3} \cdot l^{4.5} \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) A_1 \cdots A_l).$$

Note that, for $\eta_1, \dots, \eta_l$ real algebraic numbers,

$$\Lambda := \eta_1^{b_1} \cdots \eta_l^{b_l} - 1 \quad \text{and} \quad \Gamma := b_1 \log \eta_1 + \cdots + b_l \log \eta_l,$$

we have $\Lambda = e^\Gamma - 1$. It is a straight–forward exercise to show that $|\Gamma| < c^{-1}|\Lambda|$, when $|\Lambda| < c$, for all constant c in $(0, 1)$. We use this argument in several occasions without mentioning it.

3.2 Continued fractions

To lower the upper bound of the integer unknowns obtain by the above result, we will use a result from the theory of continued fractions. The following lemma is essentially a result due to Dujella and Pethő [3], for more details see Lemma 3 in [2]. For a real number X, we put $||X|| := \min\{|X - n| \,:\, n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3.3. *Let* M *and* Q *be positive integers such that* $Q > 6M$, *and* A, B, τ, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Let further* $\varepsilon := ||\mu Q|| - M||\tau Q||$. *If* $\varepsilon > 0$, *then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v *and* w *with*

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(AQ/\varepsilon)}{\log B}.$$

In practical applications Q is always the denominator of a convergent of the continued fraction of τ, though this is not formally required for the statement.

4 The Proof of Theorem 2.1

Recall that for nonnegative integer n, Binet's formula for the nth Fibonacci number say that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}, \qquad \text{where} \quad \alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 - \sqrt{5}}{2}. \tag{4.1}$$

Further, it's well–known that inequalities

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \qquad \text{hold for all} \quad n \geq 1.$$

Using the notation $x := x_\ell = 2^\ell \pm 1$ and the above inequality, we deduce from (1.1)

$$2^{n-1} > \alpha^{n-1} \geq F_n = x_\ell^a \pm x_\ell^b \pm 1 \geq x_\ell^a \left(1 - x_\ell^{b-a} - x_\ell^{-a}\right) > 2^{(\ell-1)a-1}$$

and

$$\alpha^{n-2} \leq F_n = x_\ell^a \pm x_\ell^b \pm 1 < x_\ell^a \left(1 + x_\ell^{b-a} + x_\ell^{-a}\right) < 2^{a\ell+1} \left(1 + 2^{-\ell}\right)^a < 2^{a(\ell+1)+1}.$$

Thus,

$$a(\ell-1) < n < 2a(\ell+1). \tag{4.2}$$

In the above inequality we have used the fact that $\log 2/\log\alpha < 1.45$ and

$$1.45\,(a(\ell+1)+1)+2 < 2a(\ell+1)$$

for all $a \geq 2$ and $\ell \geq 2$.

4.1 An inequality for a in terms of n

We begin bounding the gap between a and b. Since (4.1), equation (1.1) can be rewritten as

$$\frac{\alpha^n}{\sqrt{5}} - x_\ell^a = \pm x_\ell^b + \frac{\beta^n}{\sqrt{5}} \pm 1.$$

Dividing both sides of the above equality by x_ℓ^a and taking absolute values, we get

$$\left|(\sqrt{5})^{-1}\alpha^n x_\ell^{-a} - 1\right| \leq \frac{1}{x_\ell^{a-b}} + \frac{(\sqrt{5})^{-1}|\beta|^n+1}{x_\ell^a} \leq \frac{3}{2^{(\ell-1)(a-b)}} \tag{4.3}$$

where we have used the facts that $|\beta|<1$ and $x_\ell > 2^{\ell-1}$.

We use Lemma 3.2 on the left–hand side of (4.3) with the data $l := 3$, $(\eta_1, d_1) := (\sqrt{5}, -1)$, $(\eta_2, d_2) := (\alpha, n)$, $(\eta_3, d_3) := (x_\ell, -a)$ and $\Lambda_1 := (\sqrt{5})^{-1}\alpha^n x_\ell^{-a} - 1$. The quadratic field $\mathbb{L} = \mathbb{Q}(\sqrt{5})$ contains η_1, η_2, η_3, so $d_{\mathbb{L}} = 2$. We continue with the calculation of the logarithmic heights of η_1, η_2, η_3:

$$h(\eta_1) = (\log 5)/2, \quad h(\eta_2) = (\log\alpha)/2 \quad \text{and} \quad h(\eta_3) = \log(x_\ell) < \ell,$$

so we can take $A_1 = \log 5$, $A_2 = \log\alpha$ and $A_3 = 2\ell$. Furthermore, from inequalities (4.2) we take $D := n$. In order to continue with our application of Lemma 3.2, we need to show that $\Lambda_1 \neq 0$. This assertion follows from the observation that otherwise, we get the equation $\alpha^{2n} = 5x_\ell^{2a} \in \mathbb{Z}^+$, which is not possible for any $n \geq 3$ given that α is a unit in $\mathcal{O}_{\mathbb{L}}$.

Now Lemma 3.2 tells us that

$$\begin{aligned}\log|\Lambda_1| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 (1+\log 2)(1+\log n)(\log 5)(\log\alpha)(2\ell)\\ &> -2.9 \cdot 10^{12} \cdot \ell\log n\end{aligned}$$

where we have used that $1 + \log n < 1.92\log n$, for all $n \geq 3$.

Comparing with (4.3), we get

$$(\ell-1)(a-b)\log 2 < 2.9 \cdot 10^{12} \cdot \ell\log n + \log 3.$$

Therefore

$$a - b < 8.4 \cdot 10^{12} \cdot \log n. \tag{4.4}$$

Returning to equation (1.1), we can deduce

$$\left|(\sqrt{5})^{-1}\alpha^n x_\ell^{-a}\left(1 \pm x_\ell^{b-a}\right)^{-1} - 1\right| < \frac{(\sqrt{5})^{-1}|\beta|^n+1}{x_\ell^a} < \frac{2}{2^{(\ell-1)a}}. \tag{4.5}$$

We use again Lemma 3.2 to give a lower bound on the left–hand side of (4.5). We now take $l := 4$, $(\eta_1, d_1) := (\sqrt{5}, -1)$, $(\eta_2, d_2) := (\alpha, n)$, $(\eta_3, d_3) := (x_\ell, -a)$, $(\eta_4, d_4) := (1 \pm x_\ell^{b-a}, -1)$ and $\Lambda_2 := (\sqrt{5})^{-1}\alpha^n x_\ell^{-a}(1 \pm x_\ell^{b-a})^{-1} - 1$.

As before, we put $\mathbb{L} = \mathbb{Q}(\sqrt{5})$, $d_{\mathbb{L}} = 2$, $A_1 = \log 5$, $A_2 = \log \alpha$, $A_3 = 2\ell$ and $D := n$. Furthermore,

$$h(1 \pm x_\ell^{b-a}) \leq (a-b)\log(x_\ell) + \log 2 < 1.5\ell(a-b).$$

So, we can take $A_4 := 3\ell(a-b)$. By the same arguments used before to see that $\Lambda_1 \neq 0$, we conclude that $\Lambda_2 \neq 0$. Indeed, if $\Lambda_2 = 0$, then $\alpha^n = \sqrt{5}(x_\ell^a \pm x_\ell^b)$, so $\alpha^{2n} = 5(x_\ell^a \pm x_\ell^b)^2 \in \mathbb{Z}^+$, an impossible fact for all $n \geq 3$.

By Lemma 3.2, we can conclude that

$$\log|\Lambda_2| > -10^{15} \cdot \ell(a-b)\log n.$$

Combining the above inequality with (4.5), we have

$$a < 2.9 \cdot 10^{15}(a-b)\log n. \tag{4.6}$$

Thus, including the bound to $a - b$ from (4.4), we obtain an upper bound on a in terms of n.

Let us record what we have proved so far.

Lemma 4.1. *Let (n, ℓ, a, b) be a solution of Diophantine equation (1.1) with $n \geq 3$, $\ell \geq 2$ and $a > b \geq 1$, then $a(\ell-1) < n < 2a(\ell+1)$ and*

$$a < 2.5 \cdot 10^{28}(\log n)^2.$$

4.2 Absolute bounds on n, a and ℓ

For technical reasons, we assume that $\ell > 130$. Note that

$$x_\ell^a = 2^{\ell a}\left(1 + \frac{\epsilon}{2^\ell}\right)^a, \qquad \text{with} \quad \epsilon := \pm 1.$$

We set the elements

$$z_a := \frac{a\epsilon}{2^\ell} \qquad \text{and} \qquad r_a := \left(1 + \frac{\epsilon}{2^\ell}\right)^a.$$

By Lemma 4.1 and inequality (4.2), we have that

$$a < 2.6 \cdot 10^{28}(\log a)^2(\log \ell)^2, \qquad \text{or equivalently} \qquad \frac{a}{(\log a)^2} < 2.6 \cdot 10^{28}(\log \ell)^2.$$

Taking $y := a$, $m := 2$ and $T := 2.6 \cdot 10^{28}(\log \ell)^2$, and applying Lemma 3.1 in the above inequality, we obtain

$$a < 2.4 \cdot 10^{31}(\log \ell)^4. \tag{4.7}$$

So, we conclude that

$$|z_a| = \frac{a}{2^\ell} < \frac{2.4 \cdot 10^{31}(\log \ell)^4}{2^\ell} < \frac{1}{2^{\ell/10}},$$

where the last inequality holds for all $\ell > 130$. In particular, $|z_a| < 10^{-3}$.

Hence, if $\epsilon = -1$, then

$$1 > r_a = \left(1 - 2^{-\ell}\right)^a = \exp\left(a \log\left(1 - 2^{-\ell}\right)\right) \geq \exp(-2|z_a|) > 1 - 2|z_a|,$$

while if $\epsilon = 1$, then

$$1 < r_a = \left(1 + 2^{-\ell}\right)^a = \left(1 + \frac{|z_a|}{a}\right)^a < \exp(|z_a|) < 1 + 2|z_a|,$$

because $2^{-\ell}$ and $|z_a|$ are very small. Thus, in either case we have that

$$\left|x_\ell^a - 2^{\ell a}\right| < 2|z_a|2^{\ell a} = a2^{\ell(a-1)+1}. \tag{4.8}$$

We return once more to (1.1), where we use inequality (4.8), to obtain this time

$$\left|(\sqrt{5})^{-1}\alpha^n 2^{-\ell a} - 1\right| < \frac{2a}{2^\ell} + \frac{x_\ell^b}{2^{\ell a}} + \frac{(\sqrt{5})^{-1}|\beta|^n + 1}{2^{\ell a}} < \frac{4a}{2^\ell}. \tag{4.9}$$

We have used in the above inequality that $x_\ell^b = 2^{\ell b}(1 \pm 2^{-\ell})^b < 2^{\ell b}(1 + 2z_b) < 2^{\ell b + 0.1}$.

Putting again $\mathbb{L} := \mathbb{Q}(\sqrt{5})$, $(\eta_1, d_1) := (\sqrt{5}, -1)$, $(\eta_2, d_2) := (\alpha, n)$, $(\eta_3, d_3) := (2, -\ell a)$ and $\Lambda_3 := (\sqrt{5})^{-1}\alpha^n 2^{-\ell a} - 1$, we can again take $A_1 = \log 5$, $A_2 = \log\alpha$, $A_3 = 2\log 2$. By (4.2), we now take $D := 2a(\ell + 1)$. Furthermore, just as for Λ_1, we can ensure that $\Lambda_3 \neq 0$. By Lemma 3.2, we can conclude that

$$\log|\Lambda_3| > -4.2 \cdot 10^{12} \cdot \log(\ell a).$$

Comparing the above inequality with (4.9), we conclude that

$$\ell \log 2 < \log(4a) + 4.2 \cdot 10^{12} \cdot \log(\ell a) < 4.3 \cdot 10^{12} \cdot \log(\ell a). \tag{4.10}$$

Furthermore, by (4.7) we deduce that

$$\log(\ell a) < \log(2.4 \cdot 10^{31}\ell(\log\ell)^4) < 17.2\log\ell, \quad \text{for all} \quad \ell > 130.$$

Thus, we have from (4.10) that $\ell < 1.1 \cdot 10^{14}\log\ell$ which leads to $\ell < 4 \cdot 10^{15}$. Besides, by inequalities (4.2) and (4.7) we conclude that $a < 4 \cdot 10^{37}$ and $n < 3.2 \cdot 10^{53}$. We note that the above inequalities were obtained under the assumption that $\ell > 130$. However, one can see that if $\ell \leq 130$, then by (4.2) we obtain that $n < 262a$, so Lemma 4.1 leads to $a < 2.5 \cdot 10^{28}(\log(262a))^2$ which is true only for $a < 1.6 \cdot 10^{32}$ and again by (4.2), we have $n < 2.1 \cdot 10^{34}$, which are bounds on ℓ, a and n smaller than the ones above. Thus, we can state the following result.

Lemma 4.2. *Let (n, ℓ, a, b) be a solution of Diophantine equation (1.1) with $n \geq 3$, $\ell \geq 2$ and $a > b \geq 1$, then*

$$\ell < 4 \cdot 10^{15}, \qquad b < a < 4 \cdot 10^{37} \qquad \text{and} \qquad n < 3.2 \cdot 10^{53}.$$

4.3 Reductions of the absolut bounds

We note that the upper bounds given in Lemma 4.2 are too large to allow computing. Therefore, we transform inequalities (4.3), (4.5) and (4.9) in inequalities for linear forms in logarithms and use continued fractions to reduce the upper bounds on ℓ, $a - b$, a and then on n.

We take $\Gamma_3 := n\log\alpha - \ell a\log 2 - \log(\sqrt{5})$, so $e^{\Gamma_3} - 1 = \Lambda_3$. From estimates (4.7) and (4.9), we conclude that inequality $\left|e^{\Gamma_3} - 1\right| < 4a/2^\ell \leq 1/2$ holds for $\ell > 130$. Hence,

$$0 < \left|n\log\alpha - \ell a\log 2 - \log(\sqrt{5})\right| < \frac{8a}{2^\ell}.$$

Our first objective is to obtain a small upper bound to ℓ. Thus, dividing both sides of the above inequality by $\log 2$, we get

$$0 < \left| n\frac{\log \alpha}{\log 2} - \ell a - \frac{\log(\sqrt{5})}{\log 2} \right| < \frac{12a}{2^\ell}.$$

We put

$$\tau := \frac{\log \alpha}{\log 2}, \qquad \mu := -\frac{\log(\sqrt{5})}{\log 2}, \qquad A := 12a \qquad \text{and} \qquad B := 2.$$

Thus, the above inequality can be rewriten as

$$0 < |n\tau - \ell a + \mu| < AB^{-\ell}. \tag{4.11}$$

We take $M := 3.2 \cdot 10^{53}$ (an upper bound to n according to Lemma 4.2) and apply Lemma 3.3, with $(m, s, k) = (n, \ell a, \ell)$, on inequality (4.11). With the help of Mathematica, we found that $q_{111} > 1.17 \cdot 10^{55} > 6M$ is a denominator of a convergent of the continued fraction of τ such that $\epsilon := \|\mu q_{111}\| - M\,\|\tau q_{111}\| > 0.1547$. Then, from the conclusion of Lemma 3.3 and inequality (4.7), we have that

$$\ell < \frac{\log(12 \cdot a \cdot q_{111}/\epsilon)}{\log 2} < \frac{\log(12 \cdot 2.4 \cdot 10^{31}(\log \ell)^4) \cdot q_{111}/\epsilon)}{\log 2},$$

which leads to $\ell \le 310$. Returning to inequalities (4.2) and (4.7), we conclude that $a < 2.6 \cdot 10^{34}$ and so $n < 1.7 \cdot 10^{37}$. Running once more the reduction cycle on inequality (4.11), with $M := 1.7 \cdot 10^{37}$, we have that $q_{83} > 6M$ and $\epsilon > 0.4884$, with which

$$\ell \le 250, \qquad a < 2.3 \cdot 10^{34} \qquad \text{and} \qquad n < 1.2 \cdot 10^{37}. \tag{4.12}$$

Now, assume that $a - b \ge 2$ and $\Gamma_1 := n \log \alpha - a \log(x_\ell) - \log(\sqrt{5})$. From estimate (4.3), we get $\left|e^{\Gamma_1} - 1\right| < 3/2^{a-b} < 3/4$, so

$$0 < \left| n \log \alpha - a \log(x_\ell) - \log(\sqrt{5}) \right| < \frac{12}{2^{a-b}}.$$

We divide both sides by $\log(x_\ell) \ge \log 3$ (since $x_\ell \ge 3$ for all $\ell \ge 2$), to obtain

$$0 < \left| n\frac{\log \alpha}{\log(x_\ell)} - a - \frac{\log(\sqrt{5})}{\log(x_\ell)} \right| < \frac{12}{2^{a-b}\log(x_\ell)} < \frac{11}{2^{a-b}}.$$

Putting

$$\tau_\ell := \frac{\log \alpha}{\log(x_\ell)}, \qquad \mu_\ell := -\frac{\log\left(\sqrt{5}\right)}{\log(x_\ell)}, \qquad A := 11 \qquad \text{and} \qquad B := 2;$$

the last inequality leads to

$$0 < |n\tau_\ell - a + \mu_\ell| < AB^{-(a-b)}. \tag{4.13}$$

We now take $M := 1.2 \cdot 10^{37}$ which is a new upper bound on n according to (4.12), and apply Lemma 3.3 with $(m, s, k) = (n, a, a - b)$, to inequality (4.13), for all $\ell \in [2, 250]$ obtained in (4.12) too. For each τ_ℓ, we compute its continued fraction $[a_0^{(\ell)}, a_1^{(\ell)}, \ldots]$ and its convergents $p_1^{(\ell)}/q_1^{(\ell)}, p_2^{(\ell)}/q_2^{(\ell)}, \ldots$. In each case, we find an integer t_ℓ such that

$$q_{t_\ell}^{(\ell)} > 7.2 \times 10^{37} = 6M \qquad \text{and} \qquad \epsilon_\ell := ||\mu_\ell q_{t_\ell}^{(\ell)}|| - M||\tau_\ell q_{t_\ell}^{(\ell)}|| > 0.$$

A quick calculationin Mathematica show that

$$\max\left\{\left\lfloor \log\left(Aq_{t_\ell}^{(\ell)}/\epsilon_\ell\right)/\log B\right\rfloor : \ell\in[2,250]\right\}\le 145.$$

Thus, by Lemma 3.3 we have that $a-b\le 145$. Then, by inequalities (4.2) and (4.6), we get $a<4.4\cdot 10^{19}$ and $n<2.3\cdot 10^{22}$. A new cycle of reduction on ℓ and $a-b$, in inequalities (4.11) and (4.13) with $M:=2.3\cdot 10^{22}$, yield

$$\ell\le 205,\qquad a-b\le 95,\qquad a<3\cdot 10^{19}\qquad\text{and}\qquad n<1.3\cdot 10^{22}.\tag{4.14}$$

The above inequalities had been obtained assuming $a-b\ge 2$. If $a-b=1$ then by inequalities (4.2) and (4.6), we have $a<2.7\cdot 10^{17}$ and $n<1.4\cdot 10^{20}$, while a new reduction cycle on inequality (4.11) with $M:=1.4\cdot 10^{20}$ leads to $\ell\le 190$. Thus, in all case inequalities in (4.14) holds.

Finally, we assume that $a\ge 2$ and $\Gamma_2:=n\log\alpha-a\log(x_\ell)-\log\left(\sqrt{5}\left(1\pm x_\ell^{b-a}\right)\right)$. By (4.5), we have that $|e^{\Gamma_2}-1|<2/2^a\le 1/2$, then

$$0<\left|n\log\alpha-a\log(x_\ell)-\log\left(\sqrt{5}\left(1\pm x_\ell^{b-a}\right)\right)\right|<\frac{4}{2^a}.$$

Dividing both sides by $\log(x_\ell)\ge\log 3$, we get

$$0<\left|n\frac{\log\alpha}{\log(x_\ell)}-a-\frac{\log\left(\sqrt{5}\left(1\pm x_\ell^{b-a}\right)\right)}{\log(x_\ell)}\right|<\frac{4}{2^a\log(x_\ell)}<\frac{3.7}{2^a}.$$

We now put

$$\tau_\ell:=\frac{\log\alpha}{\log(x_\ell)},\qquad \mu_{\ell,a-b}:=-\frac{\log\left(\sqrt{5}\left(1\pm x_\ell^{b-a}\right)\right)}{\log(x_\ell)},\qquad A:=3.7\qquad\text{and}\qquad B:=2,$$

with which, the above inequality can be rewriten as

$$0<|n\tau_\ell-a+\mu_{\ell,a-b}|<AB^{-a}.\tag{4.15}$$

We take $M:=1.3\cdot 10^{22}$ and apply Lemma 3.3, with $(m,s,k)=(n,a,a)$, to inequality (4.15) for each $\ell\in[2,205]$ and $a-b\in[1,95]$, according to inequalities (4.14). With the help of Mathematica, we show that

$$\max\left\{\lfloor\log(Aq_{t_\ell}^{(\ell)}/\epsilon_{\ell,a-b})/\log B\rfloor : \ell\in[2,205],\quad a-b\in[1,95]\right\}\le 100.$$

Thus, by Lemma 3.3 we have that $a\le 100$. Then, from inequality (4.2), we get $n\le 41200$.

We run a last reduction cycle: from inequalities (4.11) we obtain that $\ell<50$, however, we begin this section under the assumption that $\ell>130$, so we conclude that $\ell\le 130$. Then, from inequality (4.13) we have $a-b\le 40$ and finally from inequality (4.15), we get $a\le 50$.

Below we summarize what we have obtained.

Lemma 4.3. *Let (n,ℓ,a,b) be a solution of Diophantine equation (1.1) with $n\ge 3$, $\ell\ge 2$ and $a>b\ge 1$, then*

$$\ell\le 130,\qquad b<a\le 50\qquad\text{and}\qquad n\le 13100.$$

5 Listing all the solutions

The final computational search for solutions of the equation (1.1), according to what we obtained above, requires us to look for the solutions in the set

$$\{F_n\} \cap \{x_\ell^a + \epsilon_1 x_\ell^b + \epsilon_2\} = \{5, 13, 21, 55, 89, 233\}$$

with $2 \leq \ell \leq 130$, $1 \leq b < a \leq 50$, $3 \leq n \leq 13100$ and $\epsilon_i \in \{\pm 1\}$ (for $i = 1, 2$). They are

(*i*) for Mersenne numbers $x_2 = 2^2 - 1 = 3$ and $x_3 = 2^3 - 1 = 7$,

$$\begin{aligned} 5 &= F_5 = 3^2 - 3^1 - 1, \qquad 13 = F_7 = 3^2 + 3^1 + 1, \\ 55 &= F_{10} = 3^4 - 3^3 + 1 = 7^2 + 7 - 1, \\ 89 &= F_{11} = 3^4 + 3^2 - 1, \qquad 233 = F_{13} = 3^5 - 3^2 - 1. \end{aligned}$$

(*ii*) for Fermat numbers $x_2 = 2^2 + 1 = 5$ and $x_3 = 2^3 + 1 = 9$,

$$21 = F_8 = 5^2 - 5^1 + 1, \qquad 89 = F_{11} = 9^2 + 9^1 - 1.$$

This ends the proof of our Theorem 1.

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