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THE DUAL SPACES OF THE SETS OF Λ-STRONGLY CONVERGENT AND BOUNDED SEQUENCES

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Abstract. In this paper we shall give the α–, β–, γ– and f–duals of the sets $w_0^p(\Lambda)$, $w_\infty^p(\Lambda)$, $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$. Furthermore, we shall determine the continuous dual spaces of the sets $w_0^p(\Lambda)$, $c_0^p(\Lambda)$ and $c^p(\Lambda)$.

1. Introduction

We write ω for the set of all complex sequences $x = (x_k)_{k=0}^\infty$, ϕ, l_∞, c and c_0 for the sets of all finite, bounded, convergent sequences and sequences convergent to naught, respectively, further cs, bs and l_1 for the sets of all convergent, bounded and absolutely convergent series.

By e and $e^{(n)}$ $(n \in \mathbb{N}_0)$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. For any sequence $x = (x_k)_{k=0}^\infty$, let $x^{[n]} = \sum_{k=0}^n x_k e^{(k)}$ be its n–section.

Let $X, Y \subset \omega$ and $z \in \omega$. Then we write

$$z^{-1} * X = \{x \in \omega : xz = (x_k z_k)_{k=0}^\infty \in X\}$$

and

$$M(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$$

for the *multiplier space of* X *and* Y. The sets $M(X, l_1)$, $M(X, cs)$ and $M(X, bs)$ are called the α–, β– and γ–*duals of* X, respectively.

A Fréchet subspace X of ω is called an *FK space* if it has continuous coordinates, that is if convergence in X implies coordinatewise convergence. An FK space $X \supset \phi$ is said to have *AK* if, for every sequence $x = (x_k)_{k=0}^\infty \in X$, $x^{[n]} \to x$ $(n \to \infty)$; it is said to have *AD* if ϕ is dense in X. A *BK* space is an FK space which is a Banach space.

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If X is a p–normed space then we write X^* for the set of all continuous linear functionals on X, the so–called *continuous dual of* X, with its norm $\|\cdot\|$ given by

$$\|f\| = \sup\{|f(x)| : \|x\| = 1\} \text{ for all } f \in X^*.$$

Let $X \supset \phi$ be an FK space. Then the set $X^f = \{(f(e^{(n)}))_{n=0}^\infty : f \in X^*\}$ is called the *f–dual of* X.

The sets $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ of sequences that are Λ–strongly convergent to naught, Λ–strongly convergent and Λ–strongly bounded were introduced and studied by Móricz [12]. Their β– and continuous duals were determined in [10] and [11]. In this paper, we shall extend these results to $0 < p \le 1$ where p is an index. Furthermore, we shall give the α–, γ– and f–duals of the spaces $w_0^p(\Lambda)$, $w_\infty^p(\Lambda)$, $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$.

2. Some Notations and Preliminary Results

We shall frequently apply the following inequality (cf. [8, p. 22])

$$(a+b)^p \le a^p + b^p \ (0 < p \le 1) \text{ for all } a, b \ge 0. \tag{2.1}$$

Given any infinite matrix $A = (a_{nk})_{n,k=0}^\infty$ of complex numbers and any sequence $x \in \omega$, we shall write $A_n(x) = \sum_{k=0}^\infty a_{nk}x_k$ $(n = 0, 1, \ldots)$, $A(x) = (A_n(x))_{n=0}^\infty$, provided the series converge, and $X_A = \{x \in \omega : A(x) \in X\}$.

We define the matrix Δ by $\Delta_{nk} = 1$ for $k = n$, $\Delta_{nk} = -1$ for $k = n-1$ and $\Delta_{nk} = 0$ otherwise $(n = 0, 1, \ldots)$, and use the convention that any symbol with a negative subscript has the value 0.

Given any real $p > 0$ and any sequence x, we write $|x|^p = (|x_k|^p)_{k=0}^\infty$ and

$$M_n^p(x) = \frac{1}{\mu_n^p}\sum_{k=0}^n |(\Delta(\mu x))_k|^p \ \text{ for } n = 0, 1, \ldots.$$

Let $0 < p < \infty$ and $\mu = (\mu_n)_{n=0}^\infty$ be a nondecreasing sequence of positive reals tending to infinity throughout. We shall consider the sets

$$w_0^p(\mu) = \left\{x \in \omega : \lim_{n\to\infty}\left(\frac{1}{\mu_n^p}\sum_{k=0}^n |x_k|^p\right) = 0\right\}, \qquad c_0^p(\mu) = (\mu)^{-1} * (w_0^p(\mu)))_\Delta,$$

$$w_\infty^p(\mu) = \left\{x \in \omega : \sup_n\left(\frac{1}{\mu_n^p}\sum_{k=0}^n |x_k|^p\right) < \infty\right\}, \qquad c_\infty^p(\mu) = (\mu)^{-1} * (w_\infty^p(\mu))_\Delta,$$

$$c^p(\mu) = \{x \in \omega : x - le \in c_0^p(\mu) \text{ for some } l \in \mathbb{C}\}.$$

If $p = 1$ then we omit the index p, that is we write $w_0(\mu) = w_0^1(\mu)$ etc.

The sets $w_0^p(\mu)$ and $w_\infty^p(\mu)$ are special cases of mixed normed spaces studied for instance in [1,2,5,6,9]. If $\frac{1}{\mu_n^p} = \frac{1}{n+1}$ for $n = 0, 1, \ldots$, then the sets $w_0^p(\mu)$ and $w_\infty^p(\mu)$ reduce to the sets w_0^p and w_∞^p introduced and studied by Maddox [7], and the sets $c_0^p(\mu)$,

$c^p(\mu)$ and $c^p_\infty(\mu)$ reduce to the sets $[c_0]_p$, $[c]_p$ and $[c_\infty]_p$ introduced and studied by Hyslop, Kuttner and Thorpe [3, 4]. For $p = 1$ the sets $c^p_0(\mu)$, $c^p(\mu)$ and $c^p_\infty(\mu)$ reduce to the sets $c_0(\mu)$, $c(\mu)$ and $c_\infty(\mu)$ introduced and studied by Móricz [12] and Malkowsky [10].

Obviously the sets $w^p_0(\mu)$, $w^p_\infty(\mu)$, $c^p_0(\mu)$, $c^p(\mu)$ and $c^p_\infty(\mu)$ are linear spaces and $w^p_0(\mu) \subset w^p_\infty(\mu)$, $c^p_0(\mu) \subset c^p(\mu)$ and $c^p_0(\mu) \subset c^p_\infty(\mu)$. Furthermore, we have

Lemma 1. (a) *Let $0 < p < 1$. Then $c^p(\mu) \subset c^p_\infty(\mu)$ if and only if*

$$\sup_n \frac{1}{\mu_n^p} \sum_{k=0}^{n} |(\Delta\mu)_k|^p < \infty \text{ or equivalently } e \in c^p_\infty(\mu). \tag{2.2}$$

(b) *Let $1 \le p < \infty$. Then $e \in c^p_\infty(\mu)$ and $c^p(\mu) \subset c^p_\infty(\mu)$.*
(c) *Let $0 < p < \infty$. If $x \in c^p(\mu)$, then $l \in \mathbb{C}$ with $x - le \in c^p_0(\mu)$ is unique.*
(d) *Let $X^p(\mu)$ denote any of the spaces $w^p_0(\mu)$, $w^p_\infty(\mu)$, $c^p_0(\mu)$, $c^p(\mu)$ and $c^p_\infty(\mu)$. Then $X^p(\mu) \subset X^{\tilde{p}}(\mu)$ for $0 < p \le \tilde{p}$.*
(e) *If $0 < p \le 1$, then $c^p_\infty(\mu) \subset l_\infty$.*

Proof. (a) First we assume that condition (2.2) holds. Let $x \in c^p(\mu)$ be given. Then there is $l \in \mathbb{C}$ such that $x - le \in c^p_0(\mu)$, and so $x = x - le + le \in c^p_\infty(\mu)$, since $c^p_\infty(\mu)$ is a linear space.
Conversely, if condition (2.2) is not satisfied, then we can determine an increasing sequence $(n_m)_{m=0}^\infty$ of integers such that $M^p_{n_m}(e) > m$ $(m = 0, 1, \dots)$. Then $x = e \in c^p(\mu) \setminus c^p_\infty(\mu)$, since

$$M_n(x - e) = 0 \ (n = 0, 1, \dots) \quad \text{and} \quad M^p_{n_m}(x) = M^p_{n_m}(e) > m \ (m = 0, 1, \dots).$$

(b) Now let $p \ge 1$. Since $1/p \le 1$ and $\mu_n \ge \mu_{n-1}$ for all n, we have by (2.1)

$$(M^p_n(e))^{1/p} \le M^1_n(e) = \frac{1}{\mu_n} \sum_{k=0}^{n} (\mu_k - \mu_{k-1}) = 1 \text{ for all } n = 0, 1, \dots,$$

hence $e \in c^p_\infty(\mu)$. The inclusion $c^p(\mu) \subset c^p_\infty(\mu)$ now follows as in the first part of the proof of part (a).

(c) Let $x \in c^p(\mu)$ and $l, l' \in \mathbb{C}$ such that $x - le \in c^p_0(\mu)$ and $x - l'e \in c^p_0(\mu)$. Given $\varepsilon > 0$, there is $n = n(\varepsilon) \in \mathbb{N}_0$ such that $M^p_n(x - le), M^p_n(x - l'e) < \varepsilon$. Then, for $0 < p < 1$ by inequality (2.1)

$$|l - l'|^p \le M^p_n((x - le) - (x - l'e)) \le M^p_n(x - le) + M^p_n(x - l'e) < 2\varepsilon$$

and, for $p \ge 1$ by Minkowski's inequality

$$|l - l'| \le (M^p_n((x - le) - (x - l'e)))^{1/p} \le (M^p_n(x - le))^{1/p} + (M^p_n(x - l'e))^{1/p} < 2\varepsilon^{1/p}.$$

Since $\varepsilon > 0$ was arbitrary, we have $l = l'$ in both cases.

(d) Since $p/\tilde{p} \leq 1$, we have

$$\left(\frac{1}{\mu_n^{\tilde{p}}}\sum_{k=0}^{n}|x_k|^{\tilde{p}}\right)^{p/\tilde{p}} \leq \frac{1}{\mu_n^p}\sum_{k=0}^{n}|x_k|^p \ (n=0,1,\dots).$$

From this, we obtain the inclusions $X^p(\mu) \subset X^{\tilde{p}}(\mu)$ for $X^p(\mu) = w_0^p(\mu)$ and $X^p(\mu) = w_\infty^p(\mu)$.
Since $x \in c_0^p(\mu)$ or $x \in c_\infty^p(\mu)$ if and only if $\Delta(\mu x) \in w_0^p(\mu)$ or $\Delta(\mu x) \in w_\infty^p(\mu)$, respectively, it follows that the inclusions also hold for $X^p(\mu) = c_0^p(\mu)$ or $X^p(\mu) = c_\infty^p(\mu)$. Finally, the inclusion $c^p(\mu) \subset c^{\tilde{p}}(\mu)$ holds, since $x \in c^p(\mu)$ if and only if $x - le \in c_0^p(\mu)$ for some $l \in \mathbb{C}$.

(e) First

$$|x_n| = \left|\frac{1}{\mu_n}\sum_{k=0}^{\infty}(\Delta(\mu x))_k\right| \leq M_n^1(x) \ (n=0,1,\dots)$$

implies $c_\infty(\mu) \subset l_\infty$, and so $c_\infty^p(\mu) \subset l_\infty$ for $0 < p \leq 1$ by part (d).

Following the notations introduced in [10], we say that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^\infty$ of positive reals tending to infinity is *exponentially bounded* if there are reals s and t with $0 < s \leq t < 1$ such that for some subsequence $(\lambda_{n(\nu)})_{\nu=0}^\infty$ of Λ, we have

$$s \leq \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu+1)}} \leq t \text{ for all } \nu = 0,1,\dots; \tag{2.3}$$

such a subsequence $(\lambda_{n(\nu)})_{\nu=0}^\infty$ will be called an *associated subsequence*.

If $(n(\nu))_{\nu=0}^\infty$ is a strictly increasing sequence of nonnegative integers then we shall write $K^{<\nu>}$ for the set of all integers k with $n(\nu) \leq k \leq n(\nu+1)-1$, and $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all k in $K^{<\nu>}$.

If X is a p–normed sequence space and $a \in \omega$, then we write

$$\|a\|_X^* = \sup\left\{\left|\sum_{k=0}^{\infty}a_k x_k\right| : \|x\| = 1\right\}$$

provided the term on the right exists and is finite. This is the case whenever $X \supset \phi$ is a p–normed FK space and $a \in X^\beta$ by [13, Theorem 7.2.9, p. 107].

Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be a nondecreasing exponentially bounded sequence of positive reals and $(\lambda_{n(\nu)})_{\nu=0}^\infty$ an associated subsequence throughout.

If $X^p(\Lambda)$ denotes any of the sets $w_0^p(\Lambda)$, $w_\infty^p(\Lambda)$, $c_0^p(\Lambda)$, $c^p(\Lambda)$ or $c_\infty^p(\Lambda)$ then we shall write $\tilde{X}^p(\Lambda)$ for the respective space with the sections $1/\lambda_n^p\sum_{k=0}^n\cdots$ replaced by the blocks $1/\lambda_{n(\nu+1)}^p\sum_\nu\cdots$. Further, we define

$$\|x\|_{w_\infty^p(\Lambda)} = \begin{cases} \sup_n\left(\frac{1}{\lambda_n^p}\sum_{k=0}^{n}|x_k|^p\right) & (0 < p \leq 1) \\ \sup_n\left(\frac{1}{\lambda_n^p}\sum_{k=0}^{n}|x_k|^p\right)^{1/p} & (1 \leq p < \infty), \end{cases}$$

$$\|x\|_{\tilde{w}^p_\infty(\Lambda)} = \begin{cases} \sup_\nu \left(\frac{1}{\lambda^p_{n(\nu+1)}} \sum_\nu |x_k|^p \right) & (0 < p \le 1) \\ \sup_\nu \left(\frac{1}{\lambda^p_{n(\nu+1)}} \sum_\nu |x_k|^p \right)^{1/p} & (1 \le p < \infty), \end{cases}$$
$$\|x\|_{c^p_\infty(\Lambda)} = \|\Delta(\Lambda x)\|_{w^p_\infty(\Lambda)} \quad \text{and} \quad \|x\|_{\tilde{c}^p_\infty(\Lambda)} = \|\Delta(\Lambda x)\|_{\tilde{w}^p_\infty(\Lambda)}.$$

Theorem 1. (a) *The sets $w^p_0(\mu)$ and $w^p_\infty(\mu)$ with $\|\cdot\|_{w^p_\infty(\mu)}$, and $c^p_0(\mu)$ and $c^p_\infty(\mu)$ with $\|\cdot\|_{c^p_\infty(\mu)}$ are p–normed FK spaces for $0 < p < 1$ and BK spaces for $1 \le p < \infty$, $w^p_0(\mu)$ is a closed subspace of $w^p_\infty(\mu)$, $c^p_0(\mu)$ is a closed subspace of $c^p_\infty(\mu)$, $w^p_0(\mu)$ has AK for all p and $c^p_0(\mu)$ has AK for $0 < p \le 1$.* (b) *We assume that condition* (2.2) *holds for $0 < p < 1$. Then $c^p(\mu)$ with $\|\cdot\|_{c^p_\infty(\mu)}$ is a p–normed FK space for $0 < p < 1$ and a BK space for $1 \le p < \infty$, $c^p(\mu)$ is a closed subspace of $c^p_\infty(\mu)$, and if $0 < p \le 1$, then every sequence $x = (x_k)^\infty_{k=0} \in c^p(\mu)$ has a unique representation*

$$x = le + \sum_{k=0}^\infty (x_k - l)e^{(k)} \text{ where } l \in \mathbb{C} \text{ is such that } x - le \in c^p_0(\Lambda). \tag{2.4}$$

(c) *If $X^p(\Lambda)$ and $\tilde{X}^p(\Lambda)$ denote any of the sets $w^p_0(\Lambda)$, $w^p_\infty(\Lambda)$, $c^p_0(\Lambda)$, $c^p(\Lambda)$ and $c^p_\infty(\Lambda)$, $\tilde{w}^p_0(\Lambda)$, $\tilde{w}^p_\infty(\Lambda)$, $\tilde{c}^p_0(\Lambda)$, $\tilde{c}^p(\Lambda)$ and $\tilde{c}^p_\infty(\Lambda)$, respectively, then $X^p(\Lambda) = \tilde{X}^p(\Lambda)$, $\|\cdot\|_{w^p_\infty(\Lambda)}$ and $\|\cdot\|_{\tilde{w}^p_\infty(\Lambda)}$ are equivalent on $w^p_0(\Lambda)$ and on $w^p_\infty(\Lambda)$, $\|\cdot\|_{c^p_\infty(\Lambda)}$ and $\|\cdot\|_{\tilde{c}^p_\infty(\Lambda)}$ are equivalent on $c^p_0(\Lambda)$, $c^p_\infty(\Lambda)$ and $c^p(\Lambda)$, in the case of $c^p(\Lambda)$ whenever condition* (2.2) *holds for $0 < p < 1$.*

Proof. (a) The assertions concerning the sets $w^p_0(\mu)$ and $w^p_\infty(\mu)$ were proved in [9]. From this, all the assertions concerning $c^p_0(\mu)$ and $c^p_\infty(\mu)$ follow from [13, Theorems 4.3.13 and 4.3.14, pp 63 and 46], except for the one that $c^p_0(\mu)$ has AK for $0 < p \le 1$.
To show that $c^p_0(\mu)$ has AK for $0 < p \le 1$, let $x \in c^p_0(\mu)$ and $\varepsilon > 0$ be given. Then there is an integer $m_0 \in \mathbb{N}_0$ such that $M^p_n(x) < \varepsilon/2$ for all $n \ge m_0$. Let $m \ge m_0$. Then, since $0 < p \le 1$, we conclude

$$\|x^{[m]} - x\|_{c^p_\infty(\mu)} = M^p_n(x^{[m]} - x) = \sup_{n \ge m+1} \frac{1}{\mu^p_n} \left(|\mu_{m+1}|^p |x_{m+1}|^p + \sum_{k=m+2}^n |(\Delta(\mu x))_k|^p \right)$$
$$< M^p_{m+1}(x) + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

(b) First we show that $c^p(\mu)$ is complete with $\|\cdot\|_{c^p_\infty(\mu)}$.
By Lemma 1 (a) and (b), $\|\cdot\|_{c^p_\infty(\mu)}$ is defined on $c^p(\mu)$.
Let $(x^{(m)})^\infty_{m=0}$ be a Cauchy sequence in $c^p(\mu)$. For each $m \in \mathbb{N}_0$, let $l^{(m)} \in \mathbb{C}$ denote the number for which $x^{(m)} - l^{(m)}e \in c^p_0(\mu)$. First we observe that $(x^{(m)})^\infty_{m=0}$ is a Cauchy sequence in $c^p_\infty(\mu)$, and so convergent by the completeness of $c^p_\infty(\mu)$,

$$\|x^{[m]} - x\|_{c^p_\infty(\mu)} \to 0 \quad (m \to \infty), \text{ say.} \tag{2.5}$$

We have to show $x \in c^p(\mu)$.

First we show that the sequence $(l^{(m)})_{m=0}^{\infty}$ converges.

Let $\varepsilon > 0$ be given. Since $(x^{(m)})_{m=0}^{\infty}$ is a Cauchy sequence, we may choose $M = M(\varepsilon) \in \mathbb{N}_0$ such that $\|x^{(m)} - x^{(j)}\|_{c_\infty^p(\mu)} < \varepsilon/3$ for all $m, j \geq M$. Let $m, j \geq M$. Since $x^{(m)} - l^{(m)}e, x^{(j)} - l^{(j)}e \in c_0^p(\mu)$, there is $n = n(m, j, \varepsilon) \in \mathbb{N}_0$ such that $M_n^p(x^{(m)} - l^{(m)}e), M_n^p(x^{(j)} - l^{(j)}e) < \varepsilon/3$. Then, for $0 < p < 1$ by inequality (2.1)

$$\begin{aligned}|l^{(m)} - l^{(j)}|^p &\leq M_n^p((l^{(m)} - l^{(j)})e) \leq M_n^p(x^{(m)} - l^{(m)}e) + \|x^{(m)} - x^{(j)}\|_{c_\infty^p(\mu)} M_n^p(x^{(j)} - l^{(j)}e) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,\end{aligned}$$

and, for $1 \leq p < \infty$ by Minkowski's inequality

$$\begin{aligned}|l^{(m)} - l^{(j)}| &\leq \Big(M_n^p((l^{(m)} - l^{(j)})e)\Big)^{1/p} \\ &\leq \Big(M_n^p(x^{(m)} - l^{(m)}e)\Big)^{1/p} + \|x^{(m)} - x^{(j)}\|_{c_\infty^p(\mu)} + \Big(M_n^p(x^{(j)} - l^{(j)}e)\Big)^{1/p} \\ &< 2(\varepsilon/3)^{1/p} + \varepsilon/3.\end{aligned}$$

Thus $(l^{(m)})_{m=0}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$, hence convergent,

$$l = \lim_{m\to\infty} l^{(m)}, \text{ say.} \tag{2.6}$$

Now we show $x - le \in c_0^p(\mu)$.
Let $\varepsilon > 0$ be given. By (2.5) and (2.6), there is $M \in \mathbb{N}_0$ such that $\|x^{(M)} - x\|_{c_\infty^p(\mu)} < \varepsilon/3$, and, with $C = \sup_n M_n^p(e) < \infty$ (for $0 < p < 1$ by condition (2.2)),

$$|l - l^{(M)}| < \left(\frac{\varepsilon}{3(C+1)}\right)^{1/p}.$$

Further, since $x^{(M)} - l^{(M)}e \in c_0^p(\mu)$, there is $N \in \mathbb{N}_0$ such that $M_n^p(x^{(M)} - l^{(M)}e) < \varepsilon/3$. Let $n \geq N$. Then, for $0 < p < 1$ by inequality (2.1)

$$\begin{aligned}M_n^p(x - le) &\leq M_n^p(x^{(M)} - l^{(M)}e) + \|x^{(m)} - x\|_{c_\infty^p(\mu)} + M_n^p((l - l^{(M)})e) \\ &< 2\varepsilon/3 + |l - l^{(M)}|^p M_n^p(e) < \frac{2\varepsilon}{3} + \frac{\varepsilon C}{3(C+1)} \leq \varepsilon,\end{aligned}$$

and, for $1 \leq p < \infty$ by Minkowski's inequality

$$\begin{aligned}(M_n^p(x - le))^{1/p} &< (\varepsilon/3)^{1/p} + \varepsilon/3 + \Big(M_n^p((l - l^{(M)})e)\Big)^{1/p} \\ &< (\varepsilon/3)^{1/p} + \varepsilon/3 + |l - l^{(M)}|\,(M_n^p(e))^{1/p} < 2(\varepsilon/3)^{1/p} + \varepsilon/3.\end{aligned}$$

This shows that $c^p(\mu)$ is complete. Consequently $c^p(\mu)$ is a p–normed FK space for $0 < p < 1$ and a BK space for $1 \leq p < \infty$ by [13, Corollary 4.2.2, p. 56].
Finally, let $0 < p \leq 1$ and $x = (x_k)_{k=0}^{\infty} \in c^p(\mu)$. Then, by Lemma 1 (c) there is a uniquely determined $l \in \mathbb{C}$ such that $x - le \in c^p(\mu)$. We put $y = x - le$. Since $c_0^p(\mu)$ has AK, $y = \sum_{k=0}^{\infty} y_k e^{(k)} = \sum_{k=0}^{\infty}(x_k - l)e^{(k)}$, and so the representation in (2.4) follows.

(c) Let $0 < p < \infty$.
From

$$\frac{1}{\lambda^p_{n(\nu+1)}}\sum\nolimits_\nu |x_k|^p \le \frac{1}{\lambda^p_{n(\nu+1)}}\sum_{k=0}^{n(\nu+1)} |x_k|^p \ (\nu = 0, 1, \ldots),$$

we conclude $X^p(\Lambda) \subset \tilde{X}^p(\Lambda)$.
Conversely, let $x \in \tilde{w}^p_0(\Lambda)$ and $\varepsilon > 0$ be given. Then there is an integer $\nu_0 \in \mathbb{N}_0$ such that

$$\frac{1}{\lambda^p_{n(\nu+1)}}\sum\nolimits_\nu |x_k|^p < \varepsilon \text{ for all } \nu \ge \nu_0.$$

Since $\lambda_{n(\nu)} \to \infty$ $(\nu \to \infty)$, we can choose an integer $\nu_1 > \nu_0$ such that

$$\frac{1}{\lambda^p_{n(\nu)}}\sum_{k=0}^{n(\nu_0)-1} |x_k|^p \text{ for all } \nu \ge \nu_1.$$

Let $m \ge n(\nu_1)$. Then there is an integer $\nu(m) \ge \nu_1$ such that $m \in K^{<\nu(m)>}$ and, using (2.3), we obtain

$$\begin{aligned}
\frac{1}{\lambda^p_m}\sum_{k=0}^{m} |x_k|^p &\le \frac{1}{\lambda^p_{n(\nu(m))}}\left(\sum_{k=0}^{n(\nu_0)-1} |x_k|^p + \sum_{\nu=\nu_0}^{\nu(m)} \sum\nolimits_\nu |x_k|^p\right) \\
&< \varepsilon + \left(\frac{\lambda_{n(\nu(m)+1)}}{\lambda_{n(\nu(m))}}\right)^p \frac{1}{\lambda^p_{n(\nu(m)+1)}} \sum_{\nu=\nu_0}^{\nu(m)} \lambda^p_{n(\nu+1)} \frac{1}{\lambda^p_{n(\nu+1)}} \sum\nolimits_\nu |x_k|^p \\
&\le \varepsilon + \frac{\varepsilon}{s^p}\sum_{\nu=\nu_0}^{\nu(m)} \left(t^{\nu(m)-\nu}\right)^p < \varepsilon\left(1 + \frac{1}{s^p}\frac{1}{1-t^p}\right).
\end{aligned}$$

This shows $\tilde{w}^p_0(\Lambda) \subset w^p_0(\Lambda)$. The inclusion $\tilde{w}^p_\infty(\Lambda) \subset w^p_\infty(\Lambda)$ is shown in exactly the same way. Now the identities $c^p_0(\Lambda) = \tilde{c}^p_0(\Lambda)$, $c^p(\Lambda) = \tilde{c}^p(\Lambda)$ and $c^p_\infty(\Lambda) = \tilde{c}^p_\infty(\Lambda)$ are obvious.

3. The Duals of The Sets $w^p_0(\Lambda)$ and $w^p_\infty(\Lambda)$

In this section, we shall give the duals of the sets $w^p_0(\Lambda)$ and $w^p_\infty(\Lambda)$ for $0 < p < \infty$.

Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be a nondecreasing exponentially bounded sequence of positive reals throughout and $(\lambda_{n(\nu)})_{\nu=0}^\infty$ an associated subsequence. We put

$$\mathcal{W}^p(\Lambda) = \begin{cases} \left\{a \in \omega : \sum\limits_{\nu=0}^\infty \lambda_{n(\nu+1)} \max_\nu |a_k| < \infty\right\} & (0 < p \le 1) \\ \left\{a \in \omega : \sum\limits_{\nu=0}^\infty \lambda_{n(\nu+1)} \left(\sum_\nu |a_k|^p\right)^{1/p} < \infty\right\} & (1 < p < \infty, q = \frac{p}{p-1}) \end{cases}$$

and on $\mathcal{W}^p(\Lambda)$

$$\|a\|_{\mathcal{W}^p(\Lambda)} = \begin{cases} \sum\limits_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max_\nu |a_k| & (0 < p \le 1) \\ \sum\limits_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \left(\sum_\nu |a_k|^p\right)^{1/p} & (1 < p < \infty, q = \frac{p}{p-1}). \end{cases}$$

Theorem 2. *Let $X^p(\Lambda) = w_0^p(\Lambda)$ or $X^p(\Lambda) = w_\infty^p(\Lambda)$ and † stand for α, β, γ or f. Then $(X^p(\Lambda))^\dagger = \mathcal{W}^p(\Lambda)$. The continuous dual $(w_0^p(\Lambda))^*$ of $w_0^p(\Lambda)$ is norm isomorphic to $\mathcal{W}^p(\Lambda)$ when $w_0^p(\Lambda)$ has the norm $\|\cdot\|_{\tilde{w}_\infty^p(\Lambda)}$. This means, $g \in (w_0^p(\Lambda))^*$ if and only if there is a sequence $b = (b_n)_{n=0}^\infty \in \mathcal{W}^p(\Lambda)$ such that*

$$g(y) = \sum_{n=0}^{\infty} b_n y_n \text{ for all } y \in w_0^p(\Lambda) \quad \text{and} \quad \|g\| = \|b\|_{\mathcal{W}^p(\Lambda)}.$$

Furthermore, $\|a\|^*_{\tilde{w}_\infty^p(\Lambda)} = \|a\|_{\mathcal{W}^p(\Lambda)}$ on $(w_\infty^p(\Lambda))^\beta$.

Proof. The statements of the theorem with the exception of those concerning the $\gamma-$ and $f-$duals are well known [9, Theorems 2,4,5 and 6].
(a) Since $w_0^p(\Lambda)$ has AK, we have $(w_0^p(\Lambda))^\beta = (w_0^p(\Lambda))^f$ by [13, Theorem 7.2.7 (ii), p. 106], and so $(w_0^p(\Lambda))^f = \mathcal{W}^p(\Lambda)$. Further, since an AK space obviously has AD, we also have $(w_0^p(\Lambda))^\beta = (w_0^p(\Lambda))^\gamma$ by [13, Theorem 7.2.7 (iii), p. 106], and so $(w_0^p(\Lambda))^\gamma = \mathcal{W}^p(\Lambda)$. Since $w_0^p(\Lambda)$ is a closed subspace of $w_\infty^p(\Lambda)$, it follows that $(w_\infty^p(\Lambda))^f = (w_0^p(\Lambda))^f$ by [13, Theorem 7.2.6, p.106], and so

$$(w_\infty^p(\Lambda))^f = \mathcal{W}^p(\Lambda). \tag{3.1}$$

Finally, by [13, Theorem 7.2.7 (i), p. 106], $(w_\infty^p(\Lambda))^\beta \subset (w_\infty^p(\Lambda))^\gamma \subset (w_\infty^p(\Lambda))^f$, and so by (3.1)

$$\mathcal{W}^p(\Lambda) \subset (w_\infty^p(\Lambda))^\beta \subset (w_\infty^p(\Lambda))^\gamma \subset (w_\infty^p(\Lambda))^f = \mathcal{W}^p(\Lambda),$$

hence $(w_\infty^p(\Lambda))^\gamma = \mathcal{W}^p(\Lambda)$.

4. The Duals of The Sets $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$

In this section, we shall determine the $\alpha-$, $\beta-$, $\gamma-$ and $f-$duals of the sets $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$ and the continuous duals of $c_0^p(\Lambda)$ and $c^p(\Lambda)$ for $0 < p \le 1$.

Let $\Lambda = (\lambda_n)_{n=0}^\infty$ be a nondecreasing exponentially bounded sequence of positive reals throughout.

We need the following lemma for the determination of the $\alpha-$ duals of $c_\infty^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$.

Lemma 2. *Let $X \subset l_\infty$ be a BK space such that $\sup_n \|e^{[n]}\|_{c_\infty^p(\Lambda)} < \infty$. Then $X^\alpha = l_1$.*

Proof. First we observe that $X \subset l_\infty$ implies $l_\infty^\alpha = l_1 \subset X^\alpha$.
Conversely, let $a \in X^\alpha$. For each $m \in \mathbb{N}_0$, we define the map $f_a^{(m)} : X \to \mathbb{R}$ by $f_a^{(m)}(x) = \sum_{k=0}^m |a_k x_k|$ $(x \in X)$. Then $(f_a^{(m)})_{m=0}^\infty$ is a sequence of seminorms on X which are continuous, since X is a BK space. Further $f_a^{(m)}(x) \le \sum_{k=0}^\infty |a_k x_k| = M(x) < \infty$ for all $m \in \mathbb{N}_0$ and all $x \in X$. By the uniform boundedness principle, there is a constant M_1 such that $\|f^{(m)}\| \le M_1$ for all $m \in \mathbb{N}_0$. From this and $\sup_n \|e^{[n]}\|_{c_\infty^p(\Lambda)} < \infty$, we conclude $a \in l_1$.

Theorem 3. *Let $X^p(\Lambda)$ denote any of the sets $c_0^p(\Lambda)$, $c^p(\Lambda)$ or $c_\infty^p(\Lambda)$, then $(X^p(\Lambda))^\alpha = l_1$ for $0 < p \le 1$.*

Proof. Since obviously $\sup_n \|e^{[n]}\|_{c_\infty^p(\Lambda)} \le 2$, and $c_0(\Lambda) \subset c(\Lambda) \subset c_\infty(\Lambda) \subset l_\infty$ by Lemma 1 (b) and (e), we conclude from Lemma 2

$$c_0^\alpha(\Lambda) = c^\alpha(\Lambda) = c_\infty^\alpha(\Lambda) = l_1. \tag{4.1}$$

Now we assume $a \in (c_0^p(\Lambda))^\alpha$. For each $m \in \mathbb{N}_0$, we define the map $f^{(m)} : c_0^p(\Lambda) \to \mathbb{R}$ as in the proof of Lemma 2, and again there is a constant $M > 0$ such that

$$\sum_{k=0}^\infty |a_k x_k| \le M \text{ for all } x \in c_0^p(\Lambda) \text{ with } \|x\|_{\tilde{c}_\infty^p(\Lambda)} = 1. \tag{4.2}$$

Since $1/\Lambda = (1/\lambda_n)_{n=0}^\infty \in c_0^p(\Lambda)$, we have

$$R_n = R_n(|a|/\lambda) = \sum_{k=n}^\infty \frac{|a_k|}{\lambda_k} < \infty \text{ for all } n = 0, 1, \ldots.$$

Let $\nu(m) \in \mathbb{N}_0$ be given. We define the sequence $x^{(\nu(m))}$ by

$$x_n^{(\nu(m))} = \begin{cases} \frac{1}{\lambda_n} \sum\limits_{\mu=0}^{\nu} \lambda_{n(\mu+1)} & (n \in N^{<\nu>}; \nu = 0, 1, \ldots, \nu(m)) \\ \frac{1}{\lambda_n} \sum\limits_{\mu=0}^{\nu(m)} \lambda_{n(\mu+1)} & (n \ge n(\nu(m)+1)). \end{cases}$$

Then

$$\left(\Delta(\Lambda x^{(\nu(m))})\right)_n = \begin{cases} 0 & (n \ge n(\nu(m)+1) \text{ or } n \ne n(\nu); \nu = 0, 1, \ldots, \nu(m)) \\ \lambda_{n(\nu+1)} & (n = n(\nu); \nu = 0, 1, \ldots, \nu(m)). \end{cases} \tag{4.3}$$

and

$$\textstyle\sum_\nu \left|\left(\Delta(\Lambda x^{(\nu(m))})\right)_n\right| = \begin{cases} \lambda_{n(\nu+1)} & (0 \le \nu \le \nu(m)) \\ 0 & (\nu \ge \nu(m)). \end{cases} \tag{4.4}$$

Therefore $x^{(\nu(m))} \in c_0^p(\Lambda)$ and $\|x^{(\nu(m))}\|_{\tilde{c}^p_\infty(\Lambda)} = 1$. Further, by (4.3), (4.4) and (4.2), and since $x_k^{(\nu(m))} \geq 0$ for all k,

$$\sum_{\nu=0}^{\nu(m)} \lambda_{n(\nu+1)} R_{n(\nu)} = \sum_{n=0}^{\infty} \left(\Delta(\Lambda x^{(\nu(m))})\right)_n \sum_{k=n}^{\infty} \frac{|a_k|}{\lambda_k}$$
$$= \sum_{k=0}^{\infty} \frac{|a_k|}{\lambda_k} \sum_{n=0}^{k} \left(\Delta(\Lambda x^{(\nu(m))})\right)_n = \sum_{k=0}^{\infty} |a_k||x_k^{(\nu(m))}| \leq M.$$

Since $\nu(m) \in \mathbb{N}_0$ was arbitrary, we conclude $\sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} R_{n(\nu)} < \infty$. Now

$$\sum_{k=0}^{\infty} |a_k| = \sum_{\nu=0}^{\infty} \sum\nolimits_{\nu} |a_k| \leq \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \sum\nolimits_{\nu} \frac{|a_k|}{\lambda_k} \leq \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} R_{n(\nu)} < \infty.$$

Thus $\left(c_0^p(\Lambda)\right)^\alpha \subset l_1$, and consequently, by (4.1)

$$\left(c_\infty^p(\Lambda)\right)^\alpha \subset \left(c_0^p(\Lambda)\right)^\alpha \subset l_1 = c_\infty^\alpha(\Lambda) \subset \left(c_\infty^p(\Lambda)\right)^\alpha \subset \left(c_0^p(\Lambda)\right)^\alpha,$$

hence $\left(c_\infty^p(\Lambda)\right)^\alpha = \left(c_0^p(\Lambda)\right)^\alpha = l_1$ for $0 < p \leq 1$.
Finally $c_0^p(\Lambda) \subset c^p(\Lambda)$ implies $\left(c^p(\Lambda)\right)^\alpha \subset l_1$, and $c(\Lambda) \subset c_\infty(\Lambda)$ implies $l_1 = c_\infty^\alpha(\Lambda) \subset c^\alpha(\Lambda) \subset \left(c^p(\Lambda)\right)^\alpha$, so $\left(c^p(\Lambda)\right)^\alpha = l_1$.

Now we give the β–, γ– and f–duals of the sets $c_0^p(\Lambda)$, $c^p(\Lambda)$ and $c_\infty^p(\Lambda)$ for $0 < p \leq 1$, and the continuous duals of $c_0^p(\Lambda)$ and $c^p(\Lambda)$ in some cases.

If $a \in cs$ then we shall write $R(a)$ for the sequence with $R_n(a) = \sum_{k=n}^{\infty} a_k$ $(n = 0, 1, \ldots)$. We shall frequently apply Abel's summation by parts

$$\sum_{n=0}^{m-1} a_n y_n = \sum_{n=0}^{m} R_n(a)(\Delta y)_n - R_m(a) y_m \text{ for all } m = 0, 1, \ldots. \tag{4.5}$$

If u is a sequence with $u_k \neq 0$ for all $k = 0, 1, \ldots$ then we shall write $1/u$ for the sequence with $(1/u)_k = 1/u_k$ for all k.

Theorem 4. *Let $0 < p \leq 1$. We put*

$$C_\beta(\Lambda) = \left\{ a \in \omega : \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max\nolimits_\nu \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right| < \infty \right\}$$

and

$$\|a\|_{C_\beta(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{n(\nu+1)} \max\nolimits_\nu \left| \sum_{k=n}^{\infty} \frac{a_k}{\lambda_k} \right|.$$

(a) *If $X^p(\Lambda)$ is any of the sets $c_0^p(\Lambda)$ or $c_\infty^p(\Lambda)$ and † stands for any of the symbols β, γ or f, then*

$$X^p(\Lambda)^\dagger = C_\beta(\Lambda).$$

This also holds when $X^p(\Lambda) = c(\Lambda)$ or $X^p(\Lambda) = c^p(\Lambda)$ for $0 < p < 1$ whenever condition (2.2) *is satisfied. Otherwise*

$$(c^p(\Lambda))^\beta = C_\beta(\Lambda) \cap cs \text{ and } (c^p(\Lambda))^\gamma = C_\beta(\Lambda) \cap bs.$$

(b) *The continuous dual $(c_0^p(\Lambda))^*$ of $c_0^p(\Lambda)$ is norm isomorphic to $C_\beta(\Lambda)$ when $c_0^p(\Lambda)$ has the p–norm $\|\cdot\|_{\tilde{c}_\infty^p(\Lambda)}$. Further*

$$\|a\|^*_{\tilde{c}_\infty^p(\Lambda)} = \|a\|_{C_\beta(\Lambda)} \text{ on } c_\infty^p(\Lambda). \tag{4.6}$$

(c) *We have $f \in c^*(\Lambda)$ if and only if*

$$\begin{aligned} &f(x) = l\chi_f + \sum_{n=0}^\infty a_n x_n \text{ for all } x \in c(\Lambda)\\ &\text{where } a \in C_\beta(\Lambda),\ l \in \mathbb{C} \text{ with } x - le \in c_0(\Lambda) \text{ and}\\ &\chi_f = f(e) - \sum_{n=0}^\infty a_n. \end{aligned} \tag{4.7}$$

Further, $\|f\|$ is equivalent to

$$|\chi_f| + \|a\|_{C_\beta(\Lambda)}. \tag{4.8}$$

If condition (2.2) *is satisfied, then this also holds for $c^p(\Lambda)$ $(0 < p < 1)$.*

Proof. In the case $p = 1$, the statements of the theorem concerning the β– and continuous duals can be found in [10, 11].

(a) Let $0 < p < 1$. First $c_\infty^p(\Lambda) \subset c_\infty(\Lambda)$ implies

$$(c_\infty(\Lambda))^\beta = C_\beta(\Lambda) \subset (c_\infty^p(\Lambda))^\beta.$$

Conversely, let $a \in (c_0^p(\Lambda))^\beta$. Since $c_0^p(\Lambda)$ is a p–normed FK space, the map $f_a : c_0^p(\Lambda) \to \mathbb{C}$ defined by $f_a(x) = \sum_{k=0}^\infty a_k x_k$ $(x \in c_0^p(\Lambda))$ is an element of $(c_0^p(\Lambda))^*$. We define the matrix $\Delta(\Lambda)$ by

$$\Delta_{nk}(\Lambda) = \begin{cases} -\lambda_{n-1} & (k = n-1)\\ \lambda_n & (k = n)\\ 0 & (otherwise) \end{cases} \qquad (n = 0, 1, \ldots).$$

By [13, Theorem 4.4.2, p. 66], there is $g \in (c_0^p(\Lambda))^*$ with

$$f = g \circ \Lambda(\Delta) \tag{4.9}$$

Since $w_0^p(\Lambda)$ is an FK space with AK, we have

$$b = (g(e^{(n)}))_{n=0}^\infty \in (w_0^p(\Lambda))^\beta \tag{4.10}$$

by [13, Theorem 7.2.9, p. 107].

Let $m \in \mathbb{N}_0$ be given. Then, for the sequence $x^{(m)}$ defined by

$$x_n^{(m)} = \begin{cases} 0 & (n < m) \\ \frac{1}{\lambda_n} & (n \geq m), \end{cases}$$

we have $x^{(m)} \in c_0^p(\Lambda)$ and

$$(\Delta_n(\Lambda))\,(x^{(m)}) = \begin{cases} 1 & (n = m) \\ 0 & (n \neq m) \end{cases} = e^{(m)} \in w_0^p(\Lambda),$$

and so $x^{(m)} \in c_0^p(\Lambda)$. From (4.9) and (4.10), we obtain

$$b_m = g(e^{(m)}) = g\left((\Delta(\Lambda))(x^{(m)})\right) = f(x^{(m)}) = \sum_{n=0}^{\infty} a_n x_n^{(m)} = \sum_{n=m}^{\infty} \frac{a_n}{\lambda_n} \ (m = 0, 1, \ldots),$$

hence $a \in C_\beta(\Lambda)$, since $(w_0^p(\Lambda))^\beta = \mathcal{W}^p(\Lambda)$ by Theorem 2 (a). Therefore $(c_0^p(\Lambda))^\beta \subset C_\beta(\Lambda)$. Thus we have shown $(c_0^p(\Lambda))^\beta = (c_\infty^p(\Lambda))^\beta = C_\beta(\Lambda)$ for $0 < p < 1$.
If condition (2.2) holds, then $c_0^p(\Lambda) \subset c^p(\Lambda) \subset c_\infty^p(\Lambda)$, and so $(c^p(\Lambda))^\beta = C_\beta(\Lambda)$.
The assertions concerning the $\gamma-$ and $f-$duals are proved in the same way as in Theorem 2 (a).
Now we consider the case where condition (2.2) does not hold. We assume $a \in C_\beta(\Lambda) \cap cs$. Let $x \in c^p(\Lambda)$ be given. Then there is $l \in \mathbb{C}$ such that $x - le \in \mathbb{C}$, and so $ax = a(x - le) + lae \in cs$, hence $a \in (c^p(\Lambda))^\beta$. Conversely, let $a \in (c^p(\Lambda))^\beta$. Then $a \in (c_0^p(\Lambda))^\beta = C_\beta(\Lambda)$, since $c_0^p(\Lambda) \subset c^p(\Lambda)$ implies $(c^p(\Lambda))^\beta \subset (c_0^p(\Lambda))^\beta$. Since $e \in c^p(\Lambda)$, we also have $a = ae \in cs$.

The identity $(c^p(\Lambda))^\gamma = C_\beta(\Lambda) \cap bs$ is proved in exactly the same way.

(b) Since $c_0^p(\Lambda)$ is an FK space with AK, the representation of $(c_0^p(\Lambda))^*$ follows from [13, Theorem 7.2.9, p. 107].

(c) Let $0 < p < 1$ and condition (2.2) hold.

We assume $f \in (c^p(\Lambda))^*$. Then $f_1 = f\Big|_{c_0^p(\Lambda)} \in (c_0^p(\Lambda))^*$. Given $x \in c^p(\Lambda)$, there is a sequence $a \in C_\beta(\Lambda)$ such that $f_1(x - le) = \sum_{k=0}^\infty a_k(x_k - l)$. Since $a \in C_\beta(\Lambda) = (c^p(\Lambda))^\beta$, we have $ax \in cs$ for all $x \in c^p(\Lambda)$, in particular, for $x = e \in c^p(\Lambda)$, this implies $ae = a \in cs$, and we may write

$$f(x) = l\left(f(e) - \sum_{k=0}^{\infty} a_k\right) + \sum_{k=0}^{\infty} a_k x_k.$$

Putting $\chi_f = f(e) - \sum_{k=0}^\infty a_k x_k$, we obtain the given representation.
Conversely, if f has the given represenation, then $f \in c^*(\Lambda)$, and so $f \in (c^p(\Lambda))^*$.

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