TAMKANG JOURNAL OF MATHEMATICS
Volume 54, Number 2, 175-191, 2023
DOI:10.5556/j.tkjm.54.2023.3988

Inequalities for Lerch transcendent function

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Abstract. Some fundamental inequalities for Lerch transcendent function with positive terms by utilising certain classical results due to Hölder, Čebyšev, Grüss and others, are established. Some particular cases of interest for Polylogarithm function, Hurwitz zeta function and Legendre chi function are also given.

Keywords. Lerch transcendent function, Power series, Dirichlet series, Zeta function, Polylogarithm function, Hurwitz zeta function, Legendre chi function, Discrete inequalities.

1 Introduction

The *Lerch transcendent* function is given by the series

$$\Phi(z,s,\alpha) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s}, \ \ |z|<1, \ \alpha \neq 0,-1,-2,... \tag{1.1}$$

see for instance [6, Section 1.11, p. 27] or [1, Section 25.14]. This function, defined by Mathias Lerch in 1887 in his paper [8], includes as special cases of the parameters; the Hurwitz, Riemann zeta functions and the polylogarithms, among others. Therefore the transcendent has applications ranging from number theory to physics.

The *Hurwitz zeta* function, formally defined for complex arguments s with $\mathrm{Re}(s) > 1$ and α with $\mathrm{Re}(\alpha) > 0$ by

$$\zeta(s,\alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s} \tag{1.2}$$

is a special case, given by

$$\zeta(s,\alpha) = \Phi(1,s,\alpha). \tag{1.3}$$

For $\alpha = 1$ we have the *Riemann zeta* function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1.4}$$

Received date: October 24, 2020; Published online: July 29, 2022.
2010 *Mathematics Subject Classification*. 26D15, 11M38, 11M41.
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The *polylogarithm* function $\mathrm{Li}(s,z)$ is defined by a power series in z, which is also a *Dirichlet series* in s:

$$\mathrm{Li}(s,z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1). \tag{1.5}$$

This definition is valid for arbitrary complex order s and for all complex arguments z with $|z| < 1$; it can be extended to $|z| \geq 1$ by the process of analytic continuation. The special case $s = 1$ involves the ordinary natural logarithm, $\mathrm{Li}(1,z) = -\ln(1-z)$, while the special cases $s = 2$ and $s = 3$ are called the *dilogarithm* (also referred to as *Spence's function*) and *trilogarithm* respectively.

The *Legendre chi* function is a special case, given by

$$\chi_s(z) = 2^{-s} z\Phi\left(z^2, s, 1/2\right). \tag{1.6}$$

The *Legendre chi* function is a special function whose Taylor series is also a Dirichlet series, given by

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s}. \tag{1.7}$$

Various identities include in [6, p. 27] the following:

$$\Phi(z,s,\alpha) = z^{m+1}\Phi(z,s,\alpha+m+1) + \sum_{n=0}^{m} \frac{z^n}{(n+\alpha)^s}, \tag{1.8}$$

where m is a natural number and $\alpha \neq 0, -1, -2, ...$

In this paper we obtain some fundamental inequalities for Lerch transcendent function with positive terms by utilising certain classical results due to Hölder, Čebyšev, Grüss and others. Some particular cases of interest for Polylogarithm function, Hurwitz zeta function and Legendre chi function are also given.

2 Some Convexity Results

In the following theorem we develop the convexity results of the first variable.

Theorem 2.1. *Assume that* s, $\alpha > 0$.

(i) The function $\Phi(|\cdot|, s, \alpha)$ *is convex on the open disk* $D(0,1) := \{z \in \mathbb{C} | \ |z| < 1\}$,

(ii) The function $\Phi(\cdot, s, \alpha)$ *is GG-convex on the interval* $[0,1)$, *namely it satisfies the condition*

$$\Phi\left(x^{1-t}y^t, s, \alpha\right) \leq [\Phi(x,s,\alpha)]^{1-t} [\Phi(y,s,\alpha)]^t \tag{2.1}$$

for all x, $y \in [0,1)$ *and* $t \in [0,1]$,

(iii) For all x, $y \in [0,1)$ *with* $x + y \in [0,1)$,

$$\Phi(x+y,s,\alpha) + \frac{1}{\alpha} \geq \Phi(x,s,\alpha) + \Phi(y,s,\alpha). \tag{2.2}$$

Proof. (i). Let $z,\, w\in D(0,1)$ and $t\in[0,1]$. By the convexity of $|\cdot|^n$ for $n\geq 0$ we have

$$|(1-t)\,z+tw|^n\leq(1-t)\,|z|^n+t\,|w|^n$$

for $z,\, w\in D(0,1)$ and $t\in[0,1]$.

Therefore

$$\begin{aligned}\sum_{n=0}^{m}\frac{|(1-t)\,z+tw|^n}{(n+\alpha)^s}&\leq\sum_{n=0}^{m}\frac{(1-t)\,|z|^n+t\,|w|^n}{(n+\alpha)^s}\\&=(1-t)\sum_{n=0}^{m}\frac{|z|^n}{(n+\alpha)^s}+t\sum_{n=0}^{m}\frac{|w|^n}{(n+\alpha)^s}\end{aligned}\tag{2.3}$$

for $z,\, w\in D(0,1)$, for all $m\geq 1$ and $t\in[0,1]$.

Since the series

$$\sum_{n=0}^{\infty}\frac{|(1-t)\,z+tw|^n}{(n+\alpha)^s},\quad\sum_{n=0}^{\infty}\frac{|z|^n}{(n+\alpha)^s}\text{ and }\sum_{n=0}^{\infty}\frac{|w|^n}{(n+\alpha)^s}$$

are convergent, then by taking the limit over $m\to\infty$ in (2.3), we deduce

$$\Phi\left(|(1-t)\,z+tw|\,,s,\alpha\right)\leq(1-t)\,\Phi\left((1-t)\,|z|\,,s,\alpha\right)+t\Phi\left((1-t)\,|w|\,,s,\alpha\right),$$

which proves the statement.

(ii). Let $x,\, y\in[0,1)$ and $t\in(0,1)$. For $m\geq 1$ we have

$$\begin{aligned}\sum_{n=0}^{m}\frac{\left(x^{1-t}y^t\right)^n}{(n+\alpha)^s}&=\sum_{n=0}^{m}\frac{(x^n)^{1-t}\,(y^n)^t}{(n+\alpha)^s}\\&\leq\left(\sum_{n=0}^{m}\frac{\left[(x^n)^{1-t}\right]^{1/(1-t)}}{(n+\alpha)^s}\right)^{1-t}\left(\sum_{n=0}^{m}\frac{\left[(y^n)^t\right]^{1/t}}{(n+\alpha)^s}\right)^t\\&=\left(\sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^s}\right)^{1-t}\left(\sum_{n=0}^{m}\frac{y^n}{(n+\alpha)^s}\right)^t,\end{aligned}\tag{2.4}$$

where for the first inequality we used Hölder's inequality

$$\sum_{n=0}^{m}p_na_nb_n\leq\left(\sum_{n=0}^{m}p_na_n^p\right)^{1/p}\left(\sum_{n=0}^{m}p_nb_n^q\right)^{1/q}$$

with $p_n,\, a_n,\, b_n\geq 0$ for $n\in\{0,...,m\}$, $p,\, q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, for the choices

$$p_n=\frac{1}{(n+\alpha)^s},\ a_n=(x^n)^{1-t},\ b_n=(y^n)^t$$

while $p=1/(1-t)>1$, $q=1/t>1$.

Since the series

$$\sum_{n=0}^{\infty}\frac{\left(x^{1-t}y^t\right)^n}{(n+\alpha)^s},\quad\sum_{n=0}^{\infty}\frac{x^n}{(n+\alpha)^s}\text{ and }\sum_{n=0}^{\infty}\frac{y^n}{(n+\alpha)^s}$$

are convergent, then by taking the limit over $m \to \infty$ in (2.4) we get

$$\Phi\left(x^{1-t}y^{t}, s, \alpha\right) \le [\Phi(x, s, \alpha)]^{1-t} [\Phi(y, s, \alpha)]^{t}$$

for all $x,\, y \in [0,1)$ and $t \in (0,1)$. The cases $t = 0$ and $t = 1$ are obvious.

(iii). We consider the function $f_s : [0,\infty) \to \mathbb{R}$, $f_s(t) = (t+1)^s - t^s$ we have $f_s'(t) = s\left[(t+1)^{s-1} - t^{s-1}\right]$. Observe that for $s > 1$ and $t > 0$ we have that $f_s'(t) > 0$ showing that f_s is strictly increasing on the interval $[0,\infty)$. Now for $t_0 = \frac{a}{b}$ $(b > 0, a \ge 0)$ we have $f_s(t_0) > f_s(0)$ giving that $\left(\frac{a}{b}+1\right)^s - \left(\frac{\alpha}{b}\right)^s > 1$, i.e., the inequality

$$(a+b)^s > a^s + b^s.$$

Therefore

$$(x+y)^n \ge x^n + y^n$$

for all $x,\, y \ge 0$ and $n \ge 1$.

If $x,\, y \in [0,1)$ with $x + y \in [0,1)$, then

$$\sum_{n=1}^{m} \frac{(x+y)^n}{(n+\alpha)^s} \ge \sum_{n=1}^{m} \frac{x^n}{(n+\alpha)^s} + \sum_{n=1}^{m} \frac{y^n}{(n+\alpha)^s}.$$

Since the series

$$\sum_{n=1}^{\infty} \frac{(x+y)^n}{(n+\alpha)^s},\quad \sum_{n=1}^{\infty} \frac{x^n}{(n+\alpha)^s} \text{ and } \sum_{n=1}^{\infty} \frac{y^n}{(n+\alpha)^s} \tag{2.5}$$

are convergent, then by taking the limit over $m \to \infty$ in (2.5) we get

$$\Phi(x+y, s, \alpha) - \frac{1}{\alpha} \ge \Phi(x, s, \alpha) - \frac{1}{\alpha} + \Phi(y, s, \alpha) - \frac{1}{\alpha}$$

for $x,\, y \in [0,1)$ with $x + y \in [0,1)$, and the inequality (2.2) is obtained. □

Further, we have the following convexity results for the second and third variables.

Theorem 2.2. *Assume that* $x \in (0,1)$.

(i) The function $\Phi(x, \cdot, \alpha)$ *is logarithmic convex on* $(0,\infty)$ *for all* $\alpha > 0$;

(ii) The function $\Phi(x, s, \cdot)$ *is convex on* $(0,\infty)$ *for all* $s > 0$.

Proof. (i). Let $s_1,\, s_2 > 0$ and $t \in (0,1)$. We have

$$\frac{1}{(n+\alpha)^{(1-t)s_1+ts_2}} = \left(\frac{1}{(n+\alpha)^{s_1}}\right)^{1-t} \left(\frac{1}{(n+\alpha)^{s_2}}\right)^{t}.$$

Therefore

$$\begin{aligned} \sum_{n=0}^{m} \frac{x^n}{(n+\alpha)^{(1-t)s_1+ts_2}} &= \sum_{n=0}^{m} x^n \left(\frac{1}{(n+\alpha)^{s_1}}\right)^{1-t} \left(\frac{1}{(n+\alpha)^{s_2}}\right)^{t} \\ &\le \left(\sum_{n=0}^{m} x^n \left[\left(\frac{1}{(n+\alpha)^{s_1}}\right)^{1-t}\right]^{1/(1-t)}\right)^{1-t} \end{aligned} \tag{2.6}$$

$$\times\left(\sum_{n=0}^{m} x^n\left[\left(\frac{1}{(n+\alpha)^{s_2}}\right)^t\right]^{1/t}\right)^t$$
$$=\left(\sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^{s_1}}\right)^{1-t}\left(\sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^{s_2}}\right)^t,$$

where for the first inequality we used Hölder's inequality

$$\sum_{n=0}^{m} p_n a_n b_n \le \left(\sum_{n=0}^{m} p_n a_n^p\right)^{1/p}\left(\sum_{n=0}^{m} p_n b_n^q\right)^{1/q} \tag{2.7}$$

with p_n, a_n, $b_n \ge 0$ for $n \in \{0, ..., m\}$, p, $q > 1$ and $\frac{1}{p}+\frac{1}{q}=1$, for the choices

$$p_n = x^n,\ a_n = \left(\frac{1}{(n+\alpha)^{s_1}}\right)^{1-t},\ b_n = \left(\frac{1}{(n+\alpha)^{s_2}}\right)^t$$

while $p = 1/(1-t) > 1$, $q = 1/t > 1$.

Since the series

$$\sum_{n=0}^{\infty}\frac{x^n}{(n+\alpha)^{(1-t)s_1+ts_2}},\ \sum_{n=0}^{\infty}\frac{x^n}{(n+\alpha)^{s_1}} \text{ and } \sum_{n=0}^{\infty}\frac{x^n}{(n+\alpha)^{s_2}}$$

are convergent, then by taking the limit over $m \to \infty$ in (2.6) we get

$$\Phi(x,(1-t)s_1+ts_2,\alpha) \le [\Phi(x,s_1,\alpha)]^{(1-t)}[\Phi(x,s_2,\alpha)]^t$$

s_1, $s_2 > 0$ and $t \in [0,1]$, namely the logarithmic convexity in the second variable.

(ii). Let α_1, $\alpha_2 > 0$ and $t \in [0,1]$. By the convexity of negative power function $f(u) = u^{-s}$, $u > 0$, $s > 0$ we have

$$\frac{1}{(n+(1-t)\alpha_1+t\alpha_2)^s} = \frac{1}{((1-t)(n+\alpha_1)+t(n+\alpha_2))^s}$$
$$\le (1-t)\frac{1}{(n+\alpha_1)^s}+t\frac{1}{(n+\alpha_2)^s}$$

for $n \ge 0$.

Therefore

$$\sum_{n=0}^{m}\frac{x^n}{(n+(1-t)\alpha_1+t\alpha_2)^s} \le (1-t)\sum_{n=0}^{m}\frac{x^n}{(n+\alpha_1)^s}+t\sum_{n=0}^{m}\frac{x^n}{(n+\alpha_2)^s} \tag{2.8}$$

for α_1, $\alpha_2 > 0$, $x \in (0,1)$ and $t \in [0,1]$.

Since the series

$$\sum_{n=0}^{\infty}\frac{x^n}{(n+(1-t)\alpha_1+t\alpha_2)^s},\ \sum_{n=0}^{m}\frac{x^n}{(n+\alpha_1)^s} \text{ and } \sum_{n=0}^{m}\frac{x^n}{(n+\alpha_2)^s}$$

are convergent, then by taking the limit over $m \to \infty$ in (2.6) we get

$$\Phi(x,s,(1-t)\alpha_1+t\alpha_2) \le (1-t)\Phi(x,s,\alpha_1)+t\Phi(x,s,\alpha_2)$$

and the desired convexity is proved. □

We can define the *m-truncated Lerch function* by

$$\Phi_m(z,s,\alpha) := \sum_{n=0}^{m} \frac{z^n}{(n+\alpha)^s},\ m \geq 0.$$

By the identity (1.8) we have

$$\Phi_m(z,s,\alpha) = \Phi(z,s,\alpha) - z^{m+1}\Phi(z,s,\alpha+m+1),\ m \geq 0. \tag{2.9}$$

We observe that in the proofs of Theorems 2.1 and 2.2 we proved the required inequalities for the finite case and therefore the above results remain valid if we replace Φ by Φ_m. As a consequence, we can state the following results as well:

Theorem 2.3. *a) Assume that $s,\ \alpha > 0$ and $m \geq 1$.*

(i) The function $\Phi(|\cdot|,s,\alpha) - |\cdot|^{m+1}\Phi(|\cdot|,s,\alpha+m+1)$ is convex on the open disk $D(0,1) := \{z \in \mathbb{C}|\ |z| < 1\}$,

(ii) The function $\Phi(\cdot,s,\alpha) - (\cdot)^{m+1}\Phi(\cdot,s,\alpha+m+1)$ is GG-convex on the interval $[0,1)$, namely

$$\begin{aligned}0 &\leq \Phi\left(x^{1-t}y^t,s,\alpha\right) - z^{m+1}\Phi\left(x^{1-t}y^t,s,\alpha+m+1\right) \\ &\leq \left[\Phi(x,s,\alpha) - x^{m+1}\Phi(x,s,\alpha+m+1)\right]^{1-t} \\ &\times \left[\Phi(y,s,\alpha) - y^{m+1}\Phi(y,s,\alpha+m+1)\right]^{t}\end{aligned} \tag{2.10}$$

for all $x,\ y \in [0,1)$ and $t \in [0,1]$,

(iii) For all $x,\ y \in [0,1)$ with $x+y \in [0,1)$,

$$\begin{aligned}&\Phi(x+y,s,\alpha) - \Phi(x,s,\alpha) - \Phi(y,s,\alpha) + \frac{1}{\alpha} \\ &\geq (x+y)^{m+1}\Phi(x+y,s,\alpha+m+1) - x^{m+1}\Phi(x,s,\alpha+m+1) \\ &- y^{m+1}\Phi(y,s,\alpha+m+1).\end{aligned} \tag{2.11}$$

b) Assume that $x \in (0,1)$ and $m \geq 1$.

(iv) The function $\Phi(x,\cdot,\alpha) - x^{m+1}\Phi(x,\cdot,\alpha+m+1)\,\Phi(x,\cdot,\alpha)$ is logarithmic convex on $(0,\infty)$ for all $\alpha > 0$;

(v) The function $\Phi(x,s,\cdot) - x^{m+1}\Phi(x,s,\cdot+m+1)\,\Phi(x,\cdot,\alpha)$ is convex on $(0,\infty)$ for all $s > 0$.

Remark 1. If we consider the *polylogarithm* function

$$\mathrm{Li}(s,z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s},\ \ |z| < 1,\ \alpha \neq 0,-1,-2,...$$

then, by utilising a similar argument as above, we conclude that the function $\mathrm{Li}(s,|\cdot|)$ is convex on the open disk $D(0,1)$, $\mathrm{Li}(s,\cdot)$is *GG*-convex on the interval $[0,1)$ and satisfies the superadditivity property

$$\mathrm{Li}(s,x+y) \geq \mathrm{Li}(s,x) + \mathrm{Li}(s,y) \tag{2.12}$$

for all x, $y \in [0,1)$ with $x+y \in [0,1)$. Also, the function $\mathrm{Li}(\cdot, x)$ is logarithmic convex on $(0,\infty)$ for all $x \in (0,1)$.

The *Legendre chi* is given by

$$\chi_s(z) := \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)^s}.$$

By making use of a similar argument to the one from the proofs of Theorems 2.1 and 2.2 we conclude that $\chi_s(|\cdot|)$ is convex on the open disk $D(0,1)$, χ_s is *GG*-convex on the interval $[0,1)$ and satisfies the superadditivity property

$$\chi_s(x+y) \geq \chi_s(x) + \chi_s(y) \tag{2.13}$$

for all x, $y \in [0,1)$ with $x+y \in [0,1)$. Also, the function $\chi_\cdot(x)$ is logarithmic convex on $(0,\infty)$ for all $x \in (0,1)$.

3 Further Inequalities

The following result may be stated:

Proposition 3.1. *Let α, $\beta > 1$ with $\alpha^{-1} + \beta^{-1} = 1$ and $x \in (0,1)$, $\gamma > 0$. If s, p, $q \in \mathbb{R}$ are such that $s+p+q > 0$, $s + p\alpha > 0$ and $s + q\beta > 0$, then*

$$\Phi(x, s+p+q, \gamma) \leq [\Phi(x, s+p\alpha, \gamma)]^{\frac{1}{\alpha}} [\Phi(x, s+q\beta, \gamma)]^{\frac{1}{\beta}}. \tag{3.1}$$

For $x = 1$, we get

$$\zeta(s+p+q, \gamma) \leq [\zeta(s+p\alpha, \gamma)]^{1/\alpha} [\zeta(s+q\beta, \gamma)]^{1/\beta} \tag{3.2}$$

provided s, p, $q \in \mathbb{R}$ are such that $s+p+q > 1$, $s+p\alpha > 1$ and $s+q\beta > 1$.

Proof. We use Hölder's inequality to state that:

$$\begin{aligned}
\Phi(x, s+p+q, \gamma) &= \sum_{n=0}^{\infty} \frac{x^n}{(n+\gamma)^{s+p+q}} \\
&= \sum_{n=0}^{\infty} \frac{x^n}{(n+\gamma)^s} \cdot \frac{1}{(n+\gamma)^p} \cdot \frac{1}{(n+\gamma)^q} \\
&\leq \left[\sum_{n=0}^{\infty} \frac{x^n}{(n+\gamma)^s} \cdot \left(\frac{1}{(n+\gamma)^p}\right)^{\alpha}\right]^{\frac{1}{\alpha}} \\
&\times \left[\sum_{n=0}^{\infty} \frac{x^n}{(n+\gamma)^s} \cdot \left(\frac{1}{(n+\gamma)^q}\right)^{\beta}\right]^{\frac{1}{\beta}} \\
&= \left(\sum_{n=0}^{\infty} \frac{x^n}{(n+\gamma)^{s+p\alpha}}\right)^{\frac{1}{\alpha}} \left(\sum_{n=1}^{\infty} \frac{x^n}{(n+\gamma)^{s+q\beta}}\right)^{\frac{1}{\beta}} \\
&= [\Phi(x, s+p\alpha, \gamma)]^{\frac{1}{\alpha}} [\Phi(x, s+q\beta, \gamma)]^{\frac{1}{\beta}},
\end{aligned}$$

which proves the desired inequality (3.1).

The inequality (3.2) follows in a similar way. □

Remark 2. We observe that for $\alpha = \beta = 2$, we obtain from (3.1) the following inequality

$$\Phi^2 (x, s+p+q, \gamma) \leq [\Phi (x, s+2p, \gamma)] [\Phi (x, s+2q, \gamma)] \tag{3.3}$$

provided the real numbers s, p, q satisfy the conditions $s+p+q$, $s+2p, s+2q > 0$. In its turn, the inequality (3.3) for $p = 0$, $q = 1$ and in fact (3.1), is a generalization of the following result

$$\Phi^2 (x, s+1, \gamma) \leq [\Phi (x, s, \gamma)] [\Phi (x, s+2, \gamma)] \tag{3.4}$$

provided $s > 0$.

We remark that when the *Hurwitz zeta* function reduces to ζ, one obtains from (3.2) for $\alpha = \beta = 2$ that

$$\frac{\zeta (s+1)}{\zeta (s)} \leq \frac{\zeta (s+2)}{\zeta (s+1)} \quad \text{for } s > 1. \tag{3.5}$$

This inequality that was obtained in [2] by Cerone and Dragomir is an improvement of a result due to Laforgia and Natalini [7] who proved that

$$\frac{\zeta (s+1)}{\zeta (s)} \leq \frac{s+1}{s} \cdot \frac{\zeta (s+2)}{\zeta (s+1)} \text{ for } s > 1.$$

Their arguments make use of an integral representation of the Zeta function and Turán-type inequalities.

It should be further noted that, if $s = 2n$, $n \in \mathbb{N}$, then

$$\Phi (x, 2n+1, \gamma) \leq \sqrt{\Phi (x, 2n, \gamma) \Phi (x, 2n+2, \gamma)}, \tag{3.6}$$

which in the case of the Zeta function gives by (3.5) the inequality [2]

$$\zeta (2n+1) \leq \sqrt{\zeta (2n) \zeta (2n+2)},$$

demonstrating that Zeta at the odd integers is bounded above by the geometric mean of its immediate even Zeta values.

The following result also holds:

Proposition 3.2. *If a, b, $c > 0$, $x \in (0,1)$ and $\alpha > 0$, then:*

$$\Phi (x, a, \alpha) \Phi (x, a+b+c, \alpha) \geq \Phi (x, a+b, \alpha) \Phi (x, a+c, \alpha). \tag{3.7}$$

Proof. Consider the sequence $a_n := (n+\alpha)^b$, $n \geq 0$, $\alpha > 0$, $b \in \mathbb{R}$. It is clear that a_n is increasing if $b > 0$. Therefore, the sequences $\alpha_n := \frac{1}{(n+\alpha)^b}$, $\beta_n := \frac{1}{(n+\alpha)^c}$ are both decreasing if b, $c > 0$. Utilising Čebyšev's inequality for synchronous sequences α_n, β_n with the nonnegative weights p_n,

$$\sum_{n=0}^{m} p_n \sum_{n=0}^{m} p_n \alpha_n \beta_n \geq \sum_{n=0}^{m} p_n \alpha_n \sum_{n=0}^{m} p_n \beta_n \tag{3.8}$$

we have:

$$\begin{aligned} &\Phi (x, a, \alpha) \Phi (x, a+b+c, \alpha) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{(n+\alpha)^a} \cdot \sum_{n=1}^{\infty} \frac{x^n}{(n+\alpha)^a} \cdot \frac{1}{(n+\alpha)^b} \cdot \frac{1}{(n+\alpha)^c} \end{aligned}$$

$$= \lim_{m\to\infty}\left(\sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^a}\cdot\sum_{n=1}^{m}\frac{x^n}{(n+\alpha)^a}\cdot\frac{1}{(n+\alpha)^b}\cdot\frac{1}{(n+\alpha)^c}\right)$$

$$\geq \lim_{m\to\infty}\left(\sum_{n=1}^{m}\frac{x^n}{(n+\alpha)^a}\cdot\frac{1}{(n+\alpha)^b}\cdot\sum_{n=1}^{m}\frac{x^n}{(n+\alpha)^a}\cdot\frac{1}{(n+\alpha)^c}\right)$$
$$= \sum_{n=1}^{\infty}\frac{x^n}{(n+\alpha)^a}\cdot\frac{1}{(n+\alpha)^b}\cdot\sum_{n=1}^{\infty}\frac{x^n}{(n+\alpha)^a}\cdot\frac{1}{(n+\alpha)^c}$$
$$= \Phi(x,a+b,\alpha)\,\Phi(x,a+c,\alpha),$$

and the inequality (3.7) is proved. □

Remark 3. Utilising the inequality (3.7) (for $c=b$) we can state the following result

$$\Phi(x,a,\alpha)\,\Phi(x,a+2b,\alpha)\geq\Phi^2(x,a+b,\alpha), \tag{3.9}$$

provided the numbers a, b are positive. We also remark that the choice $b=1$ will produce the same inequality (3.4).

We have:

Theorem 3.1. *Assume that p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.*

(i) For z, $w\in D(0,1)$,

$$\Phi(|zw|,s,\alpha)\leq\left[\Phi(|z|^p,s,\alpha)\right]^{1/p}\left[\Phi(|w|^q,s,\alpha)\right]^{1/q} \tag{3.10}$$

where s, $\alpha>0$;

(ii) For s, $t>0$,

$$\Phi(x,st,\alpha)\geq(1-x)\,\Phi\left(x,\frac{s^p}{p},\alpha\right)\Phi\left(x,\frac{t^p}{q},\alpha\right) \tag{3.11}$$

where $x\in(0,1)$, $\alpha>0$;

(iii) For α, $\beta>0$,

$$\Phi(x,s,\alpha\beta)\geq(1-x)\,\Phi\left(x,\frac{s}{p},\alpha^p\right)\Phi\left(x,\frac{s}{q},\beta^q\right) \tag{3.12}$$

where $x\in(0,1)$, $s>0$.

Proof. (i). By utilising Hölder's discrete inequality (2.7), we have successively that

$$\Phi(|zw|,s,\alpha)=\sum_{n=0}^{\infty}\frac{|zw|^n}{(n+\alpha)^s}=\lim_{m\to\infty}\sum_{n=0}^{m}\frac{|zw|^n}{(n+\alpha)^s}$$
$$\leq\lim_{m\to\infty}\left[\left(\sum_{n=0}^{m}\frac{|z|^{pn}}{(n+\alpha)^s}\right)^{1/p}\left(\sum_{n=0}^{m}\frac{|w|^{qn}}{(n+\alpha)^s}\right)^{1/q}\right]$$
$$=\left(\sum_{n=0}^{\infty}\frac{|z|^{pn}}{(n+\alpha)^s}\right)^{1/p}\left(\sum_{n=0}^{\infty}\frac{|w|^{qn}}{(n+\alpha)^s}\right)^{1/q}$$

$$= [\Phi(|z|^p, s, \alpha)]^{1/p} [\Phi(|w|^q, s, \alpha)]^{1/q}$$

and the inequality (3.10) is obtained.

(ii). We use Young's inequality

$$st \leq \frac{1}{p}s^p + \frac{1}{q}t^q, \ t, s > 0.$$

This implies that

$$(n+\alpha)^{st} \leq (n+\alpha)^{\frac{1}{p}s^p + \frac{1}{q}t^q} = (n+\alpha)^{\frac{1}{p}s^p} (n+\alpha)^{\frac{1}{q}t^q},$$

namely

$$\frac{x^n}{(n+\alpha)^{st}} \geq x^n \frac{1}{(n+\alpha)^{\frac{1}{p}s^p}} \frac{1}{(n+\alpha)^{\frac{1}{q}t^q}},$$

which by summation gives

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+\alpha)^{st}} \geq \sum_{n=0}^{\infty} x^n \frac{1}{(n+\alpha)^{\frac{1}{p}s^p}} \frac{1}{(n+\alpha)^{\frac{1}{q}t^q}}.$$

The sequences

$$a_n := \frac{1}{(n+\alpha)^{\frac{1}{p}s^p}} \text{ and } b_n := \frac{1}{(n+\alpha)^{\frac{1}{q}t^q}}$$

are decreasing, $p_n := x^n \geq 0$ and by Čebyšev's inequality (3.8) we get

$$\sum_{n=0}^{\infty} x^n \frac{1}{(n+\alpha)^{\frac{1}{p}s^p}} \frac{1}{(n+\alpha)^{\frac{1}{q}t^q}} \geq \frac{1}{\sum_{n=0}^{\infty} x^n} \sum_{n=0}^{\infty} x^n \frac{1}{(n+\alpha)^{\frac{1}{p}s^p}} \sum_{n=0}^{\infty} x^n \frac{1}{(n+\alpha)^{\frac{1}{q}t^q}}. \tag{3.13}$$

Since

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \ x \in (0,1),$$

hence by (3.13) we derive (3.11).

(iii). We use the elementary Hölder's inequality

$$ab + cd \leq (a^p + c^p)^{1/p} (b^q + d^q)^{1/q}, \ a, b, c, d \geq 0$$

to write that

$$n + \alpha\beta \leq \left(\left(n^{1/p}\right)^p + \alpha^p\right)^{1/p} \left(\left(n^{1/q}\right)^q + \beta^q\right)^{1/q} = (n+\alpha^p)^{1/p} (n+\beta^q)^{1/q}$$

for $\alpha, \beta > 0$ and $n \geq 0$.

This gives

$$\frac{1}{(n+\alpha\beta)^s} \geq \frac{1}{(n+\alpha^p)^{s/p} (n+\beta^q)^{s/q}}$$

and by Čebyšev's inequality (3.8) we get

$$\sum_{n=0}^{\infty} x^n \frac{1}{(n+\alpha\beta)^s} \geq \sum_{n=0}^{\infty} x^n \frac{1}{(n+\alpha^p)^{s/p}(n+\beta^q)^{s/q}} \geq \frac{1}{\sum_{n=0}^{\infty} x^n} \sum_{n=0}^{\infty} x^n \frac{1}{(n+\alpha^p)^{s/p}} \sum_{n=0}^{\infty} x^n \frac{1}{(n+\beta^q)^{s/q}},$$

which proves the desired result (3.12). □

Proposition 3.3. *With the assumptions of Proposition 3.2 we have the reverse inequality*

$$0 \leq \Phi(x,a,\alpha)\,\Phi(x,a+b+c,\alpha) - \Phi(x,a+b,\alpha)\,\Phi(x,a+c,\alpha) \leq \frac{1}{4}\frac{1}{\alpha^{b+c}}\Phi^2(x,a,\alpha). \tag{3.14}$$

Proof. We use the following Grüss type inequality

$$\left|\sum_{n=0}^{m} p_n \sum_{n=0}^{m} p_n\alpha_n\beta_n - \sum_{n=0}^{m} p_n\alpha_n \sum_{n=0}^{m} p_n\beta_n\right| \leq \frac{1}{4}(A-a)(B-b)\left(\sum_{n=0}^{m} p_n\right)^2, \tag{3.15}$$

where $a \leq \alpha_n \leq A$, $b \leq \beta_n \leq B$ for for $n \geq 0$ and the nonnegative weights p_n.

Since

$$0 \leq \alpha_n = \frac{1}{(n+\alpha)^b} \leq \frac{1}{\alpha^b},\ 0 \leq \beta_n = \frac{1}{(n+\alpha)^c} \leq \frac{1}{\alpha^c},$$

hence by (3.15)

$$\begin{aligned} 0 \leq &\sum_{n=0}^{m} \frac{x^n}{(n+\alpha)^a} \cdot \sum_{n=1}^{m} \frac{x^n}{(n+\alpha)^a} \cdot \frac{1}{(n+\alpha)^b} \cdot \frac{1}{(n+\alpha)^c} \\ &- \sum_{n=1}^{m} \frac{x^n}{(n+\alpha)^a} \cdot \frac{1}{(n+\alpha)^b} \cdot \sum_{n=1}^{m} \frac{x^n}{(n+\alpha)^a} \cdot \frac{1}{(n+\alpha)^c} \\ &\leq \frac{1}{4}\frac{1}{\alpha^b}\frac{1}{\alpha^c}\left(\sum_{n=0}^{m} \frac{x^n}{(n+\alpha)^a}\right)^2 = \frac{1}{4}\frac{1}{\alpha^{b+c}}\left(\sum_{n=0}^{m} \frac{x^n}{(n+\alpha)^a}\right)^2. \end{aligned} \tag{3.16}$$

By taking the limit over $m \to \infty$ in (3.16) we get (3.14). □

We also have:

Theorem 3.2. *Assume that $N \geq 2$ and $0 < \gamma \leq k_1, \ldots, k_N \leq \Gamma < \infty$. Then*

$$0 \leq \frac{N-1}{2}\sum_{j=1}^{N} \Phi(x,2k_j,\alpha) - \sum_{1\leq i<j\leq N} \Phi(x,k_i+k_j,\alpha) \leq \frac{N^2}{4}\left[\frac{\Phi(x,2\Gamma,\alpha)+\Phi(x,2\gamma,\alpha)}{2} - \Phi(x,\gamma+\Gamma,\alpha)\right] \tag{3.17}$$

for $x \in (0,1)$ and $\alpha > 0$.

Proof. We use the following Grüss type inequality:

$$\frac{1}{N}\sum_{j=1}^{N} z_j^2 - \left(\frac{1}{N}\sum_{j=1}^{N} z_j\right)^2 \leq \frac{1}{4}(\Gamma-\gamma)^2,$$

provided $\gamma \leq z_j \leq \Gamma$ for each $j \in \{1,\dots,N\}$.

Since $\gamma \leq k_j \leq \Gamma$ for $j \in \{1,\dots,N\}$, then

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\frac{1}{(n+\alpha)^{2k_j}} - \frac{1}{N^2}\left(\sum_{j=1}^{N}\frac{1}{(n+\alpha)^{k_j}}\right)^2 \\
&\leq \frac{1}{4}\left(\frac{1}{(n+\alpha)^{\gamma}} - \frac{1}{(n+\alpha)^{\Gamma}}\right)^2 \\
&= \frac{1}{4}\left(\frac{1}{(n+\alpha)^{2\gamma}} + \frac{1}{(n+\alpha)^{2\Gamma}} - \frac{2}{(n+\alpha)^{\gamma+\Gamma}}\right)
\end{aligned}$$

for $n \geq 1$, which gives

$$\begin{aligned}
&\frac{1}{N}\sum_{j=1}^{N}\frac{1}{(n+\alpha)^{2k_j}} - \frac{1}{N^2}\left(\sum_{j=1}^{N}\frac{1}{(n+\alpha)^{2k_j}} + 2\sum_{1\leq i<j\leq N}\frac{1}{(n+\alpha)^{k_i+k_j}}\right) \\
&\leq \frac{1}{4}\left(\frac{1}{(n+\alpha)^{2\gamma}} + \frac{1}{(n+\alpha)^{2\Gamma}} - \frac{2}{(n+\alpha)^{\gamma+\Gamma}}\right)
\end{aligned}$$

for $n \geq 1$.

Multiplying with N^2 and re-arranging, we get

$$\begin{aligned}
&\frac{N-1}{2}\sum_{j=1}^{N}\frac{1}{(n+\alpha)^{2k_j}} - \sum_{1\leq i<j\leq N}\frac{1}{(n+\alpha)^{k_i+k_j}} \\
&\leq \frac{N^2}{4}\left(\frac{1}{2}\left[\frac{1}{(n+\alpha)^{2\gamma}} + \frac{1}{(n+\alpha)^{2\Gamma}}\right] - \frac{1}{(n+\alpha)^{\gamma+\Gamma}}\right)
\end{aligned} \tag{3.18}$$

for any $n \geq 1$.

Finally, if we multiply (3.18) by $x^n \geq 0$ and sum over $n \geq 0$, we get

$$\begin{aligned}
&\frac{N-1}{2}\sum_{j=1}^{N}\left(\sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^{2k_j}}\right) - \sum_{1\leq i<j\leq N}\left(\sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^{k_i+k_j}}\right) \\
&\leq \frac{N^2}{4}\left(\frac{1}{2}\left[\sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^{2\gamma}} + \sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^{2\Gamma}}\right] - \sum_{n=0}^{m}\frac{x^n}{(n+\alpha)^{\gamma+\Gamma}}\right).
\end{aligned}$$

By taking the limit over $m \to \infty$ we get the desired inequality (3.17). □

Theorem 3.3. *The following statements hold:*

(i) For $s>0$, $\alpha>0$ and $\beta\in[0,1]$ we have

$$\begin{aligned}\Phi^2(zw,s,\alpha)&\le\Phi\left(z^{1+\beta}w^{1-\beta},s,\alpha\right)\Phi\left(z^{1-\beta}w^{1+\beta},s,\alpha\right)\\&\le\Phi\left(z^2,s,\alpha\right)\Phi\left(w^2,s,\alpha\right),\end{aligned}\tag{3.19}$$

where $z,\ w\in[0,1)$.

(ii) For $z\in[0,1)$, $\alpha>0$ and $\beta\in[0,1]$ we have

$$\begin{aligned}\Phi^2(z,s+t,\alpha)&\le\Phi(z,(1+\beta)s+(1-\beta)t,\alpha)\,\Phi(z,(1-\beta)s+(1+\beta)t,\alpha)\\&\le\Phi(z,2s,\alpha)\,\Phi(z,2t,\alpha)\end{aligned}\tag{3.20}$$

for all $s,\ t>0$.

Proof. We utilize the *Callebaut inequality* (see for instance [4, Remark 3. 31])

$$\left(\sum_{j=1}^m p_ja_jb_j\right)^2\le\sum_{j=1}^m p_ja_j^{1+\beta}b_j^{1-\beta}\sum_{j=1}^m p_ja_j^{1-\beta}b_j^{1+\beta}\le\sum_{j=1}^m p_ja_j^2\sum_{j=1}^m p_jb_j^2,\tag{3.21}$$

where $p_j,\ a_j,\ b_j\ge0$, $j\in\{1,...,n\}$ and $\beta\in[0,1]$.

(i). By (3.21) we have

$$\begin{aligned}\left(\sum_{n=0}^m\frac{(zw)^n}{(n+\alpha)^s}\right)^2&=\left(\sum_{n=0}^m\frac{z^nw^n}{(n+\alpha)^s}\right)^2\\&\le\left(\sum_{n=0}^m\frac{z^{n(1+\beta)}w^{n(1-\beta)}}{(n+\alpha)^s}\right)\left(\sum_{n=0}^m\frac{z^{n(1-\beta)}w^{n(1+\beta)}}{(n+\alpha)^s}\right)\\&=\left(\sum_{n=0}^m\frac{\left(z^{1+\beta}w^{1-\beta}\right)^n}{(n+\alpha)^s}\right)\left(\sum_{n=0}^m\frac{\left(z^{1-\beta}w^{1+\beta}\right)^n}{(n+\alpha)^s}\right)\\&\le\sum_{n=0}^m\frac{\left(z^2\right)^n}{(n+\alpha)^s}\sum_{n=0}^m\frac{\left(w^2\right)^n}{(n+\alpha)^s}\end{aligned}\tag{3.22}$$

for $z,\ w\in[0,1)$.

By taking the limit over $m\to\infty$ in (3.22) we get (3.19).

(ii). By (3.21) we have

$$\begin{aligned}\left(\sum_{n=0}^m\frac{z^n}{(n+\alpha)^{s+t}}\right)^2&=\left(\sum_{n=0}^m z^n\frac{1}{(n+\alpha)^s}\frac{1}{(n+\alpha)^t}\right)^2\\&\le\left(\sum_{n=0}^m z^n\frac{1}{(n+\alpha)^{s(1+\beta)}}\frac{1}{(n+\alpha)^{t(1-\beta)}}\right)\\&\times\left(\sum_{n=0}^m z^n\frac{1}{(n+\alpha)^{s(1-\beta)}}\frac{1}{(n+\alpha)^{t(1+\beta)}}\right)\\&\le\left(\sum_{n=0}^m z^n\frac{1}{(n+\alpha)^{2s}}\right)\left(\sum_{n=0}^m z^n\frac{1}{(n+\alpha)^{2t}}\right)\end{aligned}\tag{3.23}$$

for $z\in[0,1)$, $\alpha>0$, $\beta\in[0,1]$ and $s,\ t>0$.

By taking the limit over $m\to\infty$ in (3.23) we get (3.20). □

Remark 4. For s, $t > 1/2$ the inequality (3.20) also holds if $z = 1$ giving the following result for *Hurwitz zeta* function,

$$\begin{aligned}\zeta^2(s+t,\alpha) &\le \zeta((1+\beta)s+(1-\beta)t,\alpha)\,\zeta((1-\beta)s+(1+\beta)t,\alpha) \\ &\le \zeta(2s,\alpha)\,\zeta(2t,\alpha)\end{aligned} \tag{3.24}$$

for $\alpha > 0$. In particular, we have the inequality for *zeta* function

$$\begin{aligned}\zeta^2(s+t) &\le \zeta((1+\beta)s+(1-\beta)t)\,\zeta((1-\beta)s+(1+\beta)t) \\ &\le \zeta(2s)\,\zeta(2t)\end{aligned} \tag{3.25}$$

for s, $t > 1/2$.

By taking $a = 2s$, $b = 2t > 1$, we get from (3.24) and (3.25) that

$$\begin{aligned}&\zeta^2\left(\frac{a+b}{2},\alpha\right) \\ &\le \zeta\left(\frac{(1+\beta)a+(1-\beta)b}{2},\alpha\right)\zeta\left(\frac{(1-\beta)a+(1+\beta)b}{2},\alpha\right) \\ &\le \zeta(a,\alpha)\,\zeta(b,\alpha)\end{aligned} \tag{3.26}$$

for $\alpha > 0$ and

$$\begin{aligned}&\zeta^2\left(\frac{a+b}{2}\right) \\ &\le \zeta\left(\frac{(1+\beta)a+(1-\beta)b}{2}\right)\zeta\left(\frac{(1-\beta)a+(1+\beta)b}{2}\right) \\ &\le \zeta(a)\,\zeta(b)\end{aligned} \tag{3.27}$$

for a, $b > 1$, which is a refinement of logarithmic convexity property.

Further, by utilizing the following Hölder's type inequality obtained by Dragomir and Sándor in 1990 [5] (see also [4, Corollary 2.34]) for p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum_{k=0}^{m} m_k |x_k|^p \sum_{k=0}^{m} m_k |y_k|^q \ge \sum_{k=0}^{m} m_k |x_k y_k| \sum_{k=0}^{m} m_k |x_k|^{p-1} |y_k|^{q-1} \tag{3.28}$$

that holds for nonnegative numbers m_k and complex numbers x_k, y_k where $k \in \{0, ..., n\}$, we observe that the convergence of the series $\sum_{k=0}^{\infty} m_k |x_k|^p$, $\sum_{k=0}^{\infty} m_k |y_k|^q$ imply the convergence of the series $\sum_{k=0}^{\infty} m_k |x_k y_k|$ and $\sum_{k=0}^{\infty} m_k |x_k|^{p-1} |y_k|^{q-1}$.

Theorem 3.4. *Assume that p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) For s, $\alpha > 0$ we have

$$\Phi(zw,s,\alpha)\,\Phi\left(z^{p-1}w^{q-1},s,\alpha\right) \le \Phi(z^p,s,\alpha)\,\Phi(w^q,s,\alpha) \tag{3.29}$$

for z, $w \in [0,1)$.

(ii) For $z \in [0,1)$ we have

$$\Phi(z,s+t,\alpha)\,\Phi(z,s(p-1)+t(q-1),\alpha) \le \Phi(z,sp,\alpha)\,\Phi(z,tq,\alpha) \tag{3.30}$$

for s, t, $\alpha > 0$.

Proof. (i). From (3.28) we get

$$\sum_{n=0}^{m}\frac{(zw)^n}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{\left(z^{p-1}w^{q-1}\right)^n}{(n+\alpha)^s}=\sum_{n=0}^{m}\frac{z^nw^n}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{z^{n(p-1)}w^{n(q-1)}}{(n+\alpha)^s} \tag{3.31}$$
$$\leq\sum_{n=0}^{m}\frac{z^{np}}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{w^{nq}}{(n+\alpha)^s}$$
$$=\sum_{n=0}^{m}\frac{(z^p)^n}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{(w^q)^n}{(n+\alpha)^s}$$

for $z,\ w\in[0,1)$ and $s,\ \alpha>0$.

By taking the limit over $m\to\infty$ in (3.31) we get (3.29).

(ii). From (3.28) we also have

$$\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^{s+t}}\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^{s(p-1)+t(q-1)}} \tag{3.32}$$
$$=\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^s(n+\alpha)^t}\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^{s(p-1)}(n+\alpha)^{t(q-1)}}$$
$$\leq\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^{sp}}\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^{tq}}$$

for $z\in[0,1)$ and $s,\ t,\alpha>0$.

By taking the limit over $m\to\infty$ in (3.32) we get (3.30). □

Remark 5. For $s>1/p,\ t\geq 1/q$ with $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$ the inequality (3.30) also holds if $z=1$ giving the following result for *Hurwitz zeta* function,

$$\zeta(s+t,\alpha)\,\zeta(s(p-1)+t(q-1),\alpha)\leq\zeta(sp,\alpha)\,\zeta(tq,\alpha) \tag{3.33}$$

for $\alpha>0$. In particular, we have the inequality for *zeta* function

$$\zeta(s+t)\,\zeta(s(p-1)+t(q-1))\leq\zeta(sp)\,\zeta(tq)\,. \tag{3.34}$$

Finally, we use the following inequality obtained by S. S. Dragomir in 1984, [3] (see also [4, Theorem 2.20]):

$$\frac{\sum_{j=0}^{m}p_ja_jb_j\sum_{j=0}^{m}p_ja_j\sum_{j=0}^{m}p_jb_j}{\sum_{j=0}^{m}p_j}\leq\sum_{j=0}^{m}p_ja_j^2\sum_{j=0}^{m}p_jb_j^2, \tag{3.35}$$

that holds for the nonnegative numbers $a_j,\ b_j,\ p_j$ with $j\in\{0,...,n\}$ and $\sum_{j=0}^{m}p_j>0$.

Theorem 3.5. *The following statements hold.*

(i) For $s,\ \alpha>0$ we have

$$\Phi(zw,s,\alpha)\leq\zeta(s,\alpha)\frac{\Phi\left(z^2,s,\alpha\right)\Phi\left(w^2,s,\alpha\right)}{\Phi(z,s,\alpha)\,\Phi(w,s,\alpha)} \tag{3.36}$$

for $z,\ w\in[0,1)$.

(ii) For $z \in [0,1)$ we have

$$(1-z)\,\Phi(z,s+t,\alpha) \le \frac{\Phi(z,2s,\alpha)\,\Phi(w,2t,\alpha)}{\Phi(z,s,\alpha)\,\Phi(w,t,\alpha)} \tag{3.37}$$

for $s,\ t,\ \alpha > 0$.

Proof. (i). We have by (3.35) that

$$\begin{aligned}&\sum_{n=0}^{m}\frac{(zw)^n}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{w^n}{(n+\alpha)^s} \\ &\le \sum_{n=0}^{m}\frac{1}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{z^{2n}}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{w^{2n}}{(n+\alpha)^s},\end{aligned} \tag{3.38}$$

for $s,\ \alpha > 0$ and $z,\ w \in [0,1)$.

By taking the limit over $m \to \infty$ in (3.38) we get (3.36).

(ii). We also have by (3.35) that

$$\begin{aligned}&\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^{s+t}}\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^s}\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^t} \\ &\le \sum_{n=0}^{m}z^n\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^{2s}}\sum_{n=0}^{m}\frac{z^n}{(n+\alpha)^{2t}},\end{aligned} \tag{3.39}$$

for $s,\ t,\ \alpha > 0$ and $z \in [0,1)$.

By taking the limit over $m \to \infty$ in (3.39) we get (3.37). □

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