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MATRIX TRANSFORMATIONS INVOLVING CERTAIN BANACH SPACE VALUED SEQUENCE SPACES

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Abstract. In this paper for Banach spaces X and Y we characterize matrix classes $(\Gamma(X,\lambda), l_\infty(Y,\mu))$, $(\Gamma(X,\lambda), C(Y,\mu))$, $(\Gamma(X,\lambda),\ c_0(Y,\mu))$, $(\Gamma(X,\lambda),\ \Gamma^*(Y,\mu))$, $(l_1(X,\lambda),\ \Gamma(Y,\mu))$ and $(c_0(X,\lambda),\ c_0(Y,\mu))$ of bounded linear operators involving X- and Y-valued sequence spaces. Further as an application of the matrix class $(c_0(X,\lambda),\ c_0(Y,\mu))$ we investigate the Banach space $B(c_0(X,\lambda),\ c_0(Y,\mu))$ of all bounded linear mappings of $c_0(x,\lambda)$ to $c_0(Y,\mu)$.

1. Introduction

Let $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ be any sequences of non-zero complex numbers, X and Y be Banach spaces over the field C of complex numbers and $B(X,Y)$ denote the Banach space of all bounded linear operators from X into Y with the usual operator norm.

Let $A = (A_{nk})$, $(n,k = 1,2,3\ldots)$ be an infinite matrix of bounded linear operators A_{nk} on the Banach space X into the Banach space Y. Then for the classes of X-valued sequences $E(X)$ and Y-valued sequences $F(Y)$, we define the matrix class $(E(X),F(Y))$ by saying that

$$A = (A_{nk}) \in ((E(X),F(Y))$$

if

$$\text{for every} \qquad \overline{x} = (x_k) \in E(X),$$

$$y_n = A_n(\overline{x}) = \sum_{k=1}^{\infty} A_{nk}x_k$$

converges in the norm of Y, for each n, and the sequence $\overline{y} = (y_n) = (A_n(\overline{x}))$ belongs to $F(Y)$. In such a case $\overline{y} = A\overline{x}$ is called the A transform of $\overline{x}$.

We define

$$\Gamma(X,\lambda) = \{\overline{x} = (x_k) : x_k \in X, \|\lambda_k x_k\|^{1/k} \to 0 \text{ as } k \to \infty\}, \text{ and}$$
$$\Gamma^*(X,\lambda) = \{\overline{x} = (x_k) : x_k \in X, \sup_k \|\lambda_k x_k\|^{1/k} < \infty\}.$$

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Similarly $\Gamma(Y,\mu)$ and $\Gamma^*(Y,\mu)$ are defined.

The above defined spaces are the generalizations of Γ and Γ^* respectively, introduced by Iyer [2, 3]. Moreover $\Gamma(X,\lambda)$ and $\Gamma^*(X,\lambda)$ can be viewed as special cases of $c_0(X,\lambda,p)$ and $l_\infty(X,\lambda,p)$ respectively introduced and studied in [8, 9]. In fact $c_0(X,\lambda,(1/k)) = \Gamma(X,\lambda)$ and $l_\infty(X,\lambda,(1/k)) = \Gamma^*(X,\lambda)$. However we prefer the notations $\Gamma(X,\lambda)$ and $\Gamma^*(X,\lambda)$ in place of $c_0(X,\lambda,(1/k))$ and $l_\infty(X,\lambda,(1/k))$ for obvious reasons as we observe that they are more close to Γ and Γ^*. Analogous to $P_{\lambda,p}$ for $c_0(X,\lambda,p)$ of [8] we define

$$P_\lambda(\overline{x}) = \sup_k \|\lambda_k x_k\|^{1/k}, \quad \overline{x} = (x_k) \in \Gamma(X,\lambda) \text{ or } \Gamma^*(X,\lambda),$$

and we see that $(\Gamma(X,\lambda), P_\lambda)$ is a complete total paranomed space [cf. Theorem 2.22 of [9]] and hence $\Gamma(X,\lambda)$ is a complete linear metric space with respect to the metric induced by the paranorm. $\overline{\theta} = (\theta,\theta,\theta,\ldots)$ will denote the zero of $\Gamma(X,\lambda)$. Some other properties such as generalized Köthe-Toeplitz duals, continuous duals etc. for the space $(\Gamma(X,\lambda), P_\lambda)$ can easily be derived from the concerning results already investigated in [8] about $c_0(X,\lambda,p)$. Since $\inf_k 1/k = 0$, therefore $\Gamma^*(X,\lambda)$ is simply a complete metric space with respect to the metric induced by P_λ. This assertion follows from Theorem 2.18 of [9]. We further define

$$c_0(X,\lambda) = \{\overline{x} = (x_k) : x_k \in X, \|\lambda_k x_k\| \to 0 \text{ as } k \to \infty\},$$
$$c(X,\lambda) = \{\overline{x} = (x_k) : x_k \in X, \text{ there exists } l \in X \text{ such that } \|\lambda_k x_k - l\| \to 0 \text{ as } k \to \infty\},$$
$$l_\infty(X,\lambda) = \{\overline{x} = (x_k) : x_k \in X, \sup_k \|\lambda_k x_k\| < \infty\},$$
and $$l_1(X,\lambda) = \{\overline{x} = (x_k) : x_k \in X, \sum_{k=1}^\infty \|\lambda_k x_k\| < \infty\}.$$

As mentioned in [8] and [9] we note these spaces are particular cases of $c_0(X,\lambda,p)$, $c(X,\lambda,p)$, $l_\infty(X,\lambda,p)$ and $l(X,\lambda,p)$, when $p_k = 1$ for all $k \geq 1$. Of course $c_0(X,\lambda)$, $c(X,\lambda)$, $l_\infty(X,\lambda)$ are Banach spaces with the norm

$$\|\overline{x}\|_\lambda = \sup_k \|\lambda_k x_k\|$$

and $l_1(X,\lambda)$ is a Banach space with the norm

$$\|\overline{x}\|_\lambda = \sum_{k=1}^\infty \|\lambda_k x_k\|.$$

We recall the definition of generalized β-dual for the class $E(X)$ of X-valued sequences.

Definition 1.1. [7, p.8] Let X and Y be Banach spaces and $\overline{A} = (A_k)$ a sequence of linear, but not necessarily bounded, operators A_k on X into Y. Suppose $E(X)$ is a non-empty set of X-valued sequences. Then the β-dual of $E(X)$ is defined by $E^\beta(X) = \{\overline{A} = (A_k) : \sum_{k=1}^\infty A_k x_k$ converges in Y for all $\overline{x} \in E(X)\}$.

Definition 1.2. [7, p.5] Let $\overline{A} = (A_k)$ be a sequence in $B(X, Y)$. Then the group norm of (A_k) is defined by $\|(A_k)\| = \sup \| \sum_{k=1}^n A_k x_k\|$, where supremum is taken over all $n \geq 1$ and $x_k \in S$, (S is the closed unit sphere in X).

We refer [7] for some of the properties of group norm and while dealing with group norm in section 4, we shall use notation $R_k(\overline{A}) = (A_k, A_{k+1}, \ldots)$.

A number of results regarding the characterization of matrix classes of linear operators between some of the Banach space valued sequence spaces corresponding to scalar sequence spaces c_0, c, l_∞, l_p etc. have been obtained, for instance see [7]. Our aim in this note is to investigate some of the matrix classes of linear operators involving certain Banach space valued sequence spaces which will lead to generalizations of some of the results determined by Rao [5, 6] for scalar sequence spaces $c_0, c, l_1, \Gamma, \Gamma^*$ etc.

2. Matrix Transformations between Some Banach Space Valued Sequence Spaces

In this section we characterize matrix transformations (of bounded linear operators) between certain Banach space valued sequence spaces, viz. $(\Gamma(X, \lambda), l_\infty(Y, \mu))$, $(\Gamma(X, \lambda), c(Y, \mu))$, $(\Gamma(X, \lambda), c_0(Y, \mu))$ and $(\Gamma(X, \lambda), \Gamma^*(Y, \mu))$.

In order to establish conditions characterizing $(\Gamma(X, \lambda), c_0(Y, \mu))$ we first prove:

Lemma 2.1. $\overline{A} = (A_k) \in \Gamma^\beta(X, \lambda)$, *the generalized β-dual of* $\Gamma(X, \lambda)$, *if and only if*

$$\sup_k \|\lambda_k^{-1} A_k\|^{1/k} < \infty.$$

Proof. The sufficiency of the condition is straight forward. For the necessity, we suppose $\overline{A} = (A_k) \in \Gamma^\beta(X, \lambda)$ but $\sup_k \|\lambda_k^{-1} A_k\|^{1/k} = \infty$. Then there exists a subsequence $(k(n))$ of (k) such that $\|\lambda_{k(n)}^{-1} A_{k(n)}\| > n^{k(n)}$; $n \geq 1$ and for each $n \geq 1$ there exists $z_n \in S$ (the closed unit sphere in X) such that $2\|\lambda_{k(n)}^{-1} A_{k(n)} z_n\| > n^{k(n)}$. Now we define

$$\begin{aligned} x_k &= n^{-k(n)} \lambda_{k(n)}^{-1} z_n; \qquad \text{if } k = k(n), \quad n \geq 1 \quad \text{and} \\ &= \theta, \quad \text{otherwise.} \end{aligned}$$

Then we see that $\overline{x} = (x_k) \in \Gamma(X, \lambda)$ but $\overline{A} = (A_k) \notin \Gamma^\beta(X, \lambda)$, which leads to a contradiction. This completes the proof.

Theorem 2.2. $A = (A_{nk}) \in (\Gamma(X, \lambda),\ l_\infty(Y, \mu))$ *if and only if there exists* $M > 0$ *such that*

$$\|\mu_n \lambda_k^{-1} A_{nk}\|^{1/k} \leq M$$

independently of $n \geq 1$ *and* $k \geq 1$.

Proof. For the sufficiency, we consider $\overline{x} = (x_k) \in \Gamma(X, \lambda)$ and for $\varepsilon > 0$ we choose $0 < \eta < 1$ so that $\eta M < \frac{\varepsilon}{3+\varepsilon}$. Then there exists a K such that $\|\lambda_k x_k\|^{1/k} < \eta$ for all $k \geq K$.

Now for $y_n = \sum_{k=1}^{\infty} A_{nk} x_k$,

$$\begin{aligned} \|\mu_n y_n\| &= \|\sum_{k=1}^{\infty} \mu_n A_{nk} x_k\| \\ &\leq \sum_{k=1}^{K-1} M^k \|\lambda_k x_k\| + \frac{M\eta}{1 - M\eta} \\ &< \sum_{k=1}^{K-1} M^k \|\lambda_k x_k\| + \varepsilon, \end{aligned}$$

which implies that y_n exists for each $n \geq 1$. Moreover

$$\sup_n \|\mu_n y_n\| < \infty$$

and so $\overline{y} = (y_n) \in l_\infty(Y, \mu)$.

For the necessity, suppose $A = (A_{nk}) \in (\Gamma(X, \lambda), l_\infty(X, \lambda))$. Then for every $\overline{x} = (x_k) \in \Gamma(X, \lambda)$, $y_n = \sum_{k=1}^{\infty} A_{nk} x_k$ exists for each $n \geq 1$, and $\sup_n \|\mu_n y_n\| < \infty$. Thus by Lemma 2.1 we have that

$$\sup_k \|\mu_n \lambda_k^{-1} A_{nk}\|^{1/k} < \infty, \quad \text{for each } n \geq 1.$$

Now we denote

$$f_n(\overline{x}) = \|\sum_{k=1}^{\infty} \mu_n A_{nk} x_k\|$$

and

$$\sup_k \|\mu_n \lambda_k^{-1} A_{nk}\|^{1/k} = L(n), \quad n \geq 1.$$

Then we see that $f_n(\overline{x})$ is pointwise bounded functional as $\overline{y} = (y_n) \in l_\infty(Y, \mu)$. Further for sufficiently small $\eta > 0$ if $P_\lambda(\overline{x}) = \sup_k \|\lambda_k x_k\|^{1/k} < \eta$, for $\overline{x} = (x_k) \in \Gamma(X, \lambda)$ then we have

$$\begin{aligned} f_n(\overline{x}) &= \|\sum_{k=1}^{\infty} \mu_n A_{nk} x_k\| \\ &\leq \sum_{k=1}^{\infty} \|\mu_n \lambda_k^{-1} A_{nk}\| \|\lambda_k x_k\| \\ &< \frac{L(n)\eta}{1 - L(n)\eta} \end{aligned}$$

which implies that for each $n \geq 1$ f_n is continuous on $(\Gamma(X,\lambda),P_\lambda)$. Since $(\Gamma(X,\lambda),P_\lambda)$ is a complete linear metric space therefore by Osgood theorem (a verson of uniform boundedness principle) there exists $L > 0$ and a closed sphere which can without loss of generality be taken as $S[\overline{\theta},\delta]$ in $\Gamma(X,\lambda)$ with centre $\overline{\theta}$ and radius δ such that

$$|f_n(\overline{x})| \leq L \tag{2.1}$$

for all $\overline{x} = (x_k) \in S[\overline{\theta},\delta]$ and for all $n \geq 1$. Now for arbitrary $z \in S$ and $k \geq 1$ we take $\overline{x} = (\theta,\theta,\ldots\theta,\delta^k\lambda_k^{-1}z,\theta,\theta\ldots) \in S[\overline{\theta},\delta]$ where $\delta^k\lambda_k^{-1}z$ is at the k^{th} place. Thus by (2.1) we have

$$\|\mu_n\lambda_k^{-1}\delta^k A_{nk}z\| \leq L$$

for all $z \in S$ and for all $n \geq 1$ and $k \geq 1$, and hence

$$\|\mu_n\lambda_k^{-1}A_{nk}\|^{1/k} \leq \frac{L^{1/k}}{\delta} \leq M,$$

for all $n \geq 1$ and $k \geq 1$, where $M = \delta^{-1}\max(1,L)$. This completes the proof.

Theorem 2.3. *$A = (A_{nk}) \in (\Gamma(X,\lambda), c(Y,\mu))$ if and only if*

(i) *there exists $M > 0$ such that*

$$\|\mu_n\lambda_k^{-1}A_{nk}\|^{1/k} \leq M,$$

independently of $n \geq 1$ and $k \geq 1$; and

(ii) *$\lim_n \mu_n A_{nk}$ exists for each $k \geq 1$.*

Proof. Let (i) and (ii) hold and $\overline{x} = (x_k) \in \Gamma(X,\lambda)$. Now for $0 < \epsilon < 1$ choose η so that $0 < \eta M < \frac{\varepsilon}{3+\varepsilon}$. Then there exists K such that

$$\|\lambda_k x_k\|^{1/k} < \eta, \quad \text{for all } k \geq K.$$

Now

$$\|\mu_n y_n\| \leq \sum_{k=1}^{K-1} M^k\|\lambda_k x_k\| + \frac{M\eta}{1-M\eta} < \infty.$$

Thus y_n exists for each $n \geq 1$. Further $\lim_n \mu_n A_{nk}x$ exists for each k and each $x \in X$, so we take

$$\lim_n \mu_n A_{nk}x = A_k x$$

for each $k \geq 1$ and $x \in X$. By Banach-Steinhaus theorem, $A_k \in B(X,Y)$ for each $k \geq 1$. Thus form (i) and (ii) we easily get that

$$\|\lambda_k^{-1}A_k\|^{1/k} \leq M, \quad \text{for all } k \geq 1. \tag{2.2}$$

Now using (2.2) we easily see that $l = \sum_{k=1}^{\infty} A_k x_k$ exists in Y. Thus

$$\begin{aligned}\|\mu_n y_n - l\| &\le \sum_{k=1}^{K-1} \|(\mu_n A_{nk} - A_k)x_k\| + \sum_{k=K}^{\infty} \|\mu_n \lambda_k^{-1} A_{nk}\| \|\lambda_k x_k\| + \sum_{k=K}^{\infty} \|\lambda_k^{-1} A_k\| \|\lambda_k x_k\| \\ &\le \sum_{k=1}^{K-1} \|(\mu_n A_{nk} - A_k)x_k\| + 2\frac{M\eta}{1-M\eta} \\ &< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon\end{aligned}$$

for all sufficiently large values of n and so $y = (y_n) \in c(Y, \mu)$.

Necessity of (ii) can easily be proved by considering the sequence $\overline{x} = (\theta, \theta, \theta, \dots, \theta, x, \theta, \theta)$ taking $x \in X$ at k^{th} place, $k \ge 1$. Moreover since $(\Gamma(X, \lambda),\ c(Y, \mu)) \subset (\Gamma(X, \lambda),\ l_\infty(Y, \mu))$ therefore the necessity of (i) follows from Theorem 2.2, this completes the proof.

Theorem 2.4. *$A = (A_{nk}) \in (\Gamma(X, \lambda),\ c_0(Y, \mu))$ if and only if*

(i) *there exists $M > 0$ such that*

$$\|\mu_n \lambda_k^{-1} A_{nk}\|^{1/k} \le M.$$

independently of $n \ge 1$ and $k \ge 1$; and

(ii) $\lim_n \mu_n A_{nk} = \theta$, *for each k*

Proof. It can easily be proved along the lines of Theorem 2.3.

Theorem 2.5. *$A = (A_{nk}) \in (\Gamma(X, \lambda),\ \Gamma^*(Y, \mu))$ if and only if there exists $M > 0$ such that*

$$\|\mu_n \lambda_k^{-1} A_{nk}\|^{1/(n+k)} \le M$$

independently of $n \ge 1$ and $k \ge 1$.

Proof. For the sufficiency of the condition, consider $\overline{x} = (x_k) \in \Gamma(X, \lambda)$ and $\varepsilon > 0$ and choose η so that $0 < \eta M < \frac{\varepsilon}{1+\varepsilon}$. Then there exists K such that $\|\lambda_k x_k\|^{1/k} < \eta$, for all $k \ge K$. Now

$$\begin{aligned}\|\mu_n y_n\| &\le \sum_{k=1}^{K-1} M^{n+k} \|\lambda_k x_k\| + \sum_{k=K}^{\infty} M^{n+k} \eta^k \\ &< M^n \left\{ \sum_{k=1}^{K-1} M^k \|\lambda_k x_k\| + \varepsilon \right\} < \infty.\end{aligned}$$

Thus y_n exists for each $n \ge 1$. Further

$$\begin{aligned}\|\mu_n y_n\|^{1/n} &\le M \left\{ \sum_{k=1}^{K-1} M^k \|\lambda_k x_k\| + \varepsilon \right\}^{1/n} \\ &\le M \max \left\{ 1, \sum_{k=1}^{K-1} M^k \|\lambda_k x_k\| + \varepsilon \right\},\end{aligned}$$

and hence $\sup_n \|\mu_n y_n\|^{1/n} < \infty$. Thus $y = (y_n) \in \Gamma^*(Y, \mu)$.

Conversely let $A = (A_{nk}) \in (\Gamma(X, \lambda),\ \Gamma^*(Y, \mu))$. Then $y_n = \sum_{k=1}^{\infty} A_{nk} x_k$ exists for each $n \geq 1$ and for each $\overline{x} = (x_k) \in \Gamma(X, \lambda)$, and $\sup_n \|\mu_n y_n\|^{1/n} < \infty$. Thus by Lemma 2.1 we have that

$$\sup_k \|\mu_n \lambda_k^{-1} A_{nk}\|^{1/k} = L(n) < \infty$$

for each $n \geq 1$. Now we denote

$$f_n(\overline{x}) = \|\sum_{k=1}^{\infty} \mu_n A_{nk} x_k\|^{1/n},$$

and we see that $(f_n(\overline{x}))$ is pointwise bounded. Further if $P_\lambda(\overline{x}) = \sup_k \|\lambda_k x_k\|^{1/k} < \eta$ then f_n is continuous functional on $\Gamma(X, \lambda)$, for each $n \geq 1$. Moreover $\Gamma(X, \lambda)$ is complete linear metric space therefore by uniform boundedness principle (Osgood theorem) we can find $L > 0$ and a closed sphere $S[\overline{\theta}, \delta]$ in $\Gamma(X, \lambda)$ such that

$$|f_n(\overline{x})| \leq L$$

for all $n \geq 1$ and for all $\overline{x} \in S[\overline{\theta}, \delta]$, or

$$\|\sum_{k=1}^{\infty} \mu_n A_{nk} x_k\| \leq L^n \tag{2.3}$$

for all $n \geq 1$ and for all $\overline{x} \in S[\overline{\theta}, \delta]$. Now take $z \in S$ arbitrary and consider the sequence

$$\overline{x} = (\theta, \theta, \ldots, \theta, \delta^k \lambda_k^{-1} z, \theta, \theta, \ldots)$$

where $\delta^k \lambda_k^{-1} z$ has the entry at k^{th} place. Thus $\overline{x} \in S[\overline{\theta}, \delta]$ and so from (2.3) if follows that

$$\|\mu_n \delta^k \lambda_k^{-1} A_{nk} z\| \leq L^n$$

for all $n \geq 1$, $k \geq 1$ and $z \in S$, which implies that

$$\|\mu_n \lambda_k^{-1} A_{nk}\| \leq \frac{L^n}{\delta^k} = M^{n+k}$$

for all $n \geq 1$ and $k \geq 1$, where $M = \max(L, \delta^{-1})$. This completes the proof.

3. Characterization of $(l_1(X, \lambda), \Gamma(Y, \mu))$

In this section we determine the necessary and sufficient conditions for a matrix to map $l_1(X, \lambda)$ into $\Gamma(Y, \mu)$.

Definition 3.1. If γ and ρ are any two classes of scalar sequences then we define

$$\gamma(X) = \{\overline{x} = (x_k) : x_k \in X, \quad \text{such that } (\|x_k\|) \in \gamma\},$$
$$\gamma(X, \lambda) = \{\overline{x} = (x_k) : x_k \in X, \quad \text{such that } (\lambda_k x_k) \in \gamma(X)\}.$$

Similarly $\rho(X)$ and $\rho(X,\lambda)$ are defined.

Lemma 3.2. $A=(A_{nk})\in(\gamma(X),\rho(Y))$ *iff* $B=(\mu_n^{-1}\lambda_k A_{nk})\in(\gamma(X,\lambda),\rho(Y,\mu))$.

Proof. Let $(A_{nk})\in(\gamma(X),\rho(Y))$ then for every $\overline{x}=(x_k)\in\gamma(X)$, $y_n=\sum_{k=1}^{\infty}A_{nk}x_k$ exists for each n and $\overline{y}=(y_n)\in\rho(Y)$. Now consider $\overline{u}=(u_n)\in\gamma(X,\lambda)$ and its transform $\overline{v}=B\overline{u}$, $v_n=\sum_{k=1}^{\infty}\mu_n^{-1}\lambda_k A_{nk}u_k$. Thus $(\lambda_k u_k)\in\gamma(X)$ and so $(y_n)=(\sum_{k=1}^{\infty}A_{nk}\lambda_k u_k)_{n=1}^{\infty}\in\rho(Y)$ which implies that $(\mu_n^{-1}y_n)\in\rho(Y,\mu)$ and hence $\overline{v}=(v_n)\in\rho(Y,\mu)$. Similarly the converse may be proved. The proof is now complete.

Applying Lemma 3.2 to Theorem 4.9 of [7, p.53] we easily get

Lemma 3.3. *Let* $1\le p<\infty$. *Then* $A=(A_{nk})\in(l_1(X,\lambda),l_p(Y,\mu))$ *if and only if*

$$\sup\sum_{n=1}^{\infty}\|\mu_n\lambda_k^{-1}A_{nk}z\|^p<\infty,$$

where the supremum is taken over all $z\in S$ *and all* $k\ge 1$.

Following Lemma can easily be obtained by taking $p_k=1$ in Theorem 3.2 of [9] and considering all $A_k\in B(X,Y)$ (cf. [7, p.26]).

Lemma 3.4. *Let* $A_k\in B(X,Y)$ *for all* $k\ge 1$. *Then* $(A_k)\in l_1^{\beta}(X,\lambda)$ *if and only if* $\sup_k\|\lambda_k^{-1}A_k\|<\infty$.

Theorem 3.5. $A=(A_{nk})\in(l_1(X,\lambda),\Gamma(Y,\mu))$ *if and only if*
(i) $\sup_k\|\lambda_k^{-1}A_{nk}\|<\infty$, *for each* $n\ge 1$, *and*
(ii) $\|\mu_n\lambda_k^{-1}A_{nk}\|^{1/n}\to 0$ *as* $n\to\infty$ *uniformly in* k.

Proof. Sufficient part of the theorem can be proved easily with the help of Lemma 3.4.

Let $A=(A_{nk})\in(l_1(X,\lambda),\Gamma(Y,\mu))$. The necessity of (i) follows easily by applying Lemma 3.4 to the fact that $y_n=\sum_{k=1}^{\infty}A_{nk}x_k$ exists for each $\overline{x}=(x_k)\in l_1(X,\lambda)$. For the necessity of (ii) suppose $A=(A_{nk})\in(l_1(X,\lambda),\Gamma(Y,\mu))$ but $\|\mu_n\lambda_k^{-1}A_{nk}\|^{1/n}\nrightarrow 0$ as $n\to\infty$, uniformly in k. Thus for an $\varepsilon>0$ and a given N there exist an $n>N$ and a $k\ge 1$ such that

$$\|\mu_n\lambda_k^{-1}A_{nk}\|^{1/n}\ge\varepsilon. \tag{3.1}$$

Further since $\Gamma(Y,\mu)\subset l_1(Y,\mu)$ therefore $(A_{nk})\in(l_1(X,\lambda),l_1(Y,\mu))$ and so by Lemma 3.3, taking $p=1$ therein, we get

$$\sup\sum_{k=1}^{\infty}\|\mu_n\lambda_k^{-1}A_{nk}z\|<\infty,$$

where the supremum is taken over all $z\in S$ and all $k\ge 1$. Thus there exists $L>0$ such that

$$\|\mu_n\lambda_k^{-1}A_{nk}z\|\le\frac{L}{2},$$

for all $n \geq 1$, $k \geq 1$ and $z \in S$ and so for each $n \geq 1$

$$V_n = \sup_k \|\mu_n \lambda_k^{-1} A_{nk}\| \leq \frac{L}{2}. \tag{3.2}$$

Now by considering the sequences $\overline{x} = (\theta, \theta, \theta, \ldots \theta, z, \theta, \theta, \ldots)$ in $l_1(X, \lambda)$ where $z \in X$ is at the k^{th} place, $k \geq 1$, we get that

$$\|\mu_n \lambda_k^{-1} A_{nk} z\|^{1/n} \to 0 \text{ as } n \to \infty, \text{ for each } k \text{ and for each } z. \tag{3.3}$$

Thus in view of (3.1) we may choose $n(1)$ and $k(1)$ such that

$$\|\mu_{n(1)} \lambda_{k(1)}^{-1} A_{n(1)k(1)}\|^{1/n(1)} \geq \varepsilon \tag{3.4}$$

and so there exists $z_1 \in S$ such that

$$\|\mu_{n(1)} \lambda_{k(1)}^{-1} A_{n(1)k(1)} z_1\| \geq \frac{1}{2^{n(1)}} \|\mu_{n(1)} \lambda_{k(1)}^{-1} A_{n(1)k(1)}\|. \tag{3.5}$$

Next in view of (3.1) and (3.3) we choose $n(2) > n(1)$ sufficiently large and $k(2) > k(1)$ such that

$$\frac{L}{2^{n(2)}} < \left[\frac{\varepsilon}{8}\right]^{n(1)} \tag{3.6}$$

$$\mu_{n(2)} \lambda_{k(2)}^{-1} A_{n(2)k(2)}\|^{1/n(2)} \geq \varepsilon, \quad \text{and} \tag{3.7}$$

$$\|\mu_{n(2)} \lambda_{k(1)}^{-1} A_{n(2)k(1)}\|^{1/n(2)} < \frac{\varepsilon}{16}. \tag{3.8}$$

Now we select $z_2 \in S$ such that

$$\|\mu_{n(2)} \lambda_{k(2)} A_{n(2)k(2)} z_2\| \geq \frac{1}{2^{n(2)}} \|\mu_{n(2)} \lambda_{k(2)}^{-1} A_{n(2)k(2)}\|. \tag{3.9}$$

Further in view of (3.1) and (3.3) we choose $n(3) > n(2)$ sufficiently large and $k(3) > k(2)$ such that

$$\frac{L}{2^{n(3)}} < \left[\frac{\varepsilon}{16}\right]^{n(2)} \tag{3.10}$$

$$\|\mu_{n(3)} \lambda_{k(3)}^{-1} A_{n(3)k(3)}\|^{1/n(3)} \geq \varepsilon, \tag{3.11}$$

and

$$\begin{cases} \|\mu_{n(3)} \lambda_{k(2)}^{-1} A_{n(3)k(2)} z_2\|^{1/n(3)} < \frac{\varepsilon}{24}, \\ \|\mu_{n(3)} \lambda_{k(1)}^{-1} A_{n(3)k(1)} z_1\|^{1/n(3)} < \frac{\varepsilon}{24}. \end{cases} \tag{3.12}$$

Now we can select $z_3 \in S$ such that

$$\|\mu_{n(3)} \lambda_{k(3)}^{-1} A_{n(3)k(3)} z_3\| \geq \frac{1}{2^{n(3)}} \|\mu_{n(3)} \lambda_{k(3)}^{-1} A_{n(3)k(3)}\|. \tag{3.13}$$

Again by (3.1) and (3.3) we can choose $n(4) > n(3)$ sufficiently large and $k(4) > k(3)$, such that

$$\frac{L}{2^{n(4)}} < \left[\frac{\varepsilon}{24}\right]^{n(3)} \tag{3.14}$$

$$\|\mu_{n(4)}\lambda_{k(4)}^{-1}A_{n(4)k(4)}\|^{1/n(4)} \geq \varepsilon, \quad \text{and} \tag{3.15}$$

$$\begin{cases} \|\mu_{n(4)}\lambda_{k(3)}^{-1}A_{n(4)k(3)}z_3\|^{1/n(4)} < \frac{\varepsilon}{32}, \\ \|\mu_{n(4)}\lambda_{k(2)}^{-1}A_{n(4)k(2)}z_2\|^{1/n(4)} < \frac{\varepsilon}{32}, \\ \|\mu_{n(4)}\lambda_{k(1)}^{-1}A_{n(4)k(1)}z_1\|^{1/n(4)} < \frac{\varepsilon}{32}. \end{cases} \tag{3.16}$$

We take $z_4 \in S$ such that

$$\|\mu_{n(4)}\lambda_{k(4)}^{-1}A_{n(4)k(4)}z_4\| \geq \frac{1}{2^{n(4)}}\|\mu_{n(4)}\lambda_{k(4)}^{-1}A_{n(4)k(4)}\|, \tag{3.17}$$

and so on.

We define

$$\begin{aligned} x_k &= \frac{1}{2^{n(j)}}z_j, \quad k = k(j),\ j = 1, 2, 3\ldots,\ \text{and} \\ &= \theta, \quad \text{otherwise.} \end{aligned}$$

Now taking into consideration (3.2), (3.4), (3.5) and (3.6) we get that

$$\begin{aligned} \|\mu_{n(1)}y_{n(1)}\|^{1/n(1)} &\geq \|\mu_{n(1)}\lambda_{k(1)}^{-1}A_{n(1)k(1)}\frac{1}{2^{n(1)}}z_1\|^{1/n(1)} - \|\sum_{j=2}^{\infty}\mu_{n(1)}\lambda_{k(j)}^{-1}A_{n(1)k(j)}x_{k(j)}\|^{1/n(1)} \\ &> \frac{1}{2}\cdot\frac{1}{2}\cdot\varepsilon - \frac{\varepsilon}{8} = \frac{\varepsilon}{8} \end{aligned}$$

because by (3.4) and (3.5) we have

$$\begin{aligned} \|\mu_{n(1)}\lambda_{k(1)}^{-1}A_{n(1)k(1)}\frac{1}{2^{n(1)}}z_1\|^{1/n(1)} &\geq \frac{1}{2}\cdot\frac{1}{2}\|\mu_{n(1)}\lambda_{k(1)}^{-1}A_{n(1)k(1)}\|^{1/n(1)} \\ &\geq \frac{1}{2}\cdot\frac{1}{2}\varepsilon \end{aligned}$$

and by (3.2) and (3.6) we have

$$\begin{aligned} \|\sum_{j=2}^{\infty}\mu_{n(1)}\lambda_{k(j)}^{-1}A_{n(1)k(j)}x_{k(j)}\| &\leq \sum_{j=2}^{\infty}\|\mu_{n(1)}\lambda_{k(j)}^{-1}A_{n(1)k(j)}\frac{1}{2^{n(j)}}z_j\| \\ &\leq V_{n(1)}\sum_{j=2}^{\infty}\frac{1}{2^{n(j)}} \leq \frac{L}{2^{n(2)}} < \left[\frac{\varepsilon}{8}\right]^{n(1)} \end{aligned}$$

Further by (3.4), (3.9), (3.10) and (3.12) we get

$$\|\mu_{n(2)}Y_{n(2)}\|^{1/n(2)} \geq \|\mu_{n(2)}\lambda_{k(2)}^{-1}A_{n(2)k(2)}\frac{1}{2^{n(2)}}z_2\|^{1/n(2)} - \|\mu_{n(2)}\lambda_{k(1)}^{-1}A_{n(2)k(1)}x_{k(1)}\|^{1/n(2)}$$

$$-\|\sum_{j=3}^{\infty}\mu_{n(2)}\lambda_{k(j)}^{-1}A_{n(2)k(j)}x_{k(j)}\|^{1/n(2)}$$
$$>\frac{1}{2}\cdot\frac{1}{2}\cdot\varepsilon-\frac{\varepsilon}{16}-\frac{\varepsilon}{16}=\frac{\varepsilon}{8},$$

since (3.7) and (3.9) yield

$$\begin{aligned}\|\mu_{n(2)}\lambda_{k(2)}^{-1}A_{n(2)k(2)}\frac{1}{2^{n(2)}}z_2\|^{1/n(2)} &\geq \frac{1}{2}\cdot\frac{1}{2}\|\mu_{n(2)}\lambda_{k(2)}^{-1}A_{n(2)k(2)}\|^{1/n(2)}\\ &\geq \frac{1}{2}\cdot\frac{1}{2}\varepsilon,\end{aligned}$$

(3.8) yields

$$\begin{aligned}\|\mu_{n(2)}\lambda_{k(1)}^{-1}A_{n(2)k(1)}x_{k(1)}\| &= \|\mu_{n(2)}\lambda_{k(1)}^{-1}A_{n(2)k(1)}\frac{1}{2^{n(1)}}z_1\|\\ &< \frac{L}{2^{n(1)}}\left[\frac{\varepsilon}{16}\right]^{n(2)} < \left[\frac{\varepsilon}{16}\right]^{n(2)}\end{aligned}$$

and (3.2) and (3.10) yield,

$$\begin{aligned}\|\sum_{j=3}^{\infty}\mu_{n(2)}\lambda_{k(j)}^{-1}A_{n(2)k(j)}x_{k(j)}\| &\leq \sum_{j=3}^{\infty}\|\mu_{n(2)}\lambda_{k(j)}^{-1}A_{n(2)k(j)}\|\frac{1}{2^{n(j)}}\\ &\leq V_{n(2)}\sum_{j=3}^{\infty}\frac{1}{2^{n(j)}} \leq \frac{L}{2^{n(3)}} < \left[\frac{\varepsilon}{16}\right]^{n(2)}.\end{aligned}$$

Similarly by (3.2), (3.11), (3.12), (3.13) and (3.14) we get

$$\begin{aligned}\|\mu_{n(3)}y_{n(3)}\|^{1/n(3)} &\geq \|\mu_{n(3)}\lambda_{k(3)}^{-1}A_{n(3)k(3)}\frac{1}{2^{n(3)}}z_3\|^{1/n(3)} - \|\mu_{n(3)}\lambda_{k(2)}^{-1}A_{n(3)k(2)}x_{k(2)}\|^{1/n(3)}\\ &\quad -\|\mu_{n(3)}\lambda_{k(1)}^{-1}A_{n(3)k(1)}x_{k(1)}\|^{1/n(3)} - \|\sum_{j=4}^{\infty}\mu_{n(3)}\lambda_{k(j)}^{-1}A_{n(3)k(j)}x_{k(j)}\|^{1/n(3)}\\ &> \frac{1}{2}\cdot\frac{1}{2}\cdot\varepsilon-\frac{\varepsilon}{24}-\frac{\varepsilon}{24}-\frac{\varepsilon}{24}=\frac{\varepsilon}{8},\end{aligned}$$

since by (3.11) and (3.13)

$$\|\mu_{n(3)}\lambda_{k(3)}^{-1}A_{n(3)k(3)}\frac{1}{2^{n(3)}}z_3\|^{1/n(3)} \geq \frac{1}{2}\cdot\frac{1}{2}\|\mu_{n(3)}\lambda_{k(3)}^{-1}A_{n(3)k(3)}\|^{1/n(3)} \geq \frac{1}{2}\cdot\frac{1}{2}\varepsilon,$$

by (3.12)

$$\begin{aligned}\|\mu_{n(3)}\lambda_{k(2)}^{-1}A_{n(3)k(2)}x_{k(2)}\| &= \frac{1}{2^{n(2)}}\|\mu_{n(3)}\lambda_{k(2)}^{-1}A_{n(3)k(2)}z_2\|^{1/n(3)}\\ &\leq \frac{1}{2^{n(2)}}\left[\frac{\varepsilon}{24}\right]^{n(3)} < \left[\frac{\varepsilon}{24}\right]^{n(3)}\end{aligned}$$

and similarly

$$\|\mu_{n(3)}\lambda_{k(1)}^{-1}A_{n(3)k(1)}x_{k(1)}\| < \left[\frac{\varepsilon}{24}\right]^{n(3)},$$

and by (3.2) and (3.14)

$$\begin{aligned}\|\sum_{j=4}^{\infty}\mu_{n(3)}\lambda_{k(j)}^{-1}A_{n(3)k(j)}x_{k(j)}\| &\le \sum_{j=4}^{\infty}\|\mu_{n(3)}\lambda_{k(j)}^{-1}A_{n(3)k(j)}\|\frac{1}{2^{n(j)}}\\ &\le V_{n(3)}\sum_{j=4}^{\infty}\frac{1}{2^{n(j)}} \le \frac{L}{2^{n(4)}} < \left[\frac{\varepsilon}{24}\right]^{n(3)}\end{aligned}$$

Thus proceeding along the above lines we see that there exist sequences $(n(j))$, $(k(j))$ of integers and $\overline{x} = (x_k) \in l_1(X,\lambda)$ defined as above such that

$$\|\mu_{n(j)}y_{n(j)}\|^{1/n(j)} > \frac{\varepsilon}{8}; \quad j = 1,2,3\ldots$$

that is $\overline{y} = (y_n) \notin \Gamma(Y,\mu)$, which is a contradiction. This completes the proof.

4. Characterization of $B(c_0(X,\lambda), c_0(Y,\mu))$

In this section we obtain necessary and sufficient conditions for a matrix of linear operators to map $c_0(X,\lambda)$ into $c_0(Y,\mu)$ and as an application of this result we characterize $B(c_0(X,\lambda), c_0(Y,\mu))$, the Banach space of all bounded linear operators from the Banach space $c_0(X,\lambda)$ into the Banach space $c_0(Y,\mu)$ by obtaining matrix representation (A_{nk}) of each $A \in B(c_0(X,\lambda), c_0(Y,\mu))$.

From Theorem 3.3 [8] by taking $p_k = 1$ and $A_k \in B(X,Y)$ for all k, we easily get:

Lemma 4.1. $(A_k) \in c_0^{\beta}(X,\lambda)$ *if and only if* $\|(\lambda_k^{-1}A_k)\| < \infty$.

Definition 4.2. A normed space X-valued sequence space $(E(X),\mathcal{J})$ equipped with the linear topology $\mathcal{J}$ is said to be a GK-space if map $P_n : E(X) \to X$, $P_n(\overline{x}) = x_n$, is continuous for each n. A GK-space is said to be a GAK-space if for each $\overline{x} = (x_k)$ in $E(X)$, $s_n(\overline{x}) \to \overline{x}$ as $n \to \infty$ with respect to $\mathcal{J}$, where $s_n(\overline{x}) = (x_1, x_2, x, \ldots x_n, \theta, \theta, \ldots)$. Further $(E(X),\mathcal{J})$ is said to be a GC-space if $R_n : X \to E(X)$, $R_n(x) = \delta_n(x)$, is continuous for each n, where $\delta_n(x) = (\theta, \theta, \ldots, \theta, x, \theta \ldots)$, x at n^{th} place.

Above definitions are the generalizations of K-, AK- and C-spaces of scalar sequences (see [4]).

Following lemma can easily be proved:

Lemma 4.3. *The Banach space* $(c_0(X,\lambda), \|.\|_{\lambda})$ *is a* GK*-,* GAK*-and* GC*-space.*

Theorem 4.4. (a) $(A_{nk}) \in (c_0(X,\lambda), c_0(Y,\mu))$ *if and only if*

(i) $\sup_n \|(\mu_n\lambda_k^{-1}A_{nk})\| = H < \infty$; *and*

(ii) $\lim_n \mu_n A_{nk} = \theta$, *for each* $k \geq 1$.
(b) $A \in B(c_0(X,\lambda), c_0(Y,\mu))$ *if and only if* A *is a matrix transformation* (A_{nk}) *satisfying* (i) *and* (ii). *Moreover*

$$\|A\| = \sup_n \|(\mu_n \lambda_k^{-1} A_{nk})\|$$

Proof. (a) Let $(A_{nk}) \in (c_0(X,\lambda), c_0(Y,\mu))$. Necessity of (ii) follows easily by considering the sequences $(\theta, \theta, \theta, \ldots \theta, x, \theta, \ldots)$, $x \in X$ at k^{th} place, $k \geq 1$. We note that $c_0(X,\lambda)$ is a Banach space with $\|\overline{x}\|_\lambda = \sup_k \|\lambda_k x_k\|$, $\overline{x} = (x_k) \in c_0(X,\lambda)$. Thus for $\overline{x} \in c_0(X,\lambda)$ and each $n \geq 1$ if we define

$$T_n(\overline{x}) = \sum_{k=1}^{\infty} \mu_n A_{nk} x_k,$$

and

$$T_{n,p}(\overline{x}) = \sum_{k=1}^{p} \mu_n A_{nk} x_k, \quad p \geq 1,$$

then for each $n, T_{n,p}(\overline{x}) \to T_n(\overline{x})$ in Y as $p \to \infty$ for every $\overline{x} \in c_0(X,\lambda)$. Moreover for each n and p

$$\|T_{n,p}(\overline{x})\| \leq (\sum_{k=1}^{p} \|\mu_n \lambda_k^{-1} A_{nk}\|)\|\overline{x}\|_\lambda$$

which shows that $T_{n,p} : c_0(X,\lambda) \to Y$ is a bounded linear operator. Hence by Banach-Steinhaus theorem, $T_n : c_0(X,\lambda) \to Y$ is also a bounded linear operator for each $n \geq 1$.

Further the transform $\overline{y} = (y_n)$, where $y_n = \sum_{k=1}^{\infty} A_{nk} x_k$, of each $\overline{x} = (x_k) \in c_0(X,\lambda)$ is in $c_0(Y,\mu)$ therefore $(T_n(\overline{x}))_{n=1}^{\infty}$ is bounded for each $\overline{x} \in c_0(X,\lambda)$. Again on application of Banach-Steinhaus theorem we get $L > 0$ such that $\|T_n(\overline{x})\| \leq L\|\overline{x}\|_\lambda$, for all $n \geq 1$ and $\overline{x} \in c_0(X,\lambda)$. Now for any $n \geq 1$ if we consider $x_1, x_2, x_3 \ldots x_p \in S$, $p \geq 1$, arbitrary then $\overline{x} = (\lambda_1^{-1} x_1, \lambda_2^{-1} x_2, \lambda_3^{-1} x_3 \ldots \lambda_p^{-1} x_p, \theta, \theta \ldots) \in c_0(X,\lambda)$ with $\|\overline{x}\|_\lambda \leq 1$, and so we have

$$\|\sum_{k=1}^{p} \mu_n \lambda_k^{-1} A_{nk} x_k\| = \|T_n(\overline{x})\| \leq L\|\overline{x}\|_\lambda \leq L,$$

for all $x_1, x_2, \ldots x_p \in S$, $p \geq 1$. Thus taking supremum over these quantities we easily get

$$\|(\mu_n \lambda_k^{-1} A_{nk})\| \leq L, \quad \text{for every } n \geq 1,$$

and hence

$$\sup_n \|(\mu_n \lambda_k^{-1} A_{nk})\| \leq L.$$

This proves the necessity of (i).

Let (i) and (ii) hold. Then by Lemma 4.1 it follows that for every $\overline{x} \in c_0(X,\lambda)$ and for each $n \geq 1$, $\sum_{k=1}^{\infty} A_{nk} x_k$ converges in Y. Now for $\overline{x} \in c_0(X,\lambda)$ and $\varepsilon > 0$ there exists K such that $\|\lambda_k x_k\| < \varepsilon$, for every $k \geq K$ and in view of (ii) there exists N such that

$$\sum_{k=1}^{K-1} \|\mu_n A_{nk} x_k\| < \varepsilon, \quad \text{for all } n \geq N,$$

hence

$$\begin{aligned}\|\mu_n y_n\| &\leq \sum_{k=1}^{K-1} \|\mu_n A_{nk} x_k\| + \|\sum_{k=K}^{\infty} \mu_n \lambda_k^{-1} A_{nk}(\lambda_k x_k)\| \\ &\leq \sum_{k=1}^{K-1} \|\mu_n A_{nk} x_k\| + \|R_K(\mu_n \lambda_k^{-1} A_{nk}\| \sup_{k\geq K} \|\lambda_k x_k\| \\ &< (1+H)\varepsilon,\end{aligned}$$

for all $n \geq N$, as $\|R_K(\mu_n\lambda_k^{-1}A_{nk})\| \leq \|(\mu_n\lambda_k^{-1}A_{nk})\| \leq H$, i.e., $(y_n) \in c_0(Y,\mu)$, which proves the sufficiency of (i) and (ii).

(b) Let $A : c_0(X,\lambda) \to c_0(Y,\mu)$ be a bounded linear operator, and consider $A_{nk}x = P_n^Y oAoR_k^X(x)$. Clearly $P_n^Y : c_0(Y,\mu) \to Y$, defined by $P_n^Y(\overline{y}) = y_n$ is linear and bounded as $c_0(Y,\mu)$ is a GK-space. Again $R_k^X : X \to c_0(X,\lambda)$ defined by $R_k^X(x) = (\theta,\theta,\ldots,\theta,x,\theta,\ldots)$, x at k^{th} place. is linear and bounded since $c_0(X,\lambda)$ is a GC-space. Thus $A_{nk} : X \to Y$ is linear and bounded for each $n \geq 1$ and $k \geq 1$. Further we note that $(c_0(X,\lambda), \|\cdot\|_\lambda)$ is a GAK-space, so for each $\overline{x} \in c_0(X,\lambda)$, $s_p(\overline{x}) \to \overline{x}$ in $c_0(X,\lambda)$. Thus if $A(\overline{x}) = (y_n)$ then we have

$$\begin{aligned}y_n = P_n^Y(A(\overline{x})) &= (P_n^Y oA)(\lim_p(s_p(\overline{x})) \\ &= P_n^Y o(\lim_p A(\sum_{k=1}^{p} R_k^X(x_k))) \\ &= \lim_p \sum_{k=1}^{p} P_n^Y oAoR_k^X(x_k) = \sum_{k=1}^{\infty} A_{nk}x_k\end{aligned}$$

which shows that A determines the matrix (A_{nk}), where $A_{nk} \in B(X,Y)$, mapping $c_0(X,\lambda)$ into $c_0(Y,\mu)$ hence it will satisfy (i) and (ii).

Conversely $(A_{nk}) \in (c_0(X,\lambda), c_0(Y,\mu))$ satisfies (i) and (ii) and so clearly it determines a linear operator $A : c_0(X,\lambda) \to c_0(Y,\mu)$ such that

$$A(\overline{x}) = \left(\sum_{k=1}^{\infty} A_{nk}x_k\right)_{n=1}^{\infty}$$

Further

$$\begin{aligned}\|A(\overline{x})\|_\mu &= \sup_n \|\mu_n \sum_{k=1}^{\infty} A_{nk}x_k\| \\ &\leq \sup_n \|(\mu_n\lambda_k^{-1}A_{nk})\| \sup_k \|\lambda_k x_k\| \leq H\|\overline{x}\|_\lambda\end{aligned}$$

shows that A is bounded.

Now whether the bounded linear operator $A : c_0(X,\lambda) \to c_0(Y,\mu)$ determines a matrix $A = (A_{nk}) \in (c_0(X,\lambda), c_0(Y,\mu))$ or vice-verse, we have from what has been discussed above that

$$\|A(\overline{x})\|_\mu \le H\|x\|_\lambda$$

and so

$$\|A\| \le H. \tag{4.1}$$

On the other hand in view of (i) for $\varepsilon > 0$ there exists m such that

$$\|(\mu_m \lambda_k^{-1} A_{mk})\| > H - \frac{\varepsilon}{2}$$

and further there exist $p \ge 1$ and $x_1, x_2, \dots x_p \in S$ such that

$$\|\sum_{k=1}^{p} \mu_m \lambda_k^{-1} A_{mk} x_k\| > H - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = H - \varepsilon.$$

Now if we consider $\overline{x} = (\lambda_1^{-1}x_1, \lambda_2^{-1}x_2, \dots \lambda_p^{-1}x_p, \theta, \theta, \dots)$ in $c_0(X,\lambda)$ then $\|\overline{x}\|_\lambda \le 1$ and

$$\begin{aligned} \|A(\overline{x})\|_\mu &= \sup_n \|\mu_n \sum_{k=1}^{p} A_{nk}\lambda_k^{-1}x_k\| \\ &\ge \|\mu_m \sum_{k=1}^{p} A_{mk}\lambda_k^{-1}x_k\| > H - \varepsilon \end{aligned}$$

i.e. $\|A\| > H - \varepsilon$, for $\varepsilon > 0$ arbitrary, and so $\|A\| \ge H$. This together with (4.1) leads to

$$\|A\| = H = \sup_n \|(\mu_n \lambda_k^{-1} A_{nk})\|.$$

This completes the proof.

Finally we note that the Theorems 3.1, 3.2 and 3.3 of Das and Chaudhary [1] can easily be deduced from our Theorem 4.2.

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