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GENERALIZATION OF COMMON FIXED POINT THEOREMS FOR WEAKLY COMMUTING MAPS BY ALTERING DISTANCES

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Abstract. The main purpose of this paper is to obtain conditions for the existence of a unique common fixed point for four selfmaps on a complete metric space by altering distances between the points.

1. Introduction

Obtaining the existence and uniqueness of fixed points for selfmaps on a metric space by altering distances between the points with the use of a certain control function is an interesting aspect. In this direction, Khan, Swaleh and Sessa [1] established the existence and uniqueness of a fixed point for a single selfmap. Recently, Sastry and Babu [4] proved a fixed point theorem by altering distances between the points for a pair of selfmaps.

Pant [2] established a unique common fixed point theorem for four selfmaps by using the minimal type commutativity, contractive and continuity type conditions as follows.

Two selfmaps A and S of a metric space (X, d) are called compatible if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X.

Two selfmaps A and S of a metric space (X, d) are called R-weakly commuting at a point x in X if

$$d(ASx, SAx) \leq Rd(Ax, Sx) \text{ for some } R > 0$$

The maps A and S are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that

$$d(ASx, SAx) \leq Rd(Ax, Sx)$$

Let A and S be selfmappings of a metric space (X, d). We call A and S to be reciprocally continuous in X if

$$\lim_n ASx_n = At \text{ and } \lim_n SAx_n = St$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_n Ax_n = \lim_n Sx_n = t \text{ for some } t \text{ in } X.$$

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Pant [2] proved the following theorem.

Theorem 1.1. *Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of selfmappings of a complete metric space (X, d) such that*

i) $AX \subset TX$, $BX \subset SX$
ii) $d(Ax, By) \leq hM(x, y)$, $0 \leq h < 1$, $x, y \in X$ where

$$M(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{1}{2}[d(Ax,Ty) + d(By,Sx)]\}.$$

Suppose that (A, S) or (B, T) is a compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point.

We generalize theorem 1.1 by altering distances between the points (Theorem 2.1), using a certain control function,

$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which is continuous at zero, monotonically increasing, $\psi(2t) \leq 2\psi(t)$ and $\psi(t) = 0$ if and only if $t = 0$. (1.1.1)

We also give two examples (Examples 2.2) to show that ψ need not be subadditive.

In the rest of this paper, (X, d) is a complete metric space, $\mathbb{R}^+$ denotes the non-negative real line and $\mathbb{Z}^+$ non-negative integers.

Definition 1.2. Two selfmaps A and S of a metric space (X, d) are called weakly commuting if $d(ASx, SAx) \leq d(Ax, Sx)$ for every x in X. This condition implies that $ASx = SAx$ whenever $Ax = Sx$.

Notation 1.3. If A, B, S and T are four selfmaps of (X, d) and ψ is as in (1.1.1), we write

$$M_\psi(x,y) = \max\{\psi(d(Sx,Ty)), \psi(d(Ax,Sx)), \psi(d(By,Ty)), \\ \frac{1}{2}[\psi(d(Ax,Ty)] + \psi(d(By,Sx))]\}.$$

Definition 1.4. Two selfmaps A and S of a metric space (X, d) are called ψ-compatible if $\lim_n \psi(d(ASx_n, SAx_n)) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Ax_n = \lim_n Sx_n = t$ for some t in X.

2. Main Theorems

Theorem 2.1. *Let (A, S) and (B, T) be weakly commuting pairs of selfmaps of a complete metric space (X, d) and ψ be as in* (1.1.1)*, satisfying*

(i) $AX \subset TX$, $BX \subset SX$ *and*
(ii) *there exists h in $[0, 1)$ such that $\psi(d(Ax, By)) \leq hM_\psi(x, y)$ for all $x, y \in X$.*

Suppose that (A, S) or (B, T) is a ψ-compatible pair of reciprocally continuous mappings. Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X. Then by (i) we can define, for $n = 0, 1, 2, \dots$

$$\begin{aligned} y_{2n} &= Ax_{2n} = Tx_{2n+1} \\ y_{2n+1} &= Bx_{2n+1} = Sx_{2n+2} \end{aligned} \tag{2.1.1}$$

We now show that $\{y_n\}$ is a Cauchy sequence.
From (ii), we have

$$\begin{aligned} \psi(d(y_{2n}, y_{2n+1})) &= \psi(d(Ax_{2n}, Bx_{2n+1})) \\ &\leq hM_\psi(x_{2n}, x_{2n+1}) \\ &= h\max\{\psi(d(Sx_{2n}, Tx_{2n+1})), \psi(d(Ax_{2n}, Sx_{2n})), \\ &\qquad \psi(d(Bx_{2n+1}, Tx_{2n+1})), \frac{1}{2}\psi(d(Bx_{2n+1}, Sx_{2n}))\} \\ &= h\max\{\psi(d(y_{2n-1}, y_{2n})), \psi(d(y_{2n}, y_{2n-1})), \psi(d(y_{2n+1}, y_{2n})), \\ &\qquad \frac{1}{2}[\psi(d(y_{2n+1}, y_{2n-1}))\} \\ &= h\max\{\psi(d(y_{2n-1}, y_{2n})), \psi(d(y_{2n+1}, y_{2n})), \\ &\qquad \frac{1}{2}[\psi(d(y_{2n+1}, y_{2n-1}))\} \\ &\leq h\max\{\psi(d(y_{2n-1}, y_{2n})), \psi(d(y_{2n+1}, y_{2n})), \\ &\qquad \frac{1}{2}[\psi(d(y_{2n+1}, y_{2n}) + d(y_{2n}, y_{2n-1})]\} \\ &\leq h\max\{\psi(d(y_{2n-1}, y_{2n})), \psi(d(y_{2n+1}, y_{2n})), \\ &\qquad \frac{1}{2}(\psi(2\max\{d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n-1})\}]\}, \\ &\leq h\max\{\psi(d(y_{2n+1}, y_{2n})), \psi(d(y_{2n-1}, y_{2n})), \\ &\qquad \psi\max\{d(y_{2n+1}, y_{2n}), d(y_{2n}, y_{2n-1})\}\} \\ &= h\psi(d(y_{2n-1}, y_{2n})) \end{aligned} \tag{2.1.2}$$

In a similar way we can show that

$$\psi(d(y_{2n-1}, y_{2n})) \leq h\psi(d(y_{2n-2}, y_{2n-1})) \tag{2.1.3}$$

From (2.1.2) and (2.1.3), we get

$$\psi(d(y_n, y_{n+1})) \leq h^n\psi(d(y_0, y_1))$$

Also we have, for every positive integer p,

$$\begin{aligned} \psi(d(y_n, y_{n+p})) &\leq \psi[d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p})] \\ &\leq \psi[(1 + h + \cdots + h^{p-1})h^n d(y_0, y_1)] \\ &\leq \psi[(\frac{1}{1-h})h^n d(y_0, y_1)]. \end{aligned}$$

Now for a given $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$\psi[(\frac{1}{1-h})h^n d(y_0, y_1)] < \psi(\varepsilon) \text{ for all } n \geq N$$

This implies $d(y_n, y_{n+p}) < \varepsilon$ for all $n \geq N$.
Hence $\{y_n\}$ is a Cauchy sequence in X.
Since X is complete, there is a point z in X such that $y_n \to z$ as $n \to \infty$. Hence from (2.1.1), we have

$$\begin{aligned} y_{2n} &= Ax_{2n} = Tx_{2n+1} \to z \\ y_{2n+1} &= Bx_{2n+1} = Sx_{2n+2} \to z \end{aligned} \tag{2.1.4}$$

Now suppose that (A, S) is a ψ-compatible pair of reciprocally continuous mappings. Since A and S are reciprocally continuous, by (2.1.4), we get

$$ASx_{2n} \to Az \text{ and } SAx_{2n} \to Sz. \tag{2.1.5}$$

ψ-compatibility of A and S imply that

$$\lim_n \psi(d(ASx_{2n}, SAx_{2n}) = 0$$

We now show that $Az = Sz$. Suppose $Az \neq Sz$. Let $\varepsilon = \frac{1}{2}d(Az, Sz)$. Then there exists $N \in \mathbb{Z}^+$ such that

$$\psi(d(ASx_{2n}, SAx_{2n})) < \psi(\varepsilon) \text{ for all } n \geq N.$$

This implies that $d(ASx_{2n}, SAx_{2n}) < \varepsilon$ for all $n \geq N$. Hence by (2.1.5), $d(Az, Sz) < \varepsilon = \frac{1}{2}d(Az, Sz)$. a contradiction. Hence

$$Az = Sz \tag{2.1.6}$$

Since $AX \subset TX$, there is a point w in X such that $Tw = Az$. By (2.1.6),

$$Tw = Az = Sz. \tag{2.1.7}$$

Now, we show that $Az = Bw$. Suppose $Az \neq Bw$. Now by (ii) we have

$$\begin{aligned} \psi(d(Az, Bw)) &\leq hM_\psi(z, w) \\ &= h\psi(d(Bw, Tw)) = h\psi(d(Bw, Az)) \end{aligned}$$

a contradiction. Hence $Az = Bw$. Therefore by (2.1.7),

$$Bw = Az = Sz = Tw \tag{2.1.8}$$

Since A and S are weakly commuting, we have by (2.1.8), $ASz = SAz$ and

$$AAz = ASz = SAz = SSz \tag{2.1.9}$$

Since B and T are weakly commuting, we have

$$BBw = BTw = TBw = TTw \tag{2.1.10}$$

We now show that $AAz = Az$. Suppose $AAz \neq Az$, by (ii), we have

$$\begin{aligned}\psi(d(Az, AAz)) &= \psi(d(Bw, AAz))\\ &\leq hM_\psi(Az, w)\\ &= h\psi(d(Az, AAz)) \quad \text{(by (2.1.8) and (2.1.9))}\end{aligned}$$

a contradiction. Hence $AAz = Az$.

Also, we have, $AAz = SAz$. Therefore Az is a common fixed point for A and S. Also suppose $BBw \neq Bw$. By (ii), we have

$$\begin{aligned}\psi(d(Bw, BBw)) &= \psi(d(Az, BBw)) \quad \text{(by (2.1.8))}\\ &\leq hM_\psi(z, Bw)\\ &= h\psi(d(Bw, BBw)) \quad \text{(by (2.1.8) and (2.1.10))}\\ &< \psi(d(Bw, BBw))\end{aligned}$$

a contradiction. Hence $BBw = Bw$ and since $TBw = BBw$, we have Bw is a common fixed point for B and T. Since $Az = Bw$, we have Az is a common fixed point for A, B, S and T. Uniqueness of a common fixed point follows by (ii). The proof is similar when the pair (B, T) is assumed ψ-compatible and reciprocally continuous.

The following two examples show that ψ defined as in (1.1.1) need not be subadditive; consequently $\psi \circ d$ need not be a metric.

2.2. Examples

(i) This is an example of a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which is neither continuous nor subadditive but is continuous at zero, monotonically increasing, vanishing only at '0' and $\psi(2t) \leq 2\psi(t)$.

Define

$$\psi(t) = \begin{cases} 0 \text{ if } t = 0 \\ 2^{-n} \text{ if } 2^{-(n+1)} \leq t < 2^{-n}, \quad n = 0, 1, 2, \dots \\ 1 \text{ if } t \geq 1 \end{cases}$$

Clearly $\psi(2t) \leq 2\psi(t)$, but $\psi(s + t) \leq \psi(s) + \psi(t)$ does not hold for $s = 0.2$ and $t = 0.3$. Clearly ψ is discontinuous at 2^{-n}, $n = 0, 1, 2, \dots$.

(ii) This is an example of a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which is continuous, strictly increasing, vanishing only at '0' and $\psi(2t) = 2\psi(t)$ but not subadditive. Let k be a fixed positive real number.

Define

$$\psi(t) = \begin{cases} 0 & \text{if } t = 0 \\ (3t - \frac{1}{2})k & \text{if } \frac{1}{2} \le t \le \frac{3}{4} \\ (t+1)k & \text{if } \frac{3}{4} < t < 1 \\ 2^{n+1}\psi(\frac{t}{2^{n+1}}) & \text{if } 2^n \le t < 2^{n+1}, \ n = 0, 1, 2, \dots \\ \frac{1}{2^n}\psi(2^n t) & \text{if } \frac{1}{2^{n+1}} \le t < \frac{1}{2^n}, \ n = 1, 2, 3, \dots \end{cases}$$

Then $\psi(\frac{1}{2} + \frac{1}{4}) = \psi(\frac{3}{4}) = \frac{7}{4}k > \frac{3}{2}k = k + \frac{1}{2}k = \psi(\frac{1}{2}) + \psi(\frac{1}{4})$ so that ψ is not subadditive, while ψ has all the other properties mentioned above.

We state the following lemma, which is easy to prove. This lemma is used in the next theorem.

Lemma 2.3. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be increasing, continuous at the origin and vanishing only at zero. Then $\{t_n\} \subset \mathbb{R}^+$ and $f(t_n) \to 0$ implies $t_n \to 0$.*

In the above theorem, we replace reciprocal continuity of A and S by continuity of S and obtain result similar to Theorem 2.1,

Theorem 2.4. *Let (A, S) and (B, T) be weakly commuting pairs of selfmaps of a complete metric space (X, d) and ψ be as in* (1.1.1) *satisfying*

(i) *$AX \subset TX$, $BX \subset SX$ and*
(ii) *there exists h in $[0, 1)$ such that*

$$\psi(d(Ax, By)) \le hM_\psi(x, y) \text{ for all } x, y \in X.$$

Suppose that A and S are ψ-compatible and S is continuous. Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any point and let $\{x_n\}$ and $\{y_n\}$ be sequences in X, define, for $n = 0, 1, 2, \ldots$, by

$$\begin{aligned} y_{2n} &= Ax_{2n} = Tx_{2n+1} \\ y_{2n+1} &= Bx_{2n+1} = Sx_{2n+2} \end{aligned} \tag{2.4.1}$$

As in theorem 2.1, the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there is a point z in X such that

$$\begin{aligned} y_{2n} &= Ax_{2n} = Tx_{2n+1} \to z \\ \text{and } y_{2n+1} &= Bx_{2n+1} = Sx_{2n+2} \to z \end{aligned} \tag{2.4.2}$$

Since A and S are ψ-compatible $Ax_{2n} \to z$ and $Sx_{2n} \to z$ implies that

$$\lim_n \psi(d(SAx_{2n}, ASx_{2n})) = 0 \tag{2.4.3}$$

Since S is continuous $SAx_{2n} \to Sz$, $SSx_{2n} \to Sz$ as $n \to \infty$. Now we show that $\lim_n ASx_{2n} = Sz$. By (2.4.3),

$$\psi(d(ASx_{2n}, Sz)) \le \psi(d(ASx_{2n}, SAx_{2n}) + d(SAx_{2n}, Sz)) \to 0 \text{ as } n \to \infty.$$

Hence by above lemma $d(ASx_{2n}, Sz) \to 0$ as $n \to \infty$.

Therefore $\lim_n ASx_{2n} = Sz$. Since $AX \subset TX$, for each n, there is w_{2n} in X such that $ASx_{2n} = Tw_{2n}$. Thus $SSx_{2n} \to Sz$, $SAx_{2n} \to Sz$ $ASx_{2n} \to Sz$ and $Tw_{2n} \to Sz$ as $n \to \infty$.

We now show that $\lim_n Bw_{2n} \to Sz$. If not, there exists $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that $d(ASx_{2n_k}, Bw_{2n_k}) > \varepsilon$ and $\psi(d(ASx_{2n_k}, SAx_{2n_k})) < \varepsilon$ for all n_k. Therefore

$$\begin{aligned}
\psi(\varepsilon) &\leq \psi(d(ASx_{2n_k}, Bw_{2n_k})) \\
&\leq hM_\psi(Sx_{2n_k}, w_{2n_k})) \\
&= h\max\{\psi(d(SSx_{2n_k}, Tw_{2n_k})), \psi(d(ASx_{2n_k}, SSx_{2n_k})), \psi(d(Bw_{2n_k}, Tw_{2n_k})), \\
&\quad \frac{1}{2}[\psi(d(ASx_{2n_k}, Tw_{2n_k})) + \psi(d(Bw_{2n_k}, SSx_{2n_K}))]\} \\
&= h\max\{\psi(d(Bw_{2n_k}, Tw_{2n_k})), \frac{1}{2}\psi(d(Bw_{2n_k}, SSx_{2n_k}))\} \\
&= h\max\{\psi(d(Bw_{2n_k}, ASx_{2n_k})), \frac{1}{2}\psi(d(Bw_{2n_k}, SSx_{2n_k}))\} \\
&\leq h\max\{\psi(d(Bw_{2n_k}, ASx_{2n_k})), \frac{1}{2}\psi(d(Bw_{2n_k}, ASx_{2n_k}) + d(ASx_{2n_k}, SSx_{2n_k})]\} \\
&= h\psi(d(Bw_{2n_k}, ASx_{2n_k})) < \psi(d(Bw_{2n_k}, ASx_{2n_k}),
\end{aligned}$$

a contradiction. Hence $\lim_n Bw_{2n} = Sz$. We now show that $Az = Sz$. By (ii) we have

$$\begin{aligned}
\psi(d(Az, Bw_{2n})) &\leq hM_\psi(z, w_{2n})) \\
&= h\max\{\psi(d(Az, Sz)), \frac{1}{2}\psi(d(Az, Tw_{2n}))]\}
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
\psi(d(Az, Sz)) &\leq h\max\{\psi(d(Az, Sz)),\ \frac{1}{2}\psi(d(Az, Sz))\} \\
&= h\psi(d(Az, Sz))
\end{aligned}$$

a contradiction. Hence $Az = Sz$. Since $AX \subset TX$, there exists w in X such that $Az = Tw$. Hence $Sz = Az = Tw$. We now show that $Az = Bw$. Suppose $Az \neq Bw$, by (ii) we have

$$\begin{aligned}
\psi(d(Az, Bw)) &\leq hM_\psi(z, w) \\
&= h\max\{\psi(d(Bw, Tw)),\ \frac{1}{2}\psi(d(Bw, Sz))\} \\
&= h\max\{\psi(d(Bw, Az)),\ \frac{1}{2}\psi(d(Bw, Az))\} \\
&= h\psi(d(Bw, Az))
\end{aligned}$$

a contradiction. Hence $Az = Bw$. Thus $Sz = Az = Tw = Bw$. Since A and S are weakly commuting, we have $ASz = SAz$ and hence $AAz = ASz = SAz = SSz$ and by the weakly commuting property of B and T, we have $BBw = BTw = TBw = TTw$. The remaining part of the proof is as in Theorem 2.1.

Note. The above theorem is valid if we assume B and T are ψ-compatible and T is continuous, instead of similar restrictions on A and S.

Note. We observe that the theorem of Pant [2] is a special case of Theorem 2.1 by taking ψ as the identity function.

We conclude the paper with the following open problem:

Open problem.

Is theorem 2.4 valid if we replace continuity of S by continuity of A or continuity of T by continuity of B?

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