TAMKANG JOURNAL OF MATHEMATICS
Volume 31, Number 3, Autumn 2000

A NOTE ON SIMPSON'S INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION

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Abstract. Inequalities of Simpson's type for functions whose n-th derivative, $n \in \{0, 1, 2, 3\}$ is of bounded variation are given.

1. Introduction

One of fundamental results in numerical integration is Simpson's inequality which states if $f^{(4)}$ exists and is bounded on (a, b) then

$$\left| \int_a^b f(x)dx - \frac{b-a}{6}[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)] \right| \leq \frac{1}{2880}(b-a)^5 ||f^{(4)}||_\infty. \tag{1}$$

The disadvantage that this estimation can not be applied if the fourth derivate of f either does not exist on (a, b) or is not bounded there, was removed in result of Dragomir, [1]. Namely, in [1] the following result was proven.

Theorem A. *Let* $f : [a, b] \to \mathbf{R}$ *be a mapping of bounded variation on* $[a, b]$. *Then*

$$\left| \int_a^b f(x)dx - \frac{b-a}{6}[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)] \right| \leq \frac{1}{3}(b-a)V_a^b(f), \tag{2}$$

where $V_a^b(f)$ *denotes the total variation of* f *on the interval* $[a, b]$.

In the proof of the previous result the following identity is used:

$$-\int_a^b s(x)df(x) = \int_a^b f(x)dx - \frac{b-a}{6}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right] \tag{3}$$

where

$$s(x) = \begin{cases} x - \frac{5a+b}{6} & x \in [a, \frac{a+b}{2}) \\ x - \frac{a+5b}{6} & x \in [\frac{a+b}{2}, b]. \end{cases}$$

Received September 8, 1999.
2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 41A55.
Key words and phrases. Simpson's inequality, function of bounded variation.

The right-hand side of (3) is denoted by $R(f)$. Similar identity can be found in the book [2, p.174] but for absolute continuous function f so that on the left-hand side of (3) we have $-\int_a^b s(x)f'(x)dx$. In the same book some other identities in connection with Simpson's inequality are given. Here we give versions of those identities which are suitable for our purpose:

$$R(f)=\frac{1}{2}\int_a^{\frac{a+b}{2}}(x-a)(x-\frac{2a+b}{3})df'(x)+\frac{1}{2}\int_{\frac{a+b}{2}}^b(x-b)(x-\frac{a+2b}{3})df'(x) \tag{4}$$

$$R(f)=-\frac{1}{6}\int_a^{\frac{a+b}{2}}(x-a)^2(x-\frac{a+b}{2})df''(x)-\frac{1}{6}\int_{\frac{a+b}{2}}^b(x-b)^2(x-\frac{a+b}{2})df''(x) \tag{5}$$

$$R(f)=\frac{1}{24}\int_a^{\frac{a+b}{2}}(x-a)^3(x-\frac{a+2b}{3})df'''(x)+\frac{1}{24}\int_{\frac{a+b}{2}}^b(x-b)^3(x-\frac{2a+b}{3})df'''(x) \tag{6}$$

where f is a function such that f', f'', f''' is of bounded variation, respectively. These identities are proven using integration by parts.

In [2], result related to (2) is given. Namely, using identity (6) it is proven that if f''' is an absolutely continuous with total variation V_3, then

$$|R(f)|\le\frac{1}{1152}(b-a)^4V_3. \tag{7}$$

Here, we state results related to inequalities (2) and (7) which give an error estimate of $R(f)$ expressed by total variation of either function f or its derivatives.

2. Main Results

Theorem 1. *Let $n\in\{0,1,2,3\}$. Let f be a real function on $[a,b]$ such that $f^{(n)}$ is function of bounded varation. Then*

$$\left|\int_a^b f(x)dx-\frac{b-a}{6}\left[f(x)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\le C_n(b-a)^{n+1}V_a^b(f^{(n)}),$$

where

$$C_0=\frac{1}{3},\ C_1=\frac{1}{24},\ C_2=\frac{1}{324},C_3=\frac{1}{1152}$$

and $V_a^b(f^{(n)})$ is the total variation of function $f^{(n)}$.

Proof. For $n=0$ the proof is given in [1]. For $n=3$ the proof is similar to that one from [2].

If $n=1$, using identity (4) under notation

$$s_1(x)=\begin{cases}\frac{1}{2}(x-a)(x-\frac{2a+b}{3}), & x\in[a,\frac{a+b}{2})\\ \frac{1}{2}(x-b)(x-\frac{2b+a}{3}), & x\in[\frac{a+b}{2},b].\end{cases}$$

we have

$$|R(f)| = \left|\int_a^b s_1(x)df'(x)\right| \le \max_{x\in[a,b]} |s_1(x)| V_a^b(f').$$

Maximum of function $|\frac{1}{2}(x-a)(x-\frac{2a+b}{3})|$ on interval $[a, \frac{a+b}{2}]$ is $\frac{(b-a)^2}{24}$, and the same value is maximum of function $|\frac{1}{2}(x-b)(x-\frac{2a+b}{3})|$ on interval $[\frac{a+b}{2}, b]$. So, $\max_{x\in[a,b]} |s_1(x)| = \frac{(b-a)^2}{24}$ and

$$|R(f)| \le \frac{(b-a)^2}{24} V_a^b(f').$$

If $n = 2$ using identity (5) we have

$$|R(f)| = \left|\int_a^b s_2(x)df''\right| \le \max_{x\in[a,b]} |s_2(x)| V_a^b(f''),$$

where

$$s_2(x) = \begin{cases} -\frac{1}{6}(x-a)^2(x-\frac{a+b}{2}), & x \in [a, \frac{a+b}{2}) \\ -\frac{1}{6}(x-b)^2(x-\frac{a+b}{2}), & x \in [\frac{a+b}{2}, b]. \end{cases}$$

Global maximum investigation gives that $\max_{x\in[a,\frac{a+b}{2}]} |-\frac{1}{6}(x-a)^2(x-\frac{a+b}{2})| = \max_{x\in[\frac{a+b}{2},b]} |-\frac{1}{6}(x-b)^2(x-\frac{a+b}{2})| = \frac{(b-a)^3}{324}$.

So, the following holds

$$|R(f)| \le \frac{1}{324}(b-a)^3 V_a^b(f'').$$

As a simple consequence of the previous theorem we have the following corollary.

Corollary 1. *If f is a function on $[a,b]$ such that f''' is of bounded variation then*

$$|R(f)| \le \min_{n\in\{0,1,2,3\}} \{C_n(b-a)^{n+1} V_a^b(f^{(n)})\}.$$

Corollary 2. *Let $n \in \{0,1,2,3\}$. If f is a function such that $f^{(n)}$ is an absolute continuous function, then*

$$|R(f)| \le C_n(b-a)^{n+1}\|f^{(n+1)}\|_1,$$

where $\|g\|_1 = \int_a^b |g(x)|dx$. C_n, $n = 0,1,2,3$, are constants defined in Theorem 1.

Using inequalities of Simpson's type from Theorem 1 we can obtain the following estimation of remainder term $R(f,\sigma_n)$ in Simpson's quadrature formula

$$\int_a^b f(x)dx = A(f,\sigma_n) + R(f,\sigma_n), \tag{8}$$

where σ_n is a partition of the interval $[a,b]$, i.e.

$$\sigma_n = \{a = x_0, x_1, x_2, \ldots, x_m = b; a < x_1 < \cdots < x_{m-1} < b\}$$

and $A(f, \sigma_n)$ is equal to

$$\frac{1}{6}\sum_{i=0}^{m-1}[f(x_i) + f(x_{i+1})]h_i + \frac{2}{3}\sum_{i=0}^{m-1} f\left(\frac{x_i + x_{i+1}}{2}\right)h_i.$$

We have the following estimations for $R(f, \sigma_n)$.

Corollary 3. *Let $n \in \{0,1,2,3\}$. Let f be a function on $[a,b]$ such that $f^{(n)}$ is a function of bounded variation and σ_n be a partition of $[a,b]$. Then the remainder term $R(f,\sigma_n)$ in Simpson's quadrature formula* (8) *satisfies:*

$$|R(f,\sigma_n)| \le C_n V_a^b(f^{(n)}) \cdot \max\{h_i^{n+1} : i = 0, \ldots, n-1\}$$

where $h_i = x_{i+1} - x_i$, $i = 0, 1, \ldots, m-1$, and C_n are defined as in Theorem 1.

Remark 1. Similary we can improve results related to special means given in [1].

References

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