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BOUNDED AND POSITIVE SOLUTIONS OF DISCRETE STEADY STATE EQUATIONS

SUI SUN CHENG AND RIGOBERTO MEDINA*

Abstract. Existence of bounded and/or positive solutions of a discrete steady state equation are derived by means of the Banach contraction principle and also by a monotone method.

1. Introduction

Consider the temperature distribution of a "very long" rod. Assume that the rod is so long that it can be laid on top of the set of integers. Let $u_m^{(n)}$ be the temperature at the integral position m and integral time n of the rod. At time n, if the temperature $u_{m-1}^{(n)}$, heat will flow form the point $m-1$ to m. The amount of increase is $u_m^{(n+1)} - u_m^{(n)}$, and it is reasonable to postulate that the increase is proportional to the $u_{m-1}^{(n)} - u_m^{(n)}$, say, $r(u_{m-1}^{(n)} - u_m^{(n)})$ where r is a positive diffusion constant. Similarly, heat will flow from the point $m+1$ to m. Thus, it is reasonable that the total effect is

$$u_m^{(n+1)} - u_m^{(n)} = r\Big(u_{m-1}^{(n)} - u_m^n\Big) + r\Big(u_{m+1}^{(n)} - u_m^{(n)}\Big).$$

Such a postulate can be regarded as a discrete Newton law of cooling.

The same postulate works for the distribution of heat through a very large thin plate. Assume the plate is so large that it can be laid on top of the set of lattice points $Z^2 = \{(i,j)|i,j = 0, \pm 1, \pm 2, \ldots\}$ in the plane. Let $u_{ij}^{(n)}$ be the temperature of the plate at position (i,j) and integral time n. Then a corresponding heat equation is given by

$$\begin{aligned} u_{ij}^{(n+1)} - u_{ij}^{(n)} &= r\Big(u_{i-1,j}^{(n)} - 2u_{ij}^{(n)} + u_{i+1,j}^{(n)}\Big) + r\Big(u_{i,j-1}^{(n)} - 2u_{ij}^{(n)} + u_{i,j+1}^{(n)}\Big) \\ &= r\Big(u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j+1}^{(n)} + u_{i,j+1}^{(n)} - 4u_{ij}^{(n)}\Big), \end{aligned}$$

where $(i,j) \in Z^2$ and $n = 0, 1, 2, \ldots$.

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If the plate has an initial temperature distribution at time $n = 0$, then after a long period of time, the temperature inside the plate will stabilize, and the subsequent temperature distribution $\{u_{ij}\}$ will satisfy the steady state equation [1] (for more references on partial difference equations, the readers may consult [2, 3, 4])

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{ij} = 0, \quad (i,j) \in Z^2.$$

It is interesting to note that this equation has a bounded and also positive solution, namely $\{u_{ij}\} = \{1\}$. The question then arises as to whether the corresponding steady state equation for a nonhomogeneous thin plate, namely,

$$\alpha_{ij} u_{i-1,j} + \beta_{ij} u_{i+1,j} + \gamma_{ij} u_{i,j-1} + \delta_{ij} u_{i,j+1} - \sigma_{ij} u_{ij} + f(i,j,u_{ij}) = 0, \tag{1}$$

where α_{ij}, β_{ij}, γ_{ij}, δ_{ij}, $\sigma_{ij} \in R$ for $(i,j) \in Z^2$, has a bounded and/or positive solution.

If this equation represents a real model, it will be reasonable to expect that such is the case for appropriate conditions on $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ and δ_{ij}. In this note, we will make use of the monotone method and Banach contraction principle for deriving the existence of these solutions to (1).

First, we will establish an existence criterion for *positive* and bounded solutions of (1). Let $l(Z^2)$ be the linear space of all real double sequences of the form $x = \{x_{ij}\}_{(i,j)\in Z^2}$ endowed with the usual operations. For the sake of convenience, if $x = \{x_{ij}\}$, $y = \{y_{ij}\} \in l(Z^2)$, we write $x \le y$ if $x_{ij} \le y_{ij}$ for all $(i,j) \in Z^2$. Let $S : l(Z^2) \to l(Z^2)$ be defined as follows: for $u = \{u_{ij}\} \in l(Z^2)$,

$$(Su)_{ij} = \frac{\alpha_{ij} u_{i-1,j}}{\sigma_{ij}} + \frac{\beta_{ij} u_{i+1,j}}{\sigma_{ij}} + \frac{\gamma_{ij} u_{i,j-1}}{\sigma_{ij}} + \frac{\delta_{ij} u_{i,j+1}}{\sigma_{ij}} + \frac{f(i,j,u_{ij})}{\sigma_{ij}},$$

where we have assumed that $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \sigma_{ij} > 0$ for $(i,j) \in Z^2$.

Consider the following sequence of successive approximations: $u^{[0]} = \{B^*\}$ and

$$u^{[n+1]} = Su^{[n]}, \qquad n = 0, 1, 2, \ldots \tag{2}$$

Note that

$$u^{[1]}_{ij} = \left\{ \frac{\alpha_{ij}}{\sigma_{ij}} + \frac{\beta_{ij}}{\sigma_{ij}} + \frac{\gamma_{ij}}{\sigma_{ij}} + \frac{\delta_{ij}}{\sigma_{ij}} \right\} B^* + \frac{f(i,j,B^*)}{\sigma_{ij}}, \qquad (i,j) \in Z^2.$$

Thus, if we impose the condition

$$\left\{ \frac{\alpha_{ij}}{\sigma_{ij}} + \frac{\beta_{ij}}{\sigma_{ij}} + \frac{\gamma_{ij}}{\sigma_{ij}} + \frac{\delta_{ij}}{\sigma_{ij}} \right\} B^* + \frac{f(i,j,B^*)}{\sigma_{ij}} \le B^*, \tag{3}$$

then $u^{[1]} \le u^{[0]}$. Similarly, if we define $v^{[0]} = \{B_*\}$, and $v^{[n+1]} = Sv^{[n]}$ for $n = 0, 1, 2, \ldots$, then $v^{[0]} \le v^{[1]}$, provided that

$$\left\{ \frac{\alpha_{ij}}{\sigma_{ij}} + \frac{\beta_{ij}}{\sigma_{ij}} + \frac{\gamma_{ij}}{\sigma_{ij}} + \frac{\delta_{ij}}{\sigma_{ij}} \right\} B_* + \frac{f(i,j,B_*)}{\sigma_{ij}} \ge B_*, \qquad (i,j) \in Z^2.$$

Next, under the additional condition that f is nondecreasing in the third variable, note that $u^{[1]} \le u^{[0]}$ implies $u^{[2]} = Su^{[1]} \le Su^{[0]} = u^{[1]}$, and $v^{[0]} \le v^{[1]}$ implies $v^{[1]} \le v^{[2]}$. Furthermore, if we assume that $B_* \le B^*$, then $v^{[0]} \le u^{[0]}$, so that $v^{[1]} = Sv^{[0]} \le Su^{[0]} \le u^{[1]}$. By induction, it follows that

$$B_* = v^{[0]} \le v^{[1]} \le \cdots \le v^{[n]} \le v^{[n+1]} \le \cdots \le u^{[n+1]} \le u^{[n]} \le \cdots \le u^{[0]} = B^*.$$

It follows that for each $(i,j) \in Z^2$, $u_{ij}^{[n]}$ converges to a limit u_{ij} and $v_{ij}^{[n]}$ to v_{ij} as $n \to \infty$. By taking limits as $n \to \infty$ on both sides of (2), we see that the double sequence $u = \{u_{ij}\}$ and $v = \{v_{ij}\}$ satisfy (provided that f is continuous in the third variable)

$$w = Sw, \tag{4}$$

and hence u and v are bounded solutions of (1).

Theorem 1. *Suppose $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \sigma_{ij} > 0$ for $(i,j) \in Z^2$. Suppose further that $f : Z^2 \times R \to R$ satisfies*

$$B_* \left\{ \frac{\alpha_{ij}}{\sigma_{ij}} + \frac{\beta_{ij}}{\sigma_{ij}} + \frac{\gamma_{ij}}{\sigma_{ij}} + \frac{\delta_{ij}}{\sigma_{ij}} \right\} + \frac{f(i,j,B_*)}{\sigma_{ij}} \ge B_*, \qquad (i,j) \in Z^2,$$
$$B^* \left\{ \frac{\alpha_{ij}}{\sigma_{ij}} + \frac{\beta_{ij}}{\sigma_{ij}} + \frac{\gamma_{ij}}{\sigma_{ij}} + \frac{\delta_{ij}}{\sigma_{ij}} \right\} + \frac{f(i,j,B_*)}{\sigma_{ij}} \le B^*, \qquad (i,j) \in Z^2,$$

where $B_* \le B^*$, and for any $u, v \in R$,

$$u \le v \Rightarrow f(i,j,u) \le f(i,j,v), \quad (i,j) \in Z^2,$$

and f is continuous with respect to the third independent variable for any $(i,j) \in Z^2$. Then equation (1) has a solution bounded between B_* and B^*.

In particular, for the case where $f \equiv 0$, $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = \delta_{ij} = 1$ and $\sigma_{ij} = 4$, take $B_* = B^* > 0$, and a positive and bounded solution will result. Note further that if $B_* > 0$, then the solution v found above is positive.

If we are only interested in bounded solutions, the Banach contraction principle can also be applied. Let us first consider the simple case where $f \equiv 0$. Then (1) can be rewritten in the form

$$u_{ij} = \frac{1}{\sigma_{ij}} \{\alpha_{ij} u_{i-1,j} + \beta_{ij} u_{i+1,j} + \gamma_{ij} u_{i,j-1} + \delta_{ij} u_{i,j+1}\},$$

where we have assumed that $\sigma_{ij} \ne 0$ for $(i,j) \in Z^2$. Let Ω be the Banach subspace of all bounded sequences in $l(Z^2)$ endowed with the norm

$$\|x\| = \sup\{|x_{ij}| : (i,j) \in Z^2\}.$$

Let $T : \Omega \to l(Z^2)$ be defined as follows: for $u = \{u_{ij}\} \in \Omega$,

$$(Tu)_{ij} = \frac{1}{\sigma_{ij}} \{\alpha_{ij} u_{i-1,j} + \beta_{ij} u_{i+1,j} + \gamma_{ij} u_{i,j-1} + \delta_{ij} u_{i,j+1}\}, \qquad (i,j) \in Z^2. \tag{5}$$

If we now impose the condition that the double sequences $\{|\alpha_{ij}/\sigma_{ij}|\}$, $\{|\beta_{ij}/\sigma_{ij}\}$, $\{|\gamma_{ij}/\sigma_{ij}|\}$ and $\{|\delta_{ij}/\sigma_{ij}|\}$ are bounded, then since

$$|(Tu)_{ij}| \leq \left\{ \left|\frac{\alpha_{ij}}{\sigma_{ij}}\right| + \left|\frac{\beta_{ij}}{\sigma_{ij}}\right| + \left|\frac{\gamma_{ij}}{\sigma_{ij}}\right| + \left|\frac{\delta_{ij}}{\sigma_{ij}}\right| \right\} \|u\|,$$

hence $T\Omega$ is contained in Ω. Next, if we impose the additional condition that

$$\sup_{i,j} \left\{ \left|\frac{\alpha_{ij}}{\sigma_{ij}}\right| + \left|\frac{\beta_{ij}}{\sigma_{ij}}\right| + \left|\frac{\gamma_{ij}}{\sigma_{ij}}\right| + \left|\frac{\delta_{ij}}{\sigma_{ij}}\right| \right\} < 1, \tag{6}$$

then since

$$\|Tu - Tv\| \leq \sup_{i,j} \left\{ \left|\frac{\alpha_{ij}}{\sigma_{ij}}\right| + \left|\frac{\beta_{ij}}{\sigma_{ij}}\right| + \left|\frac{\gamma_{ij}}{\sigma_{ij}}\right| + \left|\frac{\delta_{ij}}{\sigma_{ij}}\right| \right\} \|u - v\|,$$

we see that T is a contraction mapping on Ω. Thus by means of the Banach contraction mapping principle, we see that there is some $v = \{v_{ij}\} \in \Omega$ such that $v = Tv$. But then in view of (5), v is a bounded solution of (1). Unfortunately, since $\{0\}$ is a solution of (5), the unique bounded solution can only be $\{0\}$.

However, for perturbed equations, the same principle applies, and nontrivial and bounded solutions may result.

Theorem 2. *Suppose* $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \sigma_{ij}, \lambda_{ij}$ *are defined for* $(i,j) \in Z^2$ *such that* $\sigma_{ij} \neq 0$ *and* $\lambda_{ij} \geq 0$ *for* $(i,j) \leq Z^2$*, and that*

$$\sup_{i,j} \left\{ \left|\frac{\alpha_{ij}}{\sigma_{ij}}\right| + \left|\frac{\beta_{ij}}{\sigma_{ij}}\right| + \left|\frac{\gamma_{ij}}{\sigma_{ij}}\right| + \left|\frac{\delta_{ij}}{\sigma_{ij}}\right| + \left|\frac{\lambda_{ij}}{\sigma_{ij}}\right| \right\} < 1.$$

Suppose further that $f : Z^2 \times R \to R$ *satisfies*

$$|f(i,j,u)| \leq |\sigma_{ij}|\omega(|u|), \qquad (i,j) \in Z^2, u \in R,$$

for some function $\omega : R \to R$ *which is bounded on* $[0,\infty)$*, and*

$$|f(i,j,u) - f(i,j,v)| \leq \lambda_{ij}|u - v|, \qquad (i,j) \in Z^2; u, v \in R.$$

Then the equation (1) *over* Z^2 *has a bounded solution* (*which is nontrivial if* $f(i,j,0) \neq 0$ *for some* $(i,j) \in Z^2$).

The proof follows from modifying the definition of T defined by (5) to $\bar{T}$:

$$(\bar{T}u)_{ij} = (Tu)_{ij} + \frac{1}{\sigma_{ij}} f(i,j,u_{ij}), \qquad (i,j) \in Z^2.$$

Then

$$\left|(\bar{T}u)_{ij}\right| \leq \left\{ \left|\frac{\alpha_{ij}}{\sigma_{ij}}\right| + \left|\frac{\beta_{ij}}{\sigma_{ij}}\right| + \left|\frac{\gamma_{ij}}{\sigma_{ij}}\right| + \left|\frac{\delta_{ij}}{\sigma_{ij}}\right| \right\} \|u\| + \omega(|u_{ij}|), \qquad (i,j) \in Z^2,$$

and

$$\left\|\bar{T}u - \bar{T}v\right\| \le \sup_{i,j}\left\{\left|\frac{\alpha_{ij}}{\sigma_{ij}}\right| + \left|\frac{\beta_{ij}}{\sigma_{ij}}\right| + \left|\frac{\gamma_{ij}}{\sigma_{ij}}\right| + \left|\frac{\delta_{ij}}{\sigma_{ij}}\right| + \left|\frac{\lambda_{ij}}{\sigma_{ij}}\right|\right\}\|u - v\|,$$

so that $\bar{T}$ is a contraction mapping on Ω.

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Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R.O.C.

Departmento de Ciencias Exactas, Universidad de Los Lagos, Casilla 933, Osorno, Chile.