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An exponentially fitted spline method for singularly perturbed parabolic convection-diffusion problems with large time delay

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Abstract. This paper deals with the numerical solutions of singularly perturbed parabolic convection-diffusion problems with a delay in time. It is assumed that the delay parameter is larger than the perturbation parameter. A special type of mesh is used for the temporal variable so that the shift lies on the nodal points and an exponentially fitted cubic spline method based on the Crank-Nicolson difference scheme is developed. We obtain the proposed finite difference scheme is unconditionally stable. Parameter-uniform error estimates are derived and it is shown that the method is ε-uniformly convergent of second-order accurate in both temporal direction and spatial direction. Finally, the obtained numerical results show that the method provides more accurate solutions than some other methods exist in the literature.

Keywords. Singular perturbations, parabolic differential equation,large time delay, convection-diffusion, fitted cubic splines

1 Introduction

Singularly perturbed time delay parabolic partial differential equations arise in many areas of applied mathematics and mathematical physics where a differential equation depends on small positive parameters multiplying the highest derivative terms. Singularly perturbed delay partial differential equations model physical problems for which the evaluation does not only depend on the present state of the system but also on the past history. Areas of the sciences in which governing equations involve integral terms representing the effect of the past include the study of materials with memory [1], in mathematical demography and population dynamics [24], in the study of dynamics of artificial neural networks in which there are transmission delays [33], and in mathematical finance in which inefficient markets are modelled [36]. Under the heading "Small delays can have a large effect," Kuang [24] commented on the associated risks, ignoring the delays that researchers think are small. The comment underlines the presence of small time delay in partial differential equations may result in a large effect on the solution. If we ignore the delay parameter and directly used the method to solve singularly perturbed delay partial differential equations, one may not get a solution that exactly reflect the reality of the original

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problem. The investigation here is to examine the solution when the time delay is non-zero and the effect of the parameter ε on the boundary layer is significant.

The singularly perturbed nature of equations is apparent when the magnitude of the term involving first-order derivatives is much greater than terms involving second-order derivatives. In specific situations, multiplied to the highest order derivative term have magnitudes that are much smaller than unity and such that, when these parameters tend to zero, the order of the differential equation is reduced. Then the problem has a limiting solution which is the solution of the reduced problem and the regions of non-uniform convergence lie near the boundary, which is known as boundary layers. These problems have steep gradients in the narrow intervals of space and short intervals of time. That means as the value of ε approaches zero, the error in the approximation becomes greater. Even for the one spatial dimension and one temporal variable, not all difference schemes can capture these steep variations. Therefore, for singularly perturbed problems it is desirable to construct numerical methods for which the accuracy of the approximate solution does not depend on ε, and for which the size of the error depends only on the number of mesh points used, that is, methods which converge uniformly with respect to the parameter ε. Regarding the study of uniform convergence solution, a large amount of literature exists for singularly perturbed non-delay differential equations and many assumptions have been made on the stability [9, 12, 16]. There have been extensive development studies of singular perturbation problems in the classes of singularly perturbed systems [10, 17, 18, 19, 30], singular perturbation problems with two small parameters [4, 6, 11, 27], singular perturbation problems with non-local boundary condition [20], singular perturbation problems with Robin type boundary condition [14, 21], singularly perturbed system with Robin type boundary condition [15]. However, in recent years there has been a growing interest in the numerical study of singularly perturbed partial differential equations with delay both in space and time. Various approaches for the numerical methods to solve singularly perturbed parabolic partial differential equations with delay in space are given in [5, 7, 8, 32, 34]. The present work is interested in singularly perturbed parabolic partial differential equations with a time delay. So let's take a look at some work is being done in this area. A typical real time application example of the time delay partial differential equation is the following models of a furnace used to process metal sheets

$$\frac{\partial u(x,t)}{\partial t} = \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + v[g\left(u(x,t-\tau)\right)]\left(\frac{\partial u(x,t)}{\partial x}\right) + c[f\left(u(x,t-\tau)\right) - u(x,t)]. \tag{1.1}$$

See WU [36] for more detailed literature on the existing partial differential equations with time delay models. The model problem in (1.1) is in the form of a time-delayed parabolic singular perturbation problem, which is difficult to drive its exact solution.

It is worthwhile to mention here that there exists no an exponentially fitted cubic spline finite difference method for the singularly perturbed partial differential equations with large time delay, and hence ours is the first work in this direction. Thus, the aim of this paper is to design an efficient numerical scheme based on the cubic spline method which comprises an exponentially fitted difference scheme on a uniform mesh.

Organization of the paper: In Section 2, the problem under study is formulated. In 3, some a priori estimates on the solution and its derivatives and some analytical results which are used in the convergence analysis are given. We describe an exponentially fitted cubic spline for a singularly perturbed delay parabolic partial differential equation in Section 4. We present two numerical experiments to demonstrate the applicability and efficiency of the proposed method in Section 5. We end with brief conclusions on the results obtained in the Section 6.

Notation. *For a function ϑ defined on D, the standard supremum norm has been denoted* *by* $\|.\|_\infty$ *is given by*

$$\|\vartheta\|_\infty = \sup_{(x,t)\in D} |\vartheta(x,t)|\,.$$

Through out this paper C *denotes a positive constant independent of the perturbation parameter* ε*, the mesh size and the mesh points.*

2 Problem Formulation

In this article, we consider the following singularly perturbed delay parabolic convection-diffusion initial-boundary-value problem(IBVP) with Dirichlet boundary conditions:

$$\frac{\partial u}{\partial t} - \varepsilon u_{xx}(x,t) + a(x)u_x(x,t) + b(x,t)u(x,t) = -c(x,t)u(x,t-\tau) + f(x,t), (x,t) \in D, \qquad (2.1)$$

subject to the boundary conditions

$$\begin{cases} u(0,t) & = \phi_l(t),\ \Gamma_l = \{(0,t) : 0 \le t \le T\}, \\ u(1,t) & = \phi_r(t),\ \Gamma_r = \{(1,t) : 0 \le t \le T\}, \end{cases} \qquad (2.2)$$

and the interval condition

$$u(x,t) = \phi_b(x,t), (x,t) \in \Gamma_b = [0,1] \times [-\tau, 0], \qquad (2.3)$$

for $\Omega = (0,1)$,$D = \Omega \times (0,T]$, $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$, where Γ_l and Γ_r are the left and the right side of the rectangular domain D is corresponding to $x = 0$ and $x = 1$, respectively. Also, $0 < \varepsilon \lll 1$ is a singular perturbation parameter and $\tau > 0$ represents the delay parameter. The functions $a(x), b(x,t), f(x,t)$ on $\overline{D} = [0,1] \times [0,T]$ and $\phi_b(x,t), \phi_l(t), \phi_r(t)$ on Γ are sufficiently smooth, bounded functions that satisfy $a(x) \ge \alpha > 0, b(x,t) \ge \beta > 0, (x,t) \in \overline{D}$. Also, $f(x,t)$, $\phi_l(t)$, $\phi_r(t)$ and $\phi_b(x,t)$ are smooth and bounded. The compatibility conditions

$$\begin{cases} \phi_b(0,0) = \phi_l(0), \\ \phi_b(1,0) = \phi_r(0), \end{cases} \qquad (2.4)$$

and

$$\begin{cases} \frac{d\phi_l(0)}{dt} - \varepsilon \frac{\partial^2 \phi_b(0,0)}{\partial x^2} + a(0)\frac{\partial \phi_b(0,0)}{\partial x} + b(0,0)\phi_b(0,0) = -c(0,0)\phi_b(0,-\tau) + f(0,0), \\ \frac{d\phi_r(0)}{dt} - \varepsilon \frac{\partial^2 \phi_b(1,0)}{\partial x^2} + a(1)\frac{\partial \phi_b(1,0)}{\partial x} + b(1,0)\phi_b(1,0) = -c(1,0)\phi_b(0,-\tau) + f(1,0). \end{cases} \qquad (2.5)$$

are imposed. It is clear that the solution of (2.1)-(2.3) has a boundary layer of width $O(\varepsilon)$ on Γ_r [13]. Also, the characteristics of the reduced problem of (2.1)-(2.3) are the vertical lines where x is a constant, thus the boundary layer arising in the solution is of parabolic type. In order to construct parameter-uniform numerical methods, two classical approaches are applied. The first approach is to design the mesh in a way that captures the layers, which is often known as a special mesh approach, for instance, refer to [6, 12, 16, 17, 18, 19, 30], which constructs meshes adapted to the solution of the problem. The second approach is to design an exponential fitting type [3], which have coefficients of an exponential type adapted to the singular perturbation problems and hence it reflects the behavior of the solution in the boundary layer region which is known as fitted operator method.

Most of the numerical study on singularly perturbed time delay parabolic partial differential equations has been done by using the layer adapted mesh approach rather than the fitted operator one. The study of the problem considered in this paper was started by Ansari *et al.* [2], where they discussed finite difference scheme for singularly perturbed partial differential equations on a layer-adapted mesh,which results in uniform convergence of first order in time and second order in spatial direction. Bashier and Patidar [3] designed a robust fitted operator finite difference method for the numerical solution of the singularly perturbed delay parabolic partial differential equation. The method was shown unconditionally stable and parameter-uniform convergent of order one in the temporal direction and of order two in the spatial direction. Das and Natesan [13] proposed a numerical method to solve one-dimensional singularly perturbed delay parabolic convection-diffusion problem. They proposed numerical scheme consists of the backward-Euler scheme for the time derivative, and the hybrid numerical scheme for spatial direction on Shishkin mesh. They showed that the rate of convergence is almost first order in time direction and almost second order up to a logarithm factor in spatial direction. Gowrisankar and Natesan [23] solved the singularly perturbed time delay parabolic convection-diffusion problem by the upwind scheme on Shishkin mesh. The authors proposed scheme is ε-uniform convergence of first-order in time and first-order up to a logarithmic factor in space. Kumar and Kumari [25] constructed a robust implicit unconditionally stable numerical method comprising the Crank-Nicolson method consisting of a midpoint upwind finite difference scheme on a fitted piecewise-uniform mesh condensing in the boundary layer for solving a class of parabolic singularly perturbed partial differential equations with time delay. They have shown that the proposed scheme is ε-uniform convergent of second-order accurate in the temporal direction and the first-order (up to a logarithmic factor) accurate in the spatial direction. Gelu and Duressa [21] proposed an implicit Euler method for time derivative with uniform mesh and extended cubic B-spline collocation method for space derivative on Shishkin mesh to solve singularly perturbed delay parabolic reaction-diffusion problem subject to mixed boundary conditions. The result obtained is shown to be accurate of $O\left(\Delta t + N^{-2}ln^2N\right)$ by preserving an -uniform convergence. Govindarao1 *et al.* [22] presented a higher-order parameter uniformly convergent method for a singularly perturbed delay parabolic reaction-diffusion initial-boundary-value problem. For the discretization of the time derivative, they used the implicit Euler scheme on the uniform mesh and for the spatial discretization, they used the central difference scheme on the Shishkin mesh, which provides a second-order convergence rate. To enhance the order of convergence, they applied the Richardson extrapolation technique and attains almost fourth-order convergence rate. In [29], Negero and Duressa discussed the numerical scheme comprising an exponentially fitted Tension-spline based difference scheme on a uniform mesh supported by Crank-Nicolson Method. Woldaregay and Duressa [35] considered singularly perturbed time delay problem using the Crank-Nicolson method in temporal discretization and exponentially fitted operator finite difference method in spatial discretization. They formulated scheme converges uniformly with linear order of convergence. Kumar *etal.* [26], developed graded mesh refinement approach for boundary layer originated singularly perturbed time-delayed parabolic convection diffusion problems. In [28], Negero and Duressa discussed the backward-Euler method for the time derivative and Micken's type discretization for the space derivatives, moreover authors enhanced the order of convergence by using Richardson extrapolation method.

Motivated by the work done in Kumar and Kumari [13, 23, 25] the numerical methods presented here comprise exponentially fitted cubic spline finite difference schemes on a uniform mesh. The method is based on the discretization of the time derivative at each time step, the freezing of the spatially dependent coefficients, and the piecewise analytical solution of the resulting (convection-diffusion) ordinary differential in an equally-spaced mesh. Such a solution results in a three-point, exponentially fitted finite difference equation which can easily be solved. It should also be contrasted with that proposed by Kumar and Kumari [25] who employed a midpoint upwind finite difference scheme in a piecewise uniform Shishkin mesh in spatial derivative. This work also contrasted the method in Das and Natesan [13] and Gowrisankar and

Natesan [23] who employed the upwind finite difference scheme for the spatial derivatives in a piecewise uniform Shishkin mesh.

3 Properties of continuous solution

The purpose of this section is to derive a priori bound for the solution $u(x,t)$, of 2.1-2.3 on the solution domain $\overline{D}$. These estimates contain continuous maximum principle bounds of the solution and its derivatives, and then parameter uniform bounds on the regular and singular components to analyze the proposed scheme. The detailed proofs of Lemmas 3.2, 3.3 and 3.4 are provided in [28, 13]. Let us denote differential operator $\Im$ for the differential equation in 2.1 as

$$L_\varepsilon \equiv \frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + a(x)\frac{\partial}{\partial x} + b(x,t).$$

Lemma 3.1 (Continuous maximum principle [25]). *Suppose $\psi(x,t)$ is such that $\psi(x,t) \geq 0, \forall(x,t) \in \partial D$. Then, $L_\varepsilon\psi\,(x,t) \geq 0, \forall(x,t) \in D$ implies that $\psi(x,t) \geq 0, \forall(x,t) \in \overline{D}$.*

Lemma 3.2. *For any $(x,t) \in \bar{D}$, the solution $u\,(x,t)$ of (2.2)-(2.3) is such that*

$$|u\,(x,t) - \phi_b\,(x,0)\,| \leq Ct, (x,t) \in \overline{D}.$$

where C is a constant which is independent of ε.

A direct importance of this Lemma 3.2 is the following estimate.

Lemma 3.3. *For any $(x,t) \in \bar{D}$, the estimate on the solution $u(x,t)$ of (2.2)-(2.3) satisfy the following bound:*

$$|u(x,t)| \leq C.$$

Lemma 3.4 (Stability estimate). *The uniform stability bound on the solution $u(x,t)$ of (2.2)-(2.3) satisfy:*

$$\|u\|_{\bar{D}} \leq \frac{\|L_\varepsilon u\|}{\beta} + \max\left(|\phi_b|\,, |\phi_l|\,, |\phi_r|\right).$$

where $\|.\|$ denotes the maximum norm on the domain $\bar{D}$ and β is a positive constant specified under Section 1.

Proof. By defining the barrier functions $\varphi^{\pm}$ as $\varphi^{\pm} = \frac{\|L_\varepsilon u\|}{\beta} + \max\left(|\phi_b|\,, |\phi_l|\,, |\phi_r|\right) \pm u\,(x,t)$ and applying the maximum principle, we obtain the required bound [25]. □

The solution $u(x,t)$, and its derivatives satisfy the following bounds.

Lemma 3.5. *For any non-negative integers i, j such that $0 \leq i+j \leq 5$, the solution $u(x,t)$ satisfies The exact solution $u(x,t)$ and its derivatives of problems (2.2)-(2.3), satisfy the bound:*

$$\left|\frac{\partial^{i+j}u}{\partial x^i \partial t^j}\right| \leq C\left(1 + \varepsilon^{-i}exp\left(-\alpha\,(1-x)\,/\varepsilon\right)\right), \forall(x,t) \in \bar{D}.$$

Proof. See [13]. □

4 Numerical Scheme Formulation

4.1 Temporal Discretization

We partition the time domain $\bar{\Omega} = [0,T] \times [0,1]$ with constant step size Δt through the grid points (x_m, t_n) as,

$$\bar{\Omega}_t^M = \{t_n = n\Delta t, n = 0,1,2,...,M, t_M = T, \Delta t = T/M\},$$

where M denotes the number of mesh elements in temporal direction $[0,T]$ and

$$\bar{\Omega}_t^s = \{t_n = n\Delta t, n = 0,1,2,...,s, t_s = \tau, \Delta t = \tau/s\},$$

where s is the number of mesh element in $[-\tau, 0]$. The step size Δt satisfies $\tau = s\Delta t$, where s is a positive integer and $t_n = n\Delta t, n \geq -s$.

Assume that the equation given in the form of (2.2)-(2.3) is satisfied at the point $(x, n+\frac{1}{2})^{th}$ level. At this point by keeping x fixed along the line $\{(x,t); 0 \leq t \leq T\}$ and applying fitted Crank-Nicolson's scheme, we obtain the solution u of equation (2.2)-(2.3) as:

$$(I + \frac{\Delta t}{2} L_\varepsilon^{\Delta t})U^{n+1}(x) = g^n, \tag{4.1}$$

$$\begin{cases} U^{n+1}(0) & = \phi_l(t_{n+1}),\ n = 0,...,M, \\ U^{n+1}(1) & = \phi_r(t_{n+1}),\ n = 0,...,M, \\ U^{n+1}(x) & = \phi_b(x, t_{n+1}),\ x \in (0,1)\,, -(s+1) \leq n \leq -1. \end{cases} \tag{4.2}$$

where,
$U^{n+1}(x)$ is the approximate solution to the exact value $u(x, t_n)$,

$$g^n = U^n(x) + \frac{\Delta t}{2}\left(F^n(x) + F^{n+1}(x) - L_\varepsilon^{\Delta t} U^n(x)\right), F^n(x) = c^n(x)U^{n-s}(x) + f^n(x),$$

$$L_\varepsilon^{\Delta t} U^{n+1}(x) = -\varepsilon \frac{d^2 U^{n+1}(x)}{dx^2} + a\,(x)\,\frac{dU^{n+1}(x)}{dx} + b^{n+\frac{1}{2}}(x)U^{n+1}(x). \tag{4.3}$$

In the next lemma, we state the semi-discrete maximum principle for the operator $\pounds_\varepsilon^{\Delta t}$ given in (4.3).

Lemma 4.1 (Semi-discrete maximum principle)**.** *Let $\Upsilon^{n+1}(x)$ be a sufficiently smooth function on $\bar{D}$. If $\Upsilon^{n+1}(0) \geq 0$, $\Upsilon^{n+1}(1) \geq 0$ and $L_\varepsilon^{\Delta t}\Upsilon^{n+1}(x) \geq 0$ for all $x \in D$, then $\Upsilon^{n+1}(x) \geq 0$ for all $x \in \bar{D}$.*

Proof. As the work in Kumar[25, 29], we prove the lemma by contradiction. Assume $(x^*, t^*) \in \bar{D}$ such that $\Upsilon^{n+1}(x^*) = \min_{(x)\in\bar{D}} \Upsilon^{n+1}(x) < 0$. This is clear that the point $(x^*) \notin \partial\bar{D}$, which proves that $(x^*) \in D$. Now consider

$$L_\varepsilon^{\Delta t}\Upsilon^{n+1}(x) = -\varepsilon \frac{d^2\Upsilon^{n+1}(x)}{dx^2} + a\,(x)\,\frac{d\Upsilon^{n+1}(x)}{dx} + b^{n+\frac{1}{2}}(x)\Upsilon^{n+1}(x),$$

at the point x^* the value of the operator becomes,

$$L_\varepsilon^{\Delta t}\Upsilon^{n+1}(x^*) = -\varepsilon \frac{d^2\Upsilon^{n+1}(x^*)}{dx^2} + a\,(x^*)\,\frac{d\Upsilon^{n+1}(x^*)}{dx} + b^{n+\frac{1}{2}}(x^*)\Upsilon^{n+1}(x^*). \tag{4.4}$$

We have,

$$\Upsilon_x^{n+1}(x^*) = 0, \Upsilon_{xx}^{n+1}(x^*) \geq 0, \Upsilon^{n+1}(x^*) < 0. \tag{4.5}$$

Using these properties of the Eq. 4.5 in 4.4, we get

$$L_\varepsilon^{\Delta t}\Upsilon^{n+1}(x^*) < 0,$$

which is a contradiction as $L_\varepsilon^{\Delta t}\Upsilon^{n+1}(x) > 0$ for all (x) in D. Hence the result is obtained. □

Next Lemmas gives the bounds for the local and global error in the temporal semi-discretization. Let the local error at each time step be denoted by $e_{n+1} = u(x, t_{n+1}) - U^{n+1}(x)$ for $n = 0, 1, 2, ..., M$.

Lemma 4.2 (Local error estimate)**.** *Assuming the bounds on $u(x,t)$ and its derivatives given by $\left|\frac{\partial^l u}{\partial t^l}\right| \leq C, (x,t) \in D, 0 \leq l \leq 2$. The local error estimate in the temporal direction is given by*

$$\|u(x, t_n) - U^n(x)\|_\infty \leq C(\Delta t)^3.$$

Proof. Using Taylor series expansion centering at $t_{n+\frac{1}{2}}$ we have:

$$u^{n+1}(x) = u^{n+\frac{1}{2}}(x) + \frac{\Delta t}{2}\frac{\partial u^{n+\frac{1}{2}}(x)}{\partial t} + \frac{(\Delta t)^2}{8}\frac{\partial^2 u^{n+\frac{1}{2}}(x)}{\partial t^2} + O((\Delta t)^3), \tag{4.6}$$

$$u^{n}(x) = u^{n+\frac{1}{2}}(x) - \frac{\Delta t}{2}\frac{\partial u^{n+\frac{1}{2}}(x)}{\partial t} + \frac{(\Delta t)^2}{8}\frac{\partial^2 u^{n+\frac{1}{2}}(x)}{\partial t^2} + O((\Delta t)^3). \tag{4.7}$$

Subtracting equation (4.7) from equation (4.6) gives the central difference approximation in such a point as:

$$\frac{\partial u^{n+\frac{1}{2}}(x)}{\partial t} = \frac{u^{n+1}(x) - u^n(x)}{\Delta t} + O\left((\Delta t)^2\right). \tag{4.8}$$

Applying equation (4.8) formula for time derivative in the following equation (4.9)

$$\begin{aligned}\frac{\partial u(x,t_{n+\frac{1}{2}})}{\partial t} - \varepsilon\frac{\partial^2 u(x,t_{n+\frac{1}{2}})}{\partial x^2} + a(x)\frac{\partial u(x,t_{n+\frac{1}{2}})}{\partial x} + b(x,t_{n+\frac{1}{2}})u(x,t_{n+\frac{1}{2}}) \\ = \quad c(x,t_{n+\frac{1}{2}})u(x,t_{n-s+\frac{1}{2}}) + f(x,t_{n+\frac{1}{2}})\end{aligned} \tag{4.9}$$

we obtain

$$\begin{aligned}\frac{u^{n+1}(x) - u^n(x)}{\Delta t} + O\left((\Delta t)^2\right) - \varepsilon\frac{\partial^2 u^{n+\frac{1}{2}}(x)}{\partial x^2} + a(x)\frac{\partial u^{n+\frac{1}{2}}(x)}{\partial x} + b^{n+\frac{1}{2}}(x)u^{n+\frac{1}{2}}(x) \\ = c^{n+\frac{1}{2}}(x)u^{n-s+\frac{1}{2}}(x) + f^{n+\frac{1}{2}}(x).\end{aligned} \tag{4.10}$$

For other terms of equation (4.10) related to the points (x, t_{n+1}) and(x, t_n), using fitted Crank-Nicolson's scheme can be written as:

$$\begin{aligned}&-\frac{\Delta t\varepsilon}{2}\frac{d^2u^{n+1}(x)}{dx^2} + \frac{\Delta t}{2}a\frac{du^{n+1}(x)}{dx} + (1 + \frac{\Delta t}{2}b^{n+1}(x))u^{n+1}(x) \\ &= \frac{\Delta t\varepsilon}{2}\frac{d^2u^{n}(x)}{dx^2} - \frac{\Delta t}{2}a\frac{du^{n}(x)}{dx} + (1 - \frac{\Delta t}{2}b^{n}(x))u^{n}(x) + \frac{\Delta t}{2}H^n(x) + \frac{\Delta t}{2}H^{n+1}(x),\end{aligned}$$

where $H^n(x) = c(x, t_n)u(x, t_{n-s}) + f(x, t_n)$. Since error satisfies the differential Equations, the local truncation error satisfies

$$L_\varepsilon^{\Delta t}e^{n+1} = O\left((\Delta t)^3\right), e^{n+1}(0) = 0 = e^{n+1}(1).$$

Using the semi-discrete maximum principle give $\|e_{n+1}\|_\infty \leq C(\Delta t)^3$. □

Let denote E^{n+1} be the global error estimate up to the $(n+1)th$ time step.

Theorem 4.1 (Global error estimate.). *The global error estimate at* (t_{n+1}) *is given by*

$$\|E_{n+1}\|_\infty \leq C\,(\Delta t)^2\,, \forall n = 1, 2, ..., M-1.$$

Proof. Using the results in local error estimate up to the $(n+1)th$ time step in Lemma 4.2, the global error is given as

$$\begin{aligned}
\|E_{n+1}\|_\infty &= \left\|\sum_{k=1}^{n+1} e_k\right\|_\infty \\
&\leq \|e_1\|_\infty + \|e_2\|_\infty + ... + \|e_n\|_\infty\,. \\
&\leq C_1\,(n\Delta t)\,(\Delta t)^2 \\
&\leq C_1 T\,(\Delta t)^2\,,\ \text{since}\ \ (n+1)\,(\Delta t) \leq T \\
&\leq C\,(\Delta t)^2\,,\ \ \text{denotin}\ C_1 T = C,
\end{aligned}$$

where C is constant independent of ε and Δt. □

Next, the bounds on the derivatives in x direction and asymptotic behavior with respect to ε of the solution $U^{n+1}(x)$ of the problems in (4.1) is given by the following Lemma.

Lemma 4.3. *For each* $n = 1, 2, ..., M-1$*. The solution* $U^{n+1}(x)$ *of the boundary value problem in* (4.1)-(4.2) *satisfies the bound*

$$\left|\frac{d^i U^{n+1}(x)}{dx^i}\right| \leq C\left(1 + \varepsilon^{-i}\exp\left(-\alpha\,(1-x)\,/\varepsilon\right)\right), \forall (x) \in \bar{D}, 0 \leq i \leq 4.$$

Proof. The proof is direct from Lemma 3.5. □

4.2 The Spatial Variable Discretization

In this section, we derive the cubic spline scheme on a uniform mesh $0 = x_0 < x_1 <, ..., < x_N = 1$. Now we dived the spatial domain $[0, 1]$ into N equal parts with constant mesh length $h = x_m - x_{m-1}$ for $m = 1, 2, ..., N$. The approximate solution of equations (2.1)-(2.3) is obtained using cubic spline interpolating function, which on each sub-interval $[x_m, x_{m+1}]$, denoted by $S_m(t)$. For known values $\tilde{u}_0(t), \tilde{u}_1(t), ..., \tilde{u}_N(t)$ of a function $u(t)$ at the nodal points $0 = x_0, x_1, x_2, ..., x_N = 1$, a cubic spline interpolating polynomial will have the following properties:

(i) $S_m(t)$ coincides with a polynomial of degree 3 on each $[x_m, x_{m+1}], m = 0, 1, ..., N$.

(ii) $S_m(t) \in C^2[0, 1]$,

(iii) $S_m(t) = \tilde{u}_m(t), m = 0, 1, ..., N$.

The cubic spline interpolating function is developed in [14, 15, 19] to solve singular perturbation problems using the adaptive mesh strategy. In the present study, a cubic spline interpolating function can be written in the following form:

$$\begin{aligned}
S_m(t) = {} & \frac{(x_m - x)^3}{6h}\Psi_{m-1}(t) + \frac{(x - x_{m-1})^3}{6h}\Psi_m(t) + \\
& \left(\tilde{u}_{m-1}(t) - \frac{h^2}{6}\Psi_{m-1}(t)\right)\left(\frac{x_m - x}{h}\right) + \left(\tilde{u}_m(t) - \frac{h^2}{6}\Psi_m(t)\right)\left(\frac{x - x_{m-1}}{h}\right),
\end{aligned} \tag{4.11}$$

where $\Psi_m(t) = S^{''}(x_m, t), x_m \leq x \leq x_{m+1}, 0 \leq m \leq N$.
The difference scheme is derived using this spline function which will give the approximate solution of $u(x, t)$. Differentiating Equation (4.11), we obtain

$$\begin{aligned} S_m^{'}(x,t) = -\frac{(x_m - x)^2}{2h}\Psi_{m-1}(t) + \frac{(x - x_{m-1})^2}{2h}\Psi_m(t) - \\ \left(\tilde{u}_{m-1}(t) - \frac{h^2}{6}\Psi_{m-1}(t)\right) + \left(\tilde{u}_m(t) - \frac{h^2}{6}\Psi_m(t)\right). \end{aligned} \tag{4.12}$$

As $S_m(t) \in C^2[0, 1]$ in equation (4.12), $S_m^{'}(x_m, t) = S_{m+1}^{'}(x_m, t)$ which gives,

$$\frac{h}{6}\left(\Psi_{m-1}(t) + 4\Psi_m(t) + \Psi_{m+1}(t)\right) = \frac{1}{h}\left(\tilde{u}_{m-1}(t) - 2\tilde{u}_m(t) + \tilde{u}_{m+1}(t)\right). \tag{4.13}$$

Now we define the fitting factor problem associated with equations (2.1)-(2.3) by:

$$\begin{aligned} L_\varepsilon \tilde{u}_m(t) \equiv & -\varepsilon\sigma(\rho)\tilde{u}_{xx}(x_m, t) + a_m\tilde{u}_x(t) + b_m(t)\tilde{u}_m(t) \\ & = -c_m(t)\tilde{u}_m(t-\tau) + f_m(t) - \frac{\partial \tilde{u}(x_m, t)}{\partial t}, (x, t) \in D. \end{aligned} \tag{4.14}$$

The boundary conditions (2.1)-(2.3) become

$$\begin{cases} \tilde{u}(0,t) & = \tilde{u}_0(t) = \phi_l(t),\ \Gamma_l = \{(0,t) : 0 \leq t \leq T\}, \\ \tilde{u}(1,t) & = \tilde{u}_1(t) = \phi_r(t),\ \Gamma_r = \{(1,t) : 0 \leq t \leq T\}, \end{cases} \tag{4.15}$$

and the interval condition

$$\tilde{u}(x, t) = \phi_b(x, t), (x, t) \in \Gamma b, \tag{4.16}$$

where $\sigma(\rho)$ is a fitting factor which is to be determined in such a way that the solution of (4.14) converges uniformly to the solution of (4.1) subject to the conditions (4.15)-(4.16). Now, from Equation (4.14),

$$\begin{aligned} \varepsilon\sigma(\rho)\Psi_m(t) =& a_m\tilde{u}_x(x_m, t) + b_m(t)\tilde{u}_m(t) + \\ & c_m(t)\tilde{u}_m(t-\tau) - f_m(t) + \frac{\partial \tilde{u}(x_m, t)}{\partial t}, \\ \varepsilon\sigma(\rho)\Psi_{m-1}(t) =& a_{m-1}\tilde{u}_x(x_{m-1}, t) + b_{m-1}(t)\tilde{u}_{m-1}(t) + \\ & c_{m-1}(t)\tilde{u}_{m-1}(t-\tau) - f_{m-1}(t) + \frac{\partial \tilde{u}(x_{m-1}, t)}{\partial t}, \\ \varepsilon\sigma(\rho)\Psi_{m+1}(t) =& a_{m+1}\tilde{u}_x(x_{m+1}, t) + b_{m+1}(t)\tilde{u}_{m+1}(t) + \\ & c_{m+1}(t)\tilde{u}_{m+1}(t-\tau) - f_{m+1}(t) + \frac{\partial \tilde{u}(x_{m+1}, t)}{\partial t}, \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} &\tilde{u}_x(x_m, t) \simeq \frac{\tilde{u}_{m+1}(t) - \tilde{u}_{m-1}(t)}{2h} + O\left(h^2\right), \\ &\tilde{u}_x(x_{m+1}, t) \simeq \tilde{u}_x(x_m, t) + h\tilde{u}_{xx}(x_m, t) + O\left(h^2\right) = \frac{3\tilde{u}_{m+1}(t) - 4\tilde{u}_m(t) + \tilde{u}_{m-1}(t)}{2h} + O\left(h^2\right), \\ &\tilde{u}_x(x_{m-1}, t) \simeq \tilde{u}_x(x_m, t) - h\tilde{u}_{xx}(x_m, t) + O\left(h^2\right) = \frac{-\tilde{u}_{m+1}(t) + 4\tilde{u}_m(t) - 3\tilde{u}_{m-1}(t)}{2h} + O\left(h^2\right). \end{aligned}$$

From Equations (4.17) solving for $\Psi_m(t), \Psi_{m-1}(t)$ and $\Psi_{m+1}(t)$ and substituting the values in Equation (4.13), we arrive at the following difference scheme:

$$\begin{aligned}
&\frac{1}{\sigma(\rho)}\left\{a_{m-1}\left(\frac{-\tilde{u}_{m+1}(t)+4\tilde{u}_m(t)-3\tilde{u}_{m-1}(t)}{2h}\right)\right\}+\frac{4}{\sigma(\rho)}\left\{a_m\left(\frac{\tilde{u}_{m+1}(t)-\tilde{u}_{m-1}(t)}{2h}\right)\right\}\\
&+\frac{1}{\sigma(\rho)}\left\{a_{m+1}\left(\frac{3\tilde{u}_{m+1}(t)-4\tilde{u}_m(t)+\tilde{u}_{m-1}(t)}{2h}\right)\right\}-\frac{6\varepsilon}{h^2}\left(\tilde{u}_{m-1}(t)-2\tilde{u}_m(t)+\tilde{u}_{m+1}(t)\right)+\\
&+\frac{1}{\sigma(\rho)}b_{m-1}(t)\tilde{u}_{m-1}(t)+\frac{1}{\sigma(\rho)}b_m(t)\tilde{u}_m(t)+\frac{1}{\sigma(\rho)}b_{m+1}(t)\tilde{u}_{m+1}(t)\\
&=-\frac{1}{\sigma(\rho)}c_{m-1}(t)\tilde{u}_{m-1}(t-\tau)+\frac{1}{\sigma(\rho)}f_{m-1}(t)-\frac{1}{\sigma(\rho)}\frac{\partial\tilde{u}(x_{m-1},t)}{\partial t}\\
&-\frac{1}{\sigma(\rho)}c_m(t)\tilde{u}_m(t-\tau)+\frac{1}{\sigma(\rho)}f_{m-1}(t)-\frac{1}{\sigma(\rho)}\frac{\partial\tilde{u}(x_m,t)}{\partial t}\\
&-\frac{1}{\sigma(\rho)}c_{m+1}(t)\tilde{u}_{m+1}(t-\tau)+\frac{1}{\sigma(\rho)}f_{m+1}(t)-\frac{1}{\sigma(\rho)}\frac{\partial\tilde{u}(x_{m+1},t)}{\partial t}+O\left(h^2\right).
\end{aligned} \tag{4.18}$$

4.2.1 Determining the Exponential Fitting Factor

From the theory of singular perturbations [31] it is known that the solution of (2.1)-(2.3) is of the form

$$u(x,t)=u_0(x,t)+z_0(x,t)+\varepsilon(u_1(x,t)+z_1(x,t))+O(\varepsilon),$$

where the zeroth-order asymptotic expansion $\tilde{u}(x,t)$ is given by:

$$\hat{u}(x,t)=u_0(x,t)+z_0(x,t), \tag{4.19}$$

with $u_0(x,t)$ as the solution of the reduced problem of equations (2.1)-(2.3) given by

$$a(x)\frac{\partial u_0(x,t)}{\partial x}+b(x,t)u_0(x,t)=F(x), \tag{4.20}$$

with

$$u_0(0,t)=\phi_l(t), \tag{4.21}$$

where $F(x)=c(x,t)u_0(x,t-\tau)+f(x,t)-\frac{\partial u_0(x,t)}{\partial t}, (x,t)\in D$ and $z_0(x,t)$ in (4.19) is the error correction term satisfying the differential equation

$$-\frac{d^2z_0(t)}{d\eta^2}+a(1,t)\frac{dz_0(t)}{d\eta}=0$$

with

$$z_0(1,t)=\phi_r(t)-u_0(1,t), z_0(\infty,t)=0,$$

where $\eta=\frac{1-x}{\varepsilon}$.
Using asymptotic expansion for the solution of Equations (4.20)-(4.21), it is simple to obtain $u_0(x,t)$ and using Taylor series expansion for $a(x)$ near the point $x=1$, we get

$$u(x,t)=u_0(x,t)+(\phi_r(t)-u_0(1,t))\exp\left(-a(1)\frac{(1-x)}{\varepsilon}\right). \tag{4.22}$$

Considering h is reasonably small and evaluating the result in (4.22) at x_m gives

$$\tilde{u}_m(t) = \tilde{u}_0(0,t) + (\phi_r(t) - \tilde{u}_0(1,t)) \exp\left(-a(1)\frac{(1-mh)}{\varepsilon}\right), \tag{4.23}$$

which becomes

$$\lim_{h\to 0} \tilde{u}_m(t) = \tilde{u}_0(0,t) + (\phi_r(t) - \tilde{u}_0(1,t)) \exp\left(-a(1)\left(\frac{1}{\varepsilon} - m\rho\right)\right), \tag{4.24}$$

Similarly,

$$\begin{aligned} \lim_{h\to 0} \tilde{u}_{m-1}(t) &= \tilde{u}_0(0,t) + (\phi_r(t) - \tilde{u}_0(1,t)) \exp\left(-a(1)\left(\frac{1}{\varepsilon} - (m-1)\rho\right)\right), \\ \lim_{h\to 0} \tilde{u}_{m+1}(t) &= \tilde{u}_0(0,t) + (\phi_r(t) - \tilde{u}_0(1,t)) \exp\left(-a(1)\left(\frac{1}{\varepsilon} - (m+1)\rho\right)\right). \end{aligned} \tag{4.25}$$

where $\rho = \frac{h}{\varepsilon}$.
Multiplying both sides of Equation (4.18) by h and taking the limit as $h \to 0$ give

$$\begin{aligned} &\frac{1}{\sigma(\rho)} a(1) \lim_{h\to 0}\left(\frac{-\tilde{u}_{m+1}(t) + 4\tilde{u}_m(t) - 3\tilde{u}_{m-1}(t)}{2}\right) + \\ &\frac{4}{\sigma(\rho)} a(1) \lim_{h\to 0}\left(\frac{\tilde{u}_{m+1}(t) - \tilde{u}_{m-1}(t)}{2}\right) + \\ &\frac{1}{\sigma(\rho)} \lim_{h\to 0} a(1)\left(\frac{3\tilde{u}_{m+1}(t) - 4\tilde{u}_m(t) + \tilde{u}_{m-1}(t)}{2}\right) - \\ &\frac{6}{\rho}\left(\tilde{u}_{m-1}(t) - 2\tilde{u}_m(t) + \tilde{u}_{m+1}(t)\right) = 0 \end{aligned} \tag{4.26}$$

Substituting (4.24)-(4.25) into (4.26) and simplifying give the fitting factor for the parameter ε as

$$\sigma(\rho) = \frac{\rho a(1)}{2} \coth\left(\frac{\rho a(1)}{2}\right).$$

4.3 Fully Discrete Scheme

The method developed in (4.18) together with equation (4.1) becomes,

$$\begin{aligned}
&L^{h,\Delta t}U_m^{n+1} \equiv \left(q_m^- + \frac{\Delta t}{2} r_m^-\right) U_{m-1}^{n+1} + \left(q_m^c + \frac{\Delta t}{2} r_m^c\right) U_m^{n+1} + \\
&\left(q_m^+ + \frac{\Delta t}{2} r_m^+\right) U_{m+1}^{n+1} = \left(q_m^- - \frac{\Delta t}{2} p_m^-\right) U_{m-1}^{n} + \left(q_m^c - \frac{\Delta t}{2} p_m^c\right) U_m^{n} + \\
&\left(q_m^+ - \frac{\Delta t}{2} p_m^+\right) U_{m+1}^{n} + \frac{\Delta t}{2} q_m^- \left(c_{m-1}^{n+1} H_{m-1}^{n+1} + c_{m-1}^{n} H_{m-1}^{n}\right) + \\
&\frac{\Delta t}{2} q_m^c \left(c_m^{n+1} H_m^{n+1} + c_m^{n} H_m^{n}\right) + \frac{\Delta t}{2} q_m^+ \left(c_{m+1}^{n+1} H_{m+1}^{n+1} + c_{m+1}^{n} H_{m+1}^{n}\right) + \\
&\frac{\Delta t}{2} q_m^- \left(f_{m-1}^{n+1} + f_{m-1}^{n}\right) + \frac{\Delta t}{2} q_m^c \left(f_m^{n+1} + f_m^{n}\right) + \frac{\Delta t}{2} q_m^+ \left(f_{m+1}^{n+1} + f_{m+1}^{n}\right), \\
&r_m^- = \frac{1}{\sigma(\rho)}\left(\frac{-3}{2}\frac{a_{m-1}}{h} + b_{m-1}^{n+1}\right) - \frac{2}{\sigma(\rho)}\frac{a_m}{h} + \frac{1}{\sigma_{m+1}}\frac{a_{m+1}}{2h} - \frac{6\varepsilon}{h^2}, \\
&r_m^c = \frac{2}{\sigma(\rho)}\frac{a_{m-1}}{h} + \frac{4}{\sigma(\rho)} b_m^{n+1} - \frac{2}{\sigma(\rho)}\frac{a_{m+1}}{h} + \frac{12\varepsilon}{h^2}, \\
&r_m^+ = \frac{-1}{\sigma(\rho)}\frac{a_{m-1}}{2h} + \frac{2}{\sigma_m}\frac{a_m}{h} + \frac{1}{\sigma(\rho)}\left(\frac{3}{2}\frac{a_{m+1}}{h} + b_{m+1}^{n+1}\right) - \frac{6\varepsilon}{h^2}, \\
&p_m^- = \frac{1}{\sigma(\rho)}\left(\frac{-3}{2}\frac{a_{m-1}}{h} + b_{m-1}^{n}\right) - \frac{2}{\sigma(\rho)}\frac{a_m}{h} + \frac{1}{\sigma(\rho)}\frac{a_{m+1}}{2h} - \frac{6\varepsilon}{h^2}, \\
&p_m^c = \frac{2}{\sigma(\rho)}\frac{a_{m-1}}{h} + \frac{4}{\sigma(\rho)} b_m^{n} - \frac{2}{\sigma(\rho)}\frac{a_{m+1}}{h} + \frac{12\varepsilon}{h^2}, \\
&p_m^+ = \frac{-1}{\sigma(\rho)}\frac{a_{m-1}}{2h} + \frac{2}{\sigma(\rho)}\frac{a_m}{h} + \frac{1}{\sigma(\rho)}\left(\frac{3}{2}\frac{a_{m+1}}{h} + b_{m+1}^{n}\right) - \frac{6\varepsilon}{h^2}, \\
&q_m^- = \frac{1}{\sigma(\rho)}, \quad q_m^c = \frac{4}{\sigma(\rho)}, \quad q_m^+ = \frac{1}{\sigma(\rho)},
\end{aligned} \tag{4.27}$$

with interval initial and boundary conditions

$$\begin{aligned}
&U_0^{n+1}(0) = U^{n+1}(0) = \phi_l(t_{n+1}), U_N^{n+1}(0) = U^{n+1}(1) = \phi_r(t_{n+1}), 0 \le n \le M-1, \\
&U_m^{n+1} = \phi_b(x_m, t_{n+1}), m = 1,2,3,...,N-1, -(s+1) \le n \le -1,
\end{aligned} \tag{4.28}$$

and H_m^n denotes the delayed term $U(x_m, t_n - s)$ which is evaluated as

$$H_m^n = \begin{cases} \phi_b(x_m, t_n), & \text{if } t_n < s, m = 0,1,...,M-1, \\ U_m^{n-s}, & \text{if } t_n \ge s, m = 0,1,...,M-1. \end{cases}$$

Lemma 4.4 (Discrete Comparison Principle)**.** *There exist a comparison function $\tilde{z}_m^{n+1}$ such that $L^{h,\Delta t}U_m^{n+1} \le L^{h,\Delta t}\tilde{z}_m^{n+1}, m = 1,2,...,N-1,$ and if $U_0^{n+1} \le \tilde{z}_0^{n+1}$ and $U_N^{n+1} \le \tilde{z}_N^{n+1}$ then $U_m^{n+1} \le \tilde{z}_m^{n+1}, m = 0,1,2,...,N.$*

Proof. The matrix associated with operator $L^{h,\Delta t}U_m^{n+1}$ is of size $(N+1)(N+1)$ with its entries for $m = 1, 2, ..., N-1$ are

$$\begin{aligned}
\hat{r}_m^- &= q_m^- + \frac{\Delta t}{2} r_m^- < 0, \\
\hat{r}_m^c &= q_m^c + \frac{\Delta t}{2} r_m^c > 0, \text{ as } \frac{b_m}{a_m} < \frac{3}{h}\left(\coth\left(\frac{h a_m}{2\varepsilon}\right) - 1\right), \\
r_m^+ &= q_m^+ + \frac{\Delta t}{2} r_m^+ < 0, \text{since } \coth\left(\frac{h a_m}{2\varepsilon}\right) \geq 1.
\end{aligned} \tag{4.29}$$

Therefore, the coefficient matrix of the proposed scheme, defined by (4.27)-(4.28), for the (2.1)-(2.3), satisfies the property of M-matrix. So, the inverse matrix exists and it is nonnegative. This guarantees the existence and uniqueness of the discrete solution. A similar technique has been used to show the stability for discrete operator in [18, 30]. □

An immediate consequence of the Lemma 4.4 is the following discrete stability result.

Lemma 4.5 (Discrete uniform stability estimate)**.** *The solution of the discrete scheme in* (4.27) *satisfies the bound*

$$\left|U_m^{n+1}\right| \leq \beta^{-1}\|L^{h,\Delta t} U_m^{n+1}\| + \max\left\{\left|U_0^{n+1}\right|, \left|U_N^{n+1}\right|\right\}.$$

Proof. Let $\lambda^* = \beta^{-1}\|L^{h,\Delta t} U_m^{n+1}\| + \max\left\{\left|U_0^{n+1}\right|, \left|U_N^{n+1}\right|\right\}$, and define the barrier function $\gamma^{\pm}$ by $\gamma^{\pm} = \lambda^* \pm U_m^{n+1}$. On the boundary points, we obtain

$$\begin{aligned}
\left(\gamma^{\pm}\right)_0^{n+1} &= \lambda^* \pm U_0^{n+1} = \beta^{-1}\|L^{h,\Delta t} U_m^{n+1}\| + \max\left\{\left|U_0^{n+1}\right|, \left|U_N^{n+1}\right|\right\} \pm U^{n+1}(0) \geq 0, \\
\left(\gamma^{\pm}\right)_N^{n+1} &= \lambda^* \pm U_N^{n+1} = \beta^{-1}\|L^{h,\Delta t} U_m^{n+1}\| + \max\left\{\left|U_0^{n+1}\right|, \left|U_N^{n+1}\right|\right\} \pm U^{n+1}(1) \geq 0.
\end{aligned}$$

On the discretized spatial domain $x_m, 0 < m < N$, we obtain

$$\begin{aligned}
& L^{h,\Delta t}\left(\gamma^{\pm}\right)_m^{n+1} \equiv \frac{1}{\sigma(\rho)} a_{m-1}\left(\frac{-\left(\lambda^* \pm U_{m+1}^{n+1}\right) + 4\left(\lambda^* \pm U_m^{n+1}\right) - 3\left(\lambda^* \pm U_{m-1}^{n+1}\right)}{2h}\right) + \\
& \frac{4}{\sigma(\rho)} a_m\left(\frac{\left(\lambda^* \pm U_{m+1}^{n+1}\right) - \left(\lambda^* \pm U_{m-1}^{n+1}\right)}{2h}\right) + \\
& \frac{1}{\sigma(\rho)} a_{m+1}\left(\frac{3\left(\lambda^* \pm U_{m+1}^{n+1}\right) - 4\left(\lambda^* \pm U_m^{n+1}\right) + \left(\lambda^* \pm U_{m-1}^{n+1}\right)}{2h}\right) - \\
& \frac{6\varepsilon}{h^2}\left(\left(\lambda^* \pm U_{m-1}^{n+1}\right) - 2\left(\lambda^* \pm U_m^{n+1}\right) + \left(\lambda^* \pm U_{m+1}^{n+1}\right)\right) + \\
& + \frac{1}{\sigma(\rho)} b_{m-1}^{n+1}\left(\lambda^* \pm U_{m+1}^{n+1}\right) + \frac{1}{\sigma(\rho)} b_m^{n+1}\left(\lambda^* \pm U_{m+1}^{n+1}\right) + \frac{1}{\sigma(\rho)} b_{m+1}^{n+1}\left(\lambda^* \pm U_{m+1}^{n+1}\right) \\
& = \left[\frac{1}{\sigma(\rho)} b_{m-1}^{n+1} + \frac{1}{\sigma(\rho)} b_m^{n+1} + \frac{1}{\sigma(\rho)} b_{m+1}^{n+1}\right]\left(\beta^{-1}\|L^{h,\Delta t} U_m^{n+1}\| + \max\left\{\left|U_0^{n+1}\right|, \left|U_N^{n+1}\right|\right\}\right) \pm g^{n+1} \\
& \geq 0, \text{ since } b_m^{n+1} \geq \beta > 0.
\end{aligned}$$

By the discrete comparison principle in Lemma 4.4, we obtain $\left(\gamma^{\pm}\right)_m^{n+1} \geq 0, m = 0, 1, 2, ..., N$. Hence, the required bound is obtained. □

Now, using Taylor's series approximation, we obtain the bound

$$\begin{aligned}
&\left\|\left(\frac{-U_{m+1}^{n+1}+4U_{m}^{n+1}-3U_{m-1}^{n+1}}{2h}-h\frac{d^2U_m^{n+1}}{dx^2}\right)-\frac{dU_{m-1}^{n+1}}{dx}\right\| \le CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\|,\\
&\left\|\frac{U_{m+1}^{n+1}-U_{m-1}^{n+1}}{2h}-\frac{dU_m^{n+1}}{dx}\right\| \le CN^{-2}\left\|\frac{d^2U^{n+1}(x_m)}{dx^2}\right\|,\\
&\left\|\left(\frac{3U_{m+1}^{n+1}-4U_{m}^{n+1}+U_{m-1}^{n+1}}{2h}+h\frac{d^2U_m^{n+1}}{dx^2}\right)-\frac{dU_{m+1}^{n+1}}{dx}\right\| \le CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\|,\\
&\left\|\frac{U_{m+1}^{n+1}-2U_{m}^{n+1}+U_{m-1}^{n+1}}{h^2}-\frac{d^2U_m^{n+1}}{dx^2}\right\| \le CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\|,
\end{aligned} \tag{4.30}$$

where $\left\|\frac{d^iU^{n+1}(x_m)}{dx^i}\right\| = \max_{x_0\le x\le x_N}\left|\frac{d^iU^{n+1}(x_m)}{dx^i}\right|, i=2,4.$

Theorem 4.2. *Let $a(x)$ and $f(x,t)$ be sufficiently smooth functions so that the semi-discrete solution $U^{n+1}(x)\in C^4[0,1]$ then the numerical solution U_m^{n+1} of the problems (2.1)-(2.3) satisfy the error bound*

$$\left|L^{h,\Delta t}\left(U^{n+1}(x_m)-U_m^{n+1}\right)\right| \le Ch^2\left(1+\varepsilon^{-4}\sup_{x\in(0,1)}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right) \tag{4.31}$$

Proof. Consider the truncation error that is given by

$$\begin{aligned}
&\left|L^{h,\Delta t}\left(U^{n+1}(x_m)-U_m^{n+1}\right)\right| \le \left\|-\varepsilon\left(\sigma(\rho)\frac{U_{m+1}^{n+1}-2U_m^{n+1}+U_{m-1}^{n+1}}{h^2}-\frac{d^2U_m^{n+1}}{dx^2}\right)\right\|+\\
&\left\|a_{m-1}\left(\left(\frac{-U_{m+1}^{n+1}+4U_m^{n+1}-3U_{m-1}^{n+1}}{2h}-h\frac{d^2U_m^{n+1}}{dx^2}\right)-\frac{dU_{m-1}^{n+1}}{dx}\right)\right\|+\\
&\left\|4a_m\left(\frac{U_{m+1}^{n+1}-U_{m-1}^{n+1}}{2h}-\frac{dU_m^{n+1}}{dx}\right)\right\|+\\
&\left\|a_{m+1}\left(\left(\frac{3U_{m+1}^{n+1}-4U_m^{n+1}+U_{m-1}^{n+1}}{2h}+h\frac{d^2U_m^{n+1}}{dx^2}\right)-\frac{dU_{m+1}^{n+1}}{dx}\right)\right\|.
\end{aligned} \tag{4.32}$$

□

Using the bounds given in (4.32)

$$\begin{aligned}
&\left|L^{h,\Delta t}\left(U^{n+1}(x_m)-U_m^{n+1}\right)\right| \le CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\|+CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\|+\\
&CN^{-2}\left\|\frac{d^2U^{n+1}(x_m)}{dx^2}\right\|+CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\|.
\end{aligned} \tag{4.33}$$

Substituting the bounds for the derivatives of the solution in Lemma 4.3, gives

$$\begin{aligned}
&\left|L^{h,\Delta t}\left(U^{n+1}(x_m)-U_m^{n+1}\right)\right| \leq CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\| + CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\| + \\
&CN^{-2}\left\|\frac{d^2U^{n+1}(x_m)}{dx^2}\right\| + CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\| \leq \\
&CN^{-2}\left\|\frac{d^2U^{n+1}(x_m)}{dx^2}\right\| + CN^{-2}\left\|\frac{d^4U^{n+1}(x_m)}{dx^4}\right\| \leq \\
&CN^{-2}\left(1+\varepsilon^{-2}\sup_{x\in(0,1)}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right) + CN^{-2}\left(1+\varepsilon^{-4}\sup_{x\in(0,1)}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right).
\end{aligned} \tag{4.34}$$

Since $\varepsilon^{-2} \leq \varepsilon^{-4}$, we obtain

$$\left|L^{h,\Delta t}\left(U^{n+1}(x_m)-U_m^{n+1}\right)\right| \leq CN^{-2}\left(1+\varepsilon^{-4}\sup_{x\in(0,1)}\exp\left(-\frac{\alpha(1-x)}{\varepsilon}\right)\right). \tag{4.35}$$

Lemma 4.6. *For $\varepsilon \longrightarrow 0$ and a fixed number of mesh numbers N, it holds*

$$\lim_{\varepsilon\longrightarrow 0}\max_{1\leq m\leq N-1}\frac{\exp\left(\frac{-\alpha(x_m)}{\varepsilon}\right)}{\varepsilon^i} = 0, \qquad \textit{and}$$
$$\lim_{\varepsilon\longrightarrow 0}\max_{1\leq m\leq N-1}\frac{\exp\left(\frac{-\alpha(1-x_m)}{\varepsilon}\right)}{\varepsilon^i} = 0,$$

for $i = 1, 2, 3, ...,$ where $x_m = mh, h = N^{-1}\forall m = 1, 2, ...N-1$.

Proof. The proof is given in [35]. □

Theorem 4.3. *Under the conditions of Theorem 4.2 and Lemma 4.6, the solution of the discrete schemes in (4.27) satisfies the following error bound:*

$$\sup_{0\leq\varepsilon<1}\left|U^{n+1}(x_m)-U_m^{n+1}\right| \leq Ch = CN^{-2}. \tag{4.36}$$

Proof. By using the results of Lemma 4.6 in Theorem 4.2 to the mesh function $U^{n+1}(x_m)-U_m^{n+1}$ and then using Lemma 4.5 gives the required bound. □

Theorem 4.4. *Let $u(x_m, t_{n+1})$ be the solution of the continuous problem (2.1)-(2.3) and U_m^{n+1} be the numerical solution of (4.27). Then, there exists a constant C such that the following uniform error estimate holds:*

$$\sup_{0<\varepsilon\leq 1}\max_{0\leq m\leq N, 0\leq n\leq M}\left|u(x_m, t_{n+1}) - U_m^{n+1}\right| \leq C\left(N^{-2} + (\Delta t)^2\right). \tag{4.37}$$

Proof. The proof is the consequence of Theorem 4.1 and 4.3. □

5 Numerical Examples and Discussion

In this section, we will present two numerical experiments to demonstrate the applicability of the method. Since the exact solutions of the problems are not known, the maximum pointwise errors for the examples are calculated using the following double mesh principle:

$$E_{\varepsilon}^{M,N} = \max_{0\le n\le M, 0\le m\le N} \left| U_M^N - U_{2M}^{2N} \right|$$

where M and N are the number of mesh points in x and t directions with h and k step sizes respectively. U_M^N is the approximate solution obtained using M and N number of meshes and U_{2M}^{2N} is approximate solution obtained using double number of meshes $2M$ and $2N$ with half step sizes.
For a value of M and N,the ε-uniform maximum pointwise error is calculated by the formula

$$E^{M,N} = \max_{\varepsilon} E_{\varepsilon}^{M,N}.$$

The orders of convergence for all the examples have been calculated by the formula

$$p^{M,N} = \frac{log\left|E_{\varepsilon}^{M,N}/E_{\varepsilon}^{2M,2N}\right|}{log_2}$$

For the sake of comparison with work in [25, 23, 13] our results presented as follows. For various values of ε, and N, the computed maximum pointwise errors $E_{\varepsilon}^{N,\Delta t}$, ε-uniform maximum pointwise error $E^{N,\Delta t}$ and the orders of convergence $p^{N,\Delta t}$ using the proposed scheme (4.27) are presented for Example 1 in Tables 3 and 4, and for Example 2 in Tables 5. From the results given in the Tables (Tables 3,4 and 5) we see the monotonically decreasing behavior of the computed ε-uniform errors. This ensures that the proposed exponentially fitted spline scheme (4.27) is ε-uniform convergent. It can also be seen that for a given value of ε in Table 2, the maximum point wise error decreases whereas the numerical rate of convergence increases as the number of mesh points increases which shows the convergence of the method. In the Tables (1 and 2) we confirm that the numerical methods presented in this paper are second order ε-uniform convergent.

For both examples the appearance of the boundary layers in the solutions visualized in surface plots of Figure 1 and Figure 2. Also from Figure 1 and Figure 2 one can observe the effect of the parameter ε on the boundary layer width for both examples. It is also clearly observed that the width of the boundary layer (located at the right side of D) decreases as ε decreases. The two Figures, Figure 1 and Figure 2, are plotted by taking $M = N = 64$. Figures 3 and 4 provide the solution for the two Examples for different values of time t by taking $M = N = 64$.

The maximum pointwise error on the log-log scale is plotted in Figure 1 for Example 1 and Figure 1 for Example 2. From this Figure one can observe that the error is ε-uniform second-order convergent and again confirms the effectiveness of exponentially fitted spline scheme (4.27).

Example 1. From [25] consider the following singularly perturbed delay parabolic initial boundary value problem on $D = (0, 1) \times [0, 2]$,

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (2 - x^2)\frac{\partial u}{\partial x} + (x + 1)(t + 1)u(x, t) \\
&= -u(x, t - \tau) + 10t^2 exp(-t)x(1 - x), \\
&u(x, t) = 0, (x, t) \in [0, 1] \times [-1, 0], \\
&u(0, t) = 0, u(1, t) = 0, t \in [0, 2].
\end{aligned}$$

Example 2. From [25] consider the following singularly perturbed delay parabolic initial boundary value problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \varepsilon\frac{\partial^2 u}{\partial x^2} + (2-x^2)\frac{\partial u}{\partial x} + xu(x,t)\\
&= -u(x,t-\tau) + 10t^2 exp(-t)x(1-x), (x,t) \in (0,1)\times(0,2],\\
&u(x,t) = 0, (x,t) \in [0,1]\times[-1,0]\,,\\
&u(0,t) = 0, u(1,t) = 0, t \in [0,2]\,.
\end{aligned}$$

Table 1: The maximum absolute errors $E_\varepsilon^{N,\Delta t}$ and rate of convergences $p_\varepsilon^{N,\Delta t}$ for Example 1.

$\varepsilon\downarrow$	$N=8$ $M=16$	$N=32$ $M=32$	$N=128$ $M=64$	$N=512$ $M=128$
2^{-0}	$4.1907e-04$	$9.0350e-05$	$6.3351e-05$	$3.2725e-05$
	2.2136	0.51216	0.95297	–
2^{-4}	$4.2232e-03$	$3.9477e-04$	$2.9400e-04$	$1.5743e-04$
	3.4193	0.42520	0.90111	–
2^{-8}	$7.7995e-03$	$2.2491e-03$	$4.2550e-04$	$1.5240e-04$
	1.7940	2.4021	1.4813	–
2^{-12}	$7.7995e-03$	$2.2853e-03$	$5.5611e-04$	$1.9372e-04$
	1.7710	2.0389	1.5214	–
2^{-16}	$7.7995e-03$	$2.2853e-03$	$5.5611e-04$	$1.9441e-04$
	1.7710	2.0389	1.5163	–
2^{-20}	$7.7995e-03$	$2.2853e-03$	$5.5611e-04$	$1.9441e-04$
	1.7710	2.0389	1.5163	–
2^{-24}	$7.7995e-03$	$2.2853e-03$	$5.5611e-04$	$1.9441e-04$
	1.7710	2.0389	1.5163	–
2^{-28}	$7.7995e-03$	$2.2853e-03$	$5.5611e-04$	$1.9441e-04$
	1.7710	2.0389	1.5163	–
2^{-32}	$7.7995e-03$	$2.2853e-03$	$5.5611e-04$	$1.9441e-04$
	1.7710	2.0389	1.5163	–
$E^{M,\Delta t}$	**7.7995e-03**	**2.2853e-03**	**5.5611e-04**	**1.9441e-04**
$p^{M,\Delta t}$	**1.7710**	**2.0389**	**1.5163**	**-**

6 Conclusion

In this paper, we presented an exponentially fitted finite difference scheme to solve singularly perturbed parabolic partial differential equations with a large time delay. The method is based on cubic spline. This method is unconditionally stable and is convergent with second-order accurate in both time and space. To substantiate the suitability of the proposed method, graphs have been plotted for both examples 1 and 2 for different values of the parameter ε. It is shown that the method is uniformly convergent independent of mesh parameters and perturbation parameter ε. Our numerical results are compared with the results given in Das and Natesan [13], Gowrisankar

Table 2: The maximum absolute errors $E_{\varepsilon}^{N,\Delta t}$ and rate of convergences $p_{\varepsilon}^{N,\Delta t}$ for Example 2

$\varepsilon\downarrow$	$N=8$ $M=16$	$N=32$ $M=32$	$N=128$ $M=64$	$N=512$ $M=128$
2^{-0}	$1.3691e-03$	$8.6850e-05$	$9.0325e-06$	$2.2250e-06$
	3.9786	3.2653	2.0213	–
2^{-4}	$1.0119e-02$	$1.0238e-03$	$6.7309e-05$	$8.6530e-06$
	3.3051	3.9270	2.9595	–
2^{-8}	$1.4394e-02$	$5.2090e-03$	$1.0676e-03$	$8.6567e-05$
	1.4664	2.2866	3.6244	–
2^{-12}	$1.4394e-02$	$5.2134e-03$	$1.5548e-03$	$4.0209e-04$
	1.4652	1.7455	1.9511	–
2^{-16}	$1.4394e-02$	$5.2134e-03$	$1.5548e-03$	$4.1136e-04$
	1.4652	1.7455	1.9183	–
2^{-20}	$1.4394e-02$	$5.2134e-03$	$1.5548e-03$	$4.1136e-04$
	1.4652	1.7455	1.9183	–
2^{-24}	$1.4394e-02$	$5.2134e-03$	$1.5548e-03$	$4.1136e-04$
	1.4652	1.7455	1.9183	–
2^{-28}	$1.4394e-02$	$5.2134e-03$	$1.5548e-03$	$4.1136e-04$
	1.4652	1.7455	1.9183	–
2^{-32}	$1.4394e-02$	$5.2134e-03$	$1.5548e-03$	$4.1136e-04$
	1.4652	1.7455	1.9183	–
$E^{M,\Delta t}$	**1.4394e-02**	**5.2134e-03**	**1.5548e-03**	**4.1136e-04**
$p^{M,\Delta t}$	**1.4652**	**1.7455**	**1.9183**	**-**

and Natesan [23] and Kumar and Kumari [25]. On the basis of the numerical results, it is concluded that the present method gives high accurate numerical results than those obtained by means of upwind differences in fitted mesh methods which are boundary-layer resolving, despite the fact that the exponentially fitted method is not a boundary-layer resolving one for small values of the perturbation parameter and/or when a small number of grid points is used in the calculations. Another difficulty faced with the cubic spline based scheme when fitted mesh methods is used that, it does not lead to a system of equations described by corresponding matrix to be a M-matrix, a very restrictive condition is needed on the mesh size, specially in the outer region where a coarse mesh is enough to reflect the behavior of the solution in that region. By way of contrast, exponentially fitted cubic spline method is the ease in its use and its computer implementation. The difference scheme so obtained generates a strictly diagonally dominant tridiagonal matrix. The matrix could also be made monotone under certain assumptions on the coefficients of the reaction and diffusion terms. These all offers significant advantage of our method for the singularly perturbed parabolic partial differential equation with large time delay.

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Table 3: Comparison of $E^{M,N}$ and $p^{M,N}$ for Example 1

$\varepsilon\backslash N\Delta t$	$N=16$ $\Delta t=0.1$	$N=32$ $\Delta t=0.1/2$	$N=64$ $\Delta t=0.1/2^2$	$N=128$ $\Delta t=0.1/2^3$	$N=256$ $\Delta t=0.1/2^4$	$N=512$ $\Delta t=0.1/2^5$
Present method						
2^{-8}	$4.4897e-3$	$2.3903e-3$	$1.0010e-3$	$4.1067e-4$	$1.2771e-4$	$3.5112e-5$
2^{-12}	$4.4899e-3$	$2.4265e-3$	$1.2438e-3$	$6.2688e-4$	$3.1421e-4$	$1.5389e-4$
2^{-16}	$4.4899e-3$	$2.4265e-3$	$1.2438e-3$	$6.2688e-4$	$3.1428e-4$	$1.5729e-4$
2^{-20}	$4.4899e-3$	$2.4265e-3$	$1.2438e-3$	$6.2688e-4$	$3.1428e-4$	$1.5729e-4$
2^{-24}	$4.4899e-3$	$2.4265e-3$	$1.2438e-3$	$6.2688e-4$	$3.1428e-4$	$1.5729e-4$
2^{-28}	$4.4899e-3$	$2.4265e-3$	$1.2438e-3$	$6.2688e-4$	$3.1428e-4$	$1.5729e-4$
$E^{N,\Delta t}$	**4.4899e-3**	**2.4265e-3**	**1.2438e-3**	**6.2688e-4**	**3.1428e-4**	**1.5729e-4**
$p^{N,\Delta t}$	**0.8878**	**0.9641**	**0.9885**	**0.9961**	**0.9986**	-
Method in [23]						
2^{-8}	$1.4765e-2$	$8.8375e-3$	$5.1378e-3$	$2.9210e-3$	$1.6270e-3$	$8.9215e-4$
2^{-12}	$1.6031e-2$	$9.8736e-3$	$5.8009e-3$	$3.3117e-3$	$1.8467e-3$	$1.0128e-3$
2^{-16}	$1.6114e-2$	$9.9456e-3$	$5.8507e-3$	$3.3418e-3$	$1.8638e-3$	$1.0226e-3$
2^{-20}	$1.6119e-2$	$9.9501e-3$	$5.8538e-3$	$3.3438e-3$	$1.8649e-3$	$1.0232e-3$
2^{-24}	$1.6119e-2$	$9.9504e-3$	$5.8540e-3$	$3.3439e-3$	$1.8650e-3$	$1.0232e-3$
2^{-28}	$1.6119e-2$	$9.9504e-3$	$5.8541e-3$	$3.3439e-3$	$1.8650e-3$	$1.0232e-3$
$E^{M,N}$	**1.6119e-2**	**9.9504e-3**	**5.8541e-3**	**3.3439e-3**	**1.8650e-3**	**1.0232e-3**
$p^{M,N}$	**0.6960**	**0.7653**	**0.8079**	**0.8424**	**0.8660**	-

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Table 4: Comparison of $E^{M,N}$ and $p^{M,N}$ for Example 1

$\varepsilon \downarrow$	$N = 16$ $M = 16$	$N = 32$ $M = 32$	$N = 64$ $M = 64$	$N = 128$ $M = 128$	$N = 256$ $M = 256$	$N = 512$ $M = 512$
Present Method						
2^{-8}	$4.2048e-3$	$2.2491e-3$	$1.0057e-3$	$4.1295e-4$	$1.2889e-4$	$3.5842e-5$
2^{-12}	$4.2050e-3$	$2.2853e-3$	$1.1742e-3$	$5.9247e-4$	$2.9712e-4$	$1.4537e-4$
2^{-16}	$4.2050e-3$	$2.2853e-3$	$1.1742e-3$	$5.9247e-4$	$2.9719e-4$	$1.4878e-4$
2^{-20}	$4.2050e-3$	$2.2853e-3$	$1.1742e-3$	$5.9247e-4$	$2.9719e-4$	$1.4878e-4$
2^{-24}	$4.2050e-3$	$2.2853e-3$	$1.1742e-3$	$5.9247e-4$	$2.9719e-4$	$1.4878e-4$
2^{-28}	$4.2050e-3$	$2.2853e-3$	$1.1742e-3$	$5.9247e-4$	$2.9719e-4$	$1.4878e-4$
2^{-32}	$4.2050e-3$	$2.2853e-3$	$1.1742e-3$	$5.9247e-4$	$2.9719e-4$	$1.4878e-4$
$E^{M,N}$	**4.2050e-3**	**2.2853e-3**	**1.1742e-3**	**5.9247e-4**	**2.9719e-4**	**1.4878e-4**
$p^{M,N}$	**0.8797**	**0.9607**	**0.9869**	**0.9954**	**1.0000**	-
Method in [25]						
2^{-8}	$2.59e-2$	$1.40e-2$	$6.97e-3$	$3.34e-3$	$1.59e-3$	$7.88e-4$
2^{-12}	$3.03e-2$	$1.69e-2$	$8.85e-3$	$4.48e-3$	$2.23e-3$	$1.10e-3$
2^{-16}	$3.06e-2$	$1.72e-2$	$8.99e-3$	$4.58e-3$	$2.30e-3$	$1.15e-3$
2^{-20}	$3.06e-2$	$1.72e-2$	$9.00e-3$	$4.58e-3$	$2.30e-3$	$1.15e-3$
2^{-24}	$3.06e-2$	$1.72e-2$	$9.00e-3$	$4.58e-3$	$2.30e-3$	$1.15e-3$
2^{-28}	$3.06e-2$	$1.72e-2$	$9.00e-3$	$4.58e-3$	$2.30e-3$	$1.15e-3$
2^{-32}	$3.06e-2$	$1.72e-2$	$9.00e-3$	$4.58e-3$	$2.30e-3$	$1.15e-3$
$E^{M,N}$	**3.06e-2**	**1.72e-2**	**9.00e-3**	**4.58e-3**	**2.30e-3**	**1.15e-3**
$p^{M,N}$	**0.8311**	**0.9344**	**0.9746**	**0.9937**	**1.0000**	-

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Table 5: Comparison of $E^{M,N}$ and $p^{M,N}$ for Example 2

$\varepsilon \backslash N\Delta t$	$N=16$ $\Delta t=0.1/2$	$N=32$ $\Delta t=0.1/2^2$	$N=64$ $\Delta t=0.1/2^3$	$N=128$ $\Delta t=0.1/2^4$	$N=256$ $\Delta t=0.1/2^5$	$N=512$ $\Delta t=0.1/2^6$
Present method						
10^{-6}	$9.4529e-3$	$5.3104e-3$	$2.8485e-3$	$1.5385e-3$	$7.9916e-4$	$4.0722e-4$
10^{-7}	$9.4529e-3$	$5.3104e-3$	$2.8485e-3$	$1.5385e-3$	$7.9916e-4$	$4.0722e-4$
10^{-8}	$9.4529e-3$	$5.3104e-3$	$2.8485e-3$	$1.5385e-3$	$7.9916e-4$	$4.0722e-4$
10^{-9}	$9.4529e-3$	$5.3104e-3$	$2.8485e-3$	$1.5385e-3$	$7.9916e-4$	$4.0722e-4$
10^{-10}	$9.4529e-3$	$5.3104e-3$	$2.8485e-3$	$1.5385e-3$	$7.9916e-4$	$4.0722e-4$
$E^{M,N}$	**9.2016e-3**	**5.3104e-3**	**2.8485e-3**	**1.5385e-3**	**7.9916e-4**	**4.0722e-4**
$p^{M,N}$	**0.7931**	**0.8986**	**0.8887**	**0.9449**	**0.9727**	-
Method in [13]						
10^{-6}	$4.9484e-2$	$3.3202e-2$	$2.1164e-2$	$1.3319e-2$	$7.9342e-3$	$4.5863e-3$
10^{-7}	$4.9485e-2$	$3.3203e-2$	$2.1165e-2$	$1.3319e-2$	$7.9345e-3$	$4.5865e-3$
10^{-8}	$4.9485e-2$	$3.3203e-2$	$2.1165e-2$	$1.3319e-2$	$7.9344e-3$	$4.5864e-3$
10^{-9}	$4.9485e-2$	$3.3203e-2$	$2.1165e-2$	$1.3320e-2$	$7.9345e-3$	$4.5859e-3$
10^{-10}	$4.9485e-2$	$3.3202e-2$	$2.1164e-2$	$1.3319e-2$	$7.9243e-3$	$4.5813e-3$
$E^{M,N}$	**4.9485e-2**	**3.3203e-2**	**2.1165e-2**	**1.3320e-2**	**7.9345e-3**	**4.5865e-3**
$p^{M,N}$	**0.5757**	**0.6496**	**0.6681**	**0.7474**	**0.7908**	-

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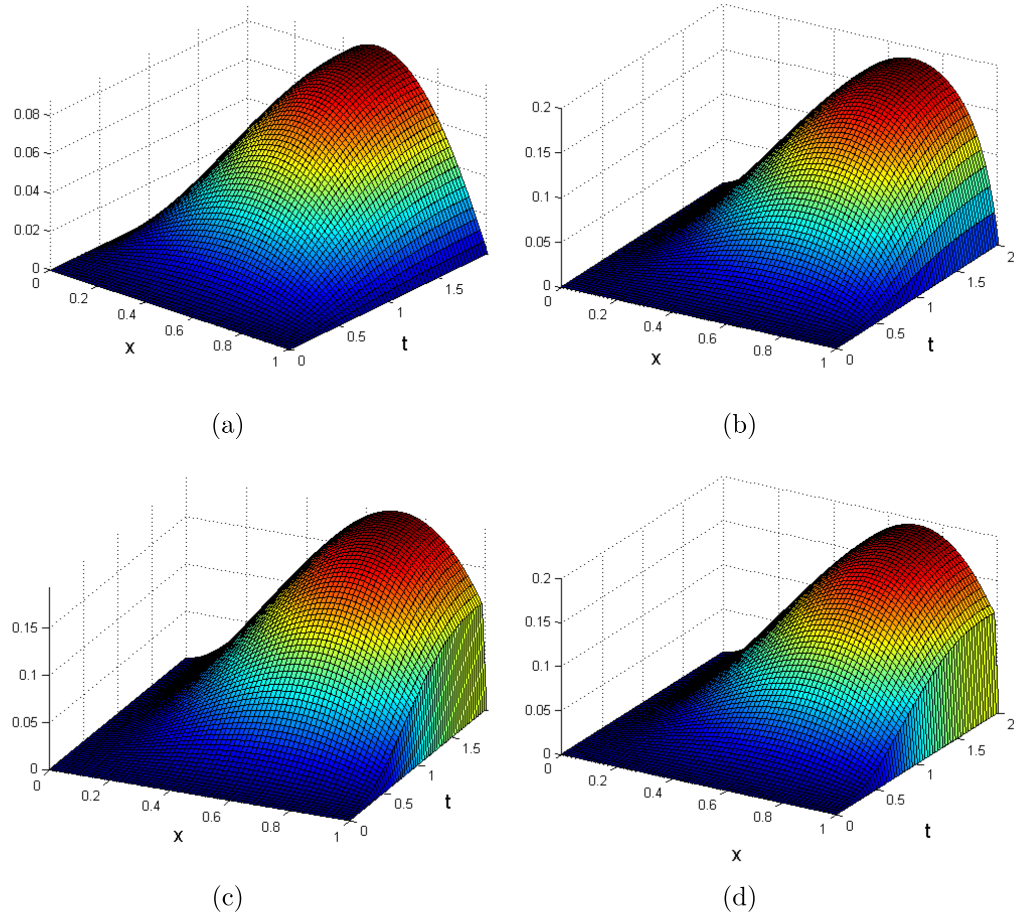

Figure 1: Numerical solution for Example 1 for different values of ε and t **a** $\varepsilon = 1$,**b** $\varepsilon = 2^{-4}$,**c** $\varepsilon = 2^{-8}$, **d** $\varepsilon = 2^{-12}$.

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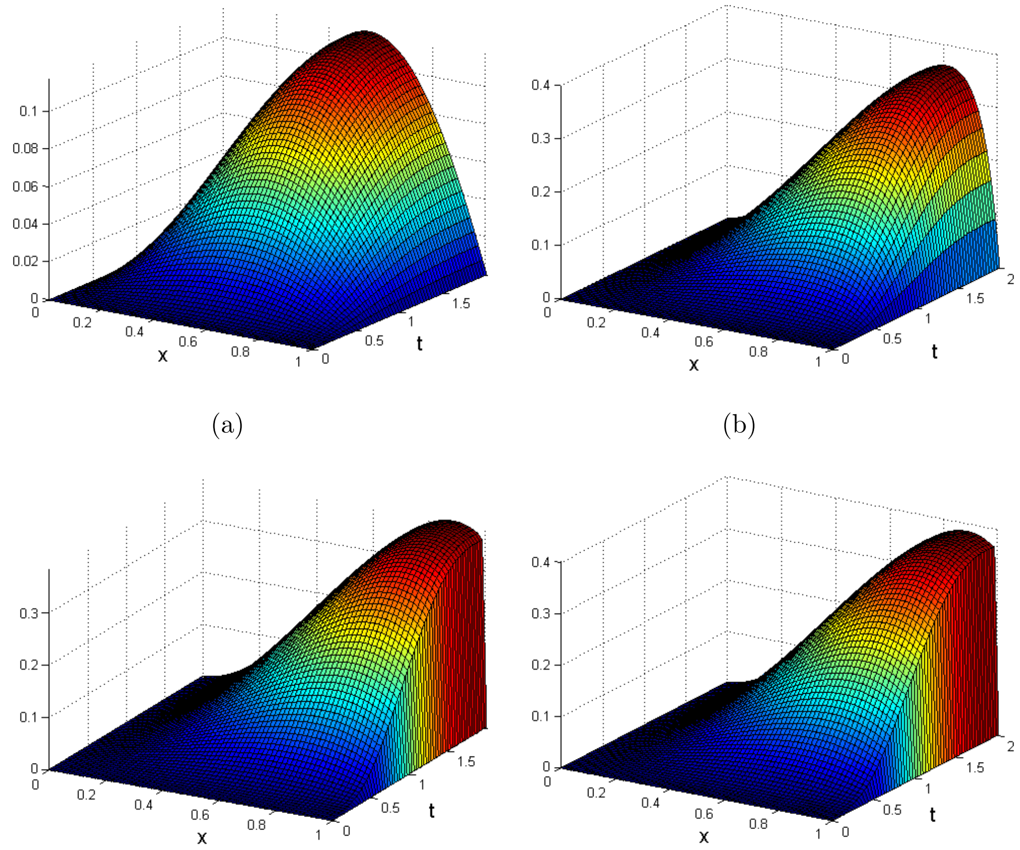

(a) (b)

(c) (d)

Figure 2: Numerical solution for Example 2 for different values of ε and t **a** $\varepsilon = 1$,**b** $\varepsilon = 2^{-4}$,**c** $\varepsilon = 2^{-8}$, **d** $\varepsilon = 2^{-12}$.

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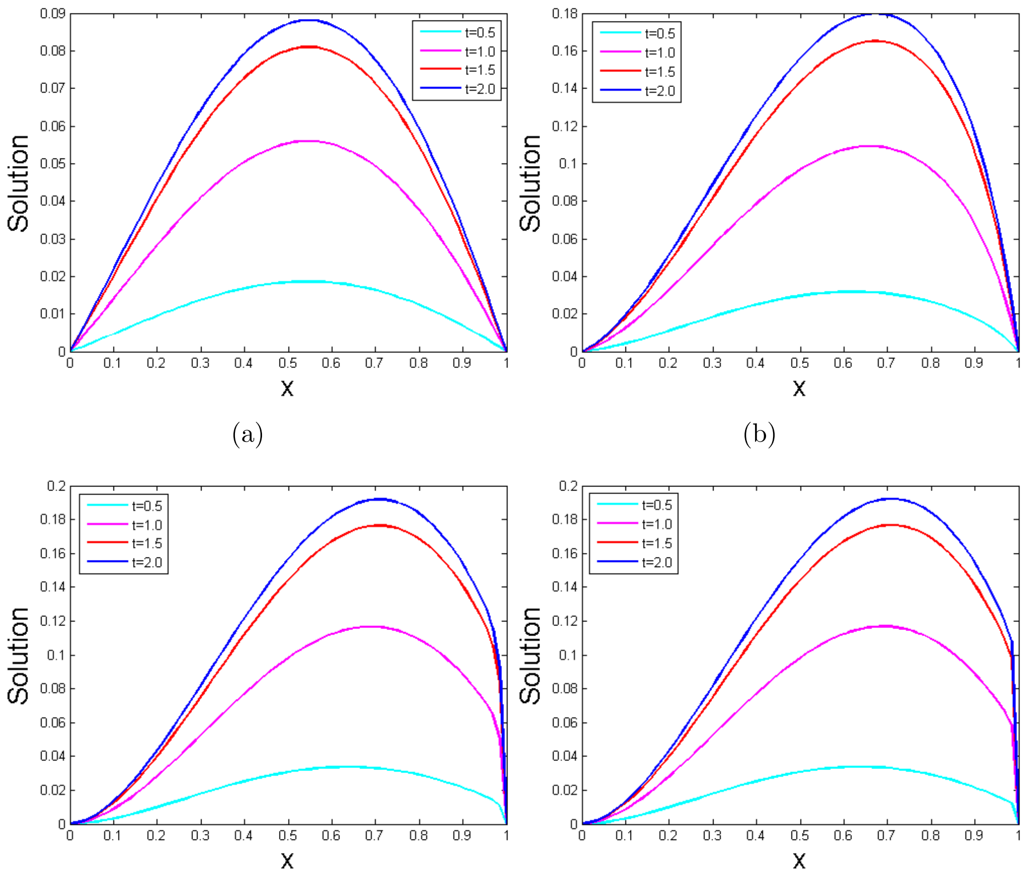

(a) (b)

(c) (d)

Figure 3: Numerical solution for Example 1 for different values of ε and t **a** $\varepsilon = 1$,**b** $\varepsilon = 0.1$,**c** $\varepsilon = 0.01$, **d** $\varepsilon = 0.005$.

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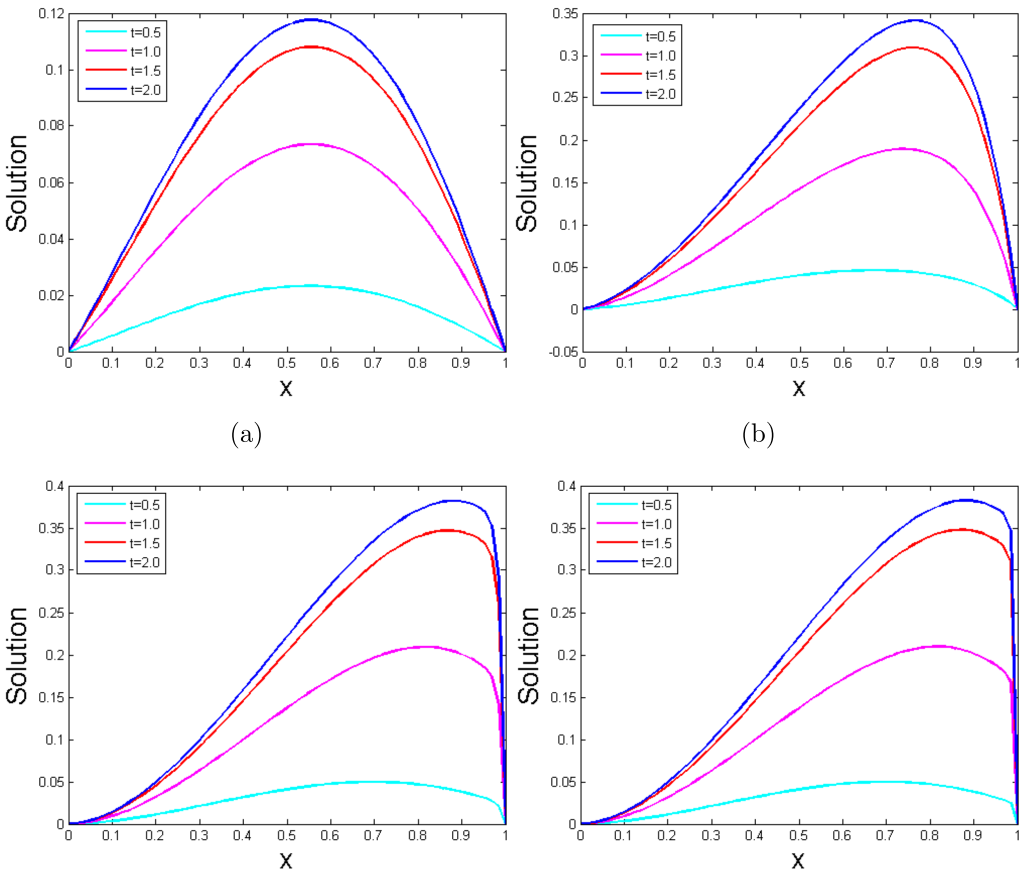

(a) (b)

(c) (d)

Figure 4: Numerical solution for Example 2 for different values of ε and t **a** $\varepsilon = 1$,**b** $\varepsilon = 0.1$,**c** $\varepsilon = 0.01$, **d** $\varepsilon = 0.005$.

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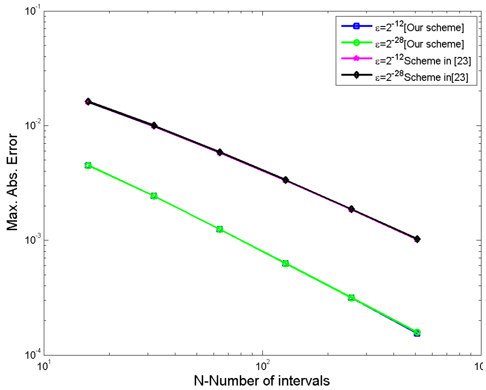
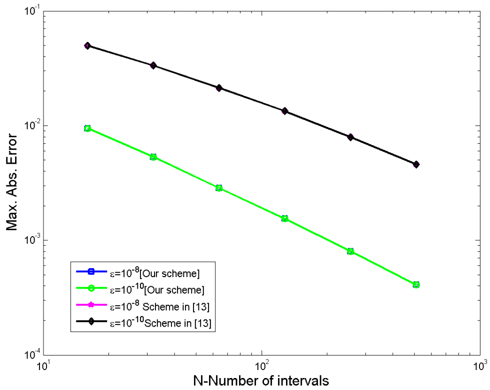


(a)



(b)

Figure 5: Visualization of the order of convergence through log-log plot in **(a)** for Example 1 and in **(b)** for Example 2 and comparison with the other schemes.

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