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SOME FIXED POINT THEOREMS IN FUZZY METRIC SPACE

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Abstract. In this paper we prove common fixed point theorems in fuzzy metric spaces employing the notion of reciprocal continuity. Moreover we have to show that in the context of reciprocal continuity the notion of compatibility and semi-compatibility of maps becomes equivalent. Our result improves recent results of Singh & Jain [13] in the sense that all maps involved in the theorems are discontinuous even at common fixed point.

1. Introduction

After Zadeh [16] introduced the concept of fuzzy sets in 1965, many authors have extensively developed the theory of fuzzy sets and its applications.

Specially to mention, fuzzy metric spaces were introduced by Deng [3], Erceg [4], Kaleva and Seikkala [8], Kramosil and Michalek [10]. In this paper we use the concept of fuzzy metric space introduced by Kramosil and Michalek [10] and modified by George and Veeramani [5] to obtain Hausdorff topology for this kind of fuzzy metric space.

Recently B.Singh et al [13] introduced the notion of semi-compatible maps in fuzzy metric space and compared this notion with the notion of compatible map, compatible map of type (α), compatible map of type (β) and obtain some fixed point theorems in complete fuzzy metric space in the sense of Grabiec [6].

In the present paper we prove a fixed point theorems in complete fuzzy metric space by replacing continuity condition with a weaker condition called reciprocal continuity. Employing the notion of reciprocal continuity of mappings we can widen the scope of many interesting fixed point theorems in fuzzy metric spaces, Menger spaces as well as intuitionistic fuzzy metric spaces.

2. Preliminaries and Notations

In this section we recall some definitions and known results in fuzzy metric space.

Definition 1.([13]) A triangular norm $*$ (shortly t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a,b,c,d \in [0,1]$ the following conditions are satisfied:

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(1) $a * 1 = a$;
(2) $a * b = b * a$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(4) $a * (b * c) = (a * b) * c$.

Definition 2.([13]) The 3-triple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$ and $s, t > 0$:

(FM-1) $M(x, y, 0) = 0$,
(FM-2) $M(x, y, t) = 1$ for all $t > 0$ iff $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $M(x, y, t) * M(y, z, t) \geq M(x, z, t + s)$,
(FM-5) $M(x, y, .) : [0,\infty) \to [0, 1]$ is left continuous
(FM-6) $\lim_{t\to\infty} M(x, y, t) = 1$.

In the definition of George and Veeramani [5], M is a fuzzy set on $X^2 \times (0,\infty)$ and (FM-1), (FM-2), (FM-5) are replaced, respectively, with (GV-1), (GV-2), (GV-5) below (the axiom (GV-2) is reformulated as in [7, Remark 1]):

(GV-1) $M(x, y, 0) > 0 \quad \forall\ t > 0$.
(GV-2) $M(x, x, t) = 1 \quad \forall\ t > 0$ and $x \neq y \Rightarrow M(x, y, t) < 1 \quad \forall\ t > 0$
(GV-5) $M(x, y, .) : (0,\infty) \to [0, 1]$ is continuous $\forall\ x, y \in X$.

Example 1.([5]) Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all $x, y \in X$ and $t > 0$, $M(x, y, t) = \frac{t}{t+d(x,y)}$.

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d the standard fuzzy metric.

Definition 3.([6]) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to converge to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for each $t > 0$.

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a Cauchy sequence if and only if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$.

Definition 4.([11]) Two self maps A and B of a fuzzy metric space $(X, M, *)$ are said to be weak-compatible if they commute at their coincidence points, i.e. $Ax = Bx$ implies $ABx = BAx$.

Definition 5.([13]) A pair (A, S) of self maps of a fuzzy metric space $(X, M, *)$ is said to be semi-compatible if $lim_{n\to\infty} ASx_n = Sx$ whenever there exists a sequence $\{x_n\} \in X$ such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x$ for some $x \in X$.

Definition 6.([12]) A pair (A, S) of self maps of a fuzzy metric space $(X, M, *)$ is said to be reciprocal continuous if $lim_{n\to\infty} ASx_n = Ax$ and $lim_{n\to\infty} SAx_n = Sx$, whenever there exists a sequence $\{x_n\} \in X$ such that $lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$, for some $x \in X$.

If A and S are both continuous then they are obviously reciprocally continuous but the converse need not be true(see example[1]).

Lemma 1.([11]) *If for all* $x, y \in X$, $t > 0$ *and* $0 < k < 1$, $M(x, y, kt) \geq M(x, y, t)$, *then* $x = y$.

Lemma 2.([6]) $M(x, y, .)$ *is non-decreasing for all* x, y *in* X.

Proof. Suppose $M(x, y, t) > M(x, y, s)$ for some $0 < t < s$. Then $M(x, y, t) * M(y, y, s-t) \leq M(x, y, s) < M(x, y, t)$. By (FM-2), $M(y, y, s-t) = 1$, and thus $M(x, y, t) \leq M(x, y, s) < M(x, y, t)$ a contradiction.

In the following proposition we have to show that in the context of reciprocal continuity the notion of compatibility and semi-compatibility of maps becomes equivalent.

Proposition 1. *Let* f *and* g *be two self maps on a fuzzy metric space* $(X, M, *)$. *Assume that* (f, g) *is reciprocal continuous then* (f, g) *is semi-compatible if and only if* (f, g) *is compatible.*

Proof. Let $\{x_n\}$ be a sequence in X such that $fx_n \to z$ and $gx_n \to z$ since pair of maps (f, g) is reciprocally continuous then we have

$$\lim_{n\to\infty} fgx_n = fz \text{ and } \lim_{n\to\infty} gfx_n = gz \tag{1}$$

Suppose that (f, g) is semi-compatible. Then we have,

$$\lim_{n\to\infty} M(fgx_n, gz, t/2) = 1 \tag{2}$$

Now, we have,

$$M(fgx_n, gfx_n, t) \geq M(fgx_n, gz, t/2) * M(gz, gfx_n, t/2)$$

Letting $n \to \infty$ we get,

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1 * 1 = 1.$$

Thus f and g are compatible maps.
Conversely, suppose (f, g) is compatible & reciprocal continuous, then for $t > 0$ we have,

$$\lim_{n\to\infty} M(fgx_n, gfx_n, t/2) = 1 \text{ for all } x_n \in X \tag{3}$$

Now,

$$\lim_{n\to\infty} M(fgx_n, gz, t) \geq \lim_{n\to\infty} (M(fgx_n, gfx_n, t/2) * M(gfx_n, gz, t/2)) = 1 * 1 = 1$$

i.e

$$\lim_{n\to\infty} M(fgx_n, gz, t) = 1.$$

Thus f and g are semi-compatible. This completes the proof.

In [13] B. Singh et al proved the following theorem:

Theorem 1. *Let* A, B, S *and* T *be self maps on a complete fuzzy metric space* $(X, M, *)$ *satisfying:*

(1) *$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$*
(2) *one of A or B is continuous*
(3) *(A,S) is semi-compatible and (B,T) is weak compatible,*
(4) *for all $x, y \in X$ and $t > 0$, $M(Ax, By, t) \geq \Phi(M(Sx, Ty, t))$, where $\Phi : [0,1] \to [0,1]$ is a continuous function such that $\Phi(t) > t$ for each $0 < t < 1$.*

Then A, B, S and T have a unique common fixed point.

3. Main Result

In the following theorem we replace the continuity condition by weaker notion of reciprocal continuity to get more general form of result 4.1, 4.2 and 4.9 of [13].

Theorem 2. *Let A, B, S and T be self maps on a complete fuzzy metric space $(X, M, *)$ where $*$ is a continuous t-norm defined by $a * b = \min\{a, b\}$ satisfying:*

(2.1) *$A(X) \subseteq T(X)$,$B(X) \subseteq S(X)$,*
(2.2) *(B,T) is weak compatible,*
(2.3) *for all $x, y \in X$ and $t > 0$, $M(Ax, By, t) \geq \Phi(M(Sx, Ty, t))$, where $\Phi : [0,1] \to [0,1]$ is a continuous function such that $\Phi(1) = 1$, $\Phi(0) = 0$ and $\Phi(a) > a$ for each $0 < a < 1$.*

If (A,S) is semi-compatible pair of reciprocal continuous maps then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be any arbitrary point. Then for which there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Thus we can construct a sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, 3, \ldots\ldots$
By contractive condition we get,

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq \Phi(M(Sx_{2n}, Tx_{2n+1}, t)) \\ &= \Phi(M(y_{2n}, y_{2n+1}, t)) \\ &> M(y_{2n}, y_{2n+1}, t). \end{aligned}$$

Similarly we get,

$$M(y_{2n+2}, y_{2n+3}, t) > M(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$\begin{aligned} M(y_{n+1}, y_n, t) &\geq \Phi(M(y_n, y_{n-1}, t)) \\ &> M(y_n, y_{n-1}, t). \qquad \text{(Since } \Phi(a) > a) \end{aligned}$$

Therefore $\{M(y_{n+1}, y_n, t)\}$ is an increasing sequence of positive real numbers in $[0,1]$ and tends to limit $l \leq 1$. We claim that $l = 1$. If $l < 1$ then $M(y_{n+1}, y_n, t) \geq M(y_n, y_{n-1}, t)$. On letting $n \to \infty$ we get,

$$\lim_{n\to\infty} M(y_{n+1}, y_n, t) \geq \Phi(\lim_{n\to\infty} M(y_n, y_{n-1}, t))$$

i.e. $l \geq \Phi(l) = l$ (Since $\Phi(a) > a$), a contradiction. Now for any positive integer p,

$$M(y_n, y_{n+p}, t) \geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \cdots * M(y_{n+p-1}, y_{n+p}, t/p).$$

Letting $n \to \infty$ we get,

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * 1 * \cdots * 1 = 1.$$

Thus,

$$\lim_{n\to\infty} M(y_n, y_{n+p}, t) = 1.$$

Thus $\{y_n\}$ is a Cauchy sequence in X. Since X is complete metric space $\{y_n\}$ converges to a point z (say) in X. Hence the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Now since A and S are reciprocal continuous and semi-compatible then we have $\lim_{n\to\infty} ASx_{2n} = Az$, $\lim_{n\to\infty} SAx_{2n} = Sz$, and $\lim_{n\to\infty} M(ASx_{2n}, Sz, t) = 1$. Therefore we get $Az = Sz$. Now we will show $Az = z$. For this suppose $Az \neq z$. Then by contractive condition we get,

$$M(Az, Bx_{2n+1}, t) \geq \Phi(M(Sz, Tx_{2n+1}, t))$$

Letting $n \to \infty$, we get,

$$M(Az, z, t) \geq \Phi(M(Az, z, t)) > M(Az, z, t),$$

a contradiction, thus $z = Az = Sz$. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Az = Tu$. Putting $x = x_{2n}$, $y = u$ in (3) we get,

$$M(Ax_{2n}, Bu, t) \geq \Phi(M(Sx_{2n}, Tu, t)).$$

Letting $n \to \infty$, we get,

$$M(z, Bu, t) \geq \Phi(M(z, z, t)) = \Phi(1) = 1,$$

i.e. $z = Bu = Tu$ and the weak-compatibility of (B, T) gives $TBu = BTu$, i.e. $Tz = Bz$. Again by contractive condition and assuming $Az \neq Bz$, we get $Az = Bz = z$. Hence finally we get $z = Az = Bz = Sz = Tz$, i.e. z is a common fixed point of A, B, S and T. The uniqueness follows from contractive condition. This completes the proof.

Now we prove an another common fixed point theorem with different contractive condition:

Theorem 3. *Let A, B, S and T be self maps on a complete fuzzy metric space $(X, M, *)$ satisfying:*

(3.1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,
(3.2) (B, T) *is weak compatible,*
(3.3) *for all $x, y \in X$ and $t > 0$,*

$$M(Ax, By, t) \geq \phi\{\min(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), M(Ax, Ty, t))\},$$

where $\Phi : [0,1] \to [0,1]$ *is a continuous function such that* $\Phi(1) = 1$, $\Phi(0) = 0$ *and* $\Phi(a) > a$ *for each* $0 < a < 1$. *If* (A, S) *is semi-compatible pair of reciprocal continuous maps then* A, B, S *and* T *have a unique common fixed point.*

Proof. Let $x_o \in X$ be any arbitrary point. Then for which there exists $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Thus we can construct a sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n = 0, 1, 2, 3, \ldots\ldots$. By contractive condition we get,

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, t) &= M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq \Phi\{\min(M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), \\ &\quad M(Bx_{2n+1}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, t))\} \\ &= \Phi\{\min(M(y_{2n-1}, y_{2n}, t), M(y_{2n}, y_{2n-1}, t), \\ &\quad M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n}, t))\} \\ &= \Phi\{\min(M(y_{2n-1}, y_{2n}, t), M(y_{2n+1}, y_{2n}, t))\} \\ &= \Phi\{M(y_{2n-1}, y_{2n}, t)\} \\ &> M(y_{2n-1}, y_{2n}, t), \quad (\text{Since } \Phi(a) > a \text{ for each } 0 < a < 1) \end{aligned}$$

Similarly we get,

$$M(y_{2n+2}, y_{2n+3}, t) > M(y_{2n+1}, y_{2n+2}, t).$$

In general,

$$M(y_{n+1}, y_n, t) \geq \Phi(M(y_n, y_{n-1}, t)) > M(y_n, y_{n-1}, t).$$

Therefore $\{M(y_{n+1}, y_n, t)\}$ is an increasing sequence of positive real numbers in $[0,1]$ and tends to limit $l \leq 1$ then by the same technique of above theorem we can easily show that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete metric space $\{y_n\}$ converges to a point z (say) in X. Hence the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Now since A and S are reciprocal continuous and semi-compatible then we have $\lim_{n\to\infty} ASx_{2n} = Az$, $\lim_{n\to\infty} SAx_{2n} = Sz$, and $\lim_{n\to\infty} M(ASx_{2n}Sz, t) = 1$. Therefore we get $Az = Sz$. Now we will show $Az = z$. For this suppose $Az \neq z$. Then by (3.2.2), we get a contradiction, thus $Az = z$. Hence by similar techniques of above theorem we can easily show that z is a common fixed point of A, B, S and T i.e. $z = Az = Bz = Sz = Tz$. Uniqueness of fixed point can be easily verify by contractive condition. This completes the proof.

We now give an example which not only illustrate our Theorem 2.1 but also shows that the notion of reciprocal continuity of maps is weaker than the continuity of maps.

Example 1. Let (X, d) be usual metric space where $X = [2, 20]$ and M be the usual fuzzy metric on $(X, M, *)$ where $* = t_{\min}$ be the induced fuzzy metric space with $M(x, y, t) = \frac{t}{t+d(x,y)}$ for $x, y \in X$, $t > 0$. We define mappings A, B, S and T by

$$A2 = 2, \quad Ax = 3 \text{ if } x > 2$$

$$S2 = 2,\ \ Sx = 6 \ \text{ if } \ x > 2$$
$$Bx = 2 \ \text{ if } \ x = 2 \ \text{ or } \ > 5,\ \ Bx = 6 \ \text{ if } \ 2 < x \le 5$$
$$Tx = 2,\ \ Tx = 12 \ \text{ if } \ 2 < x \le 5,\ \ Tx = \frac{(x+1)}{3} \ \text{ if } \ x > 5$$

Then A, B, S and T satisfy all the conditions of the above theorem with $\Phi(a) = \frac{7a}{(3a+4)} > a$, where $a = 1/1 + d(Sx, Ty)/t$ and have a unique common fixed point $x = 2$. It may be noted that in this example $A(X) = \{2,3\} \subseteq T(X) = [2,7] \bigcup \{12\}$ and $B(X) = \{2,6\} \subseteq S(X) = \{2,6\}$.

Also A and S are reciprocally continuous compatible mappings. But neither A nor S is continuous not even at fixed point $x = 2$. The mapping B and T are non-compatible but weak -compatible since they commute at their co-incidence points. To see B and T are non-compatible, let us consider the sequence $\{x_n\}$ in X defined by $\{x_n\} = \{5 + \frac{1}{n}\}$; $n \ge 1$. Then, $\lim_{n\to\infty} Tx_n = 2$,$\lim_{n\to\infty} Bx_n = 2$, $\lim_{n\to\infty} TBx_n = 2$ and $BTx_n = 6$. Hence B and T are non-compatible.

Remark 1. The maps A, B, S and T are discontinuous even at the common fixed point $x = 2$.

Remark 2. The known common fixed point theorems involving a collection of maps in fuzzy metric spaces require one of the mapping in compatible pair to be continuous. For example in the main result of R. Chug and et al [2] assume one of the mapping A, B, S or T to be continuous similarly B.Singh et al [13, 14] and M. S. Khan et al [9] assume one of the mappings in compatible pairs of maps is continuous. The present theorem however does not require any of the mappings to be continuous and hence all the results mentioned above can be further improved in the spirit of our Theorem 3.1.

Remark 3. It may be mention that in view of Theorem 3.1, we have not been able to decide the following question: Whether Theorem 3.1 & 3.2 would remain true if '$*$' is replace by any arbitrary continuous t-norm?

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