NOTE ON INTEGRAL CLOSURES OF SEMIGROUP RINGS

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Abstract. Let \( S \) be a subsemigroup which contains 0 of a torsion-free abelian (additive) group. Then \( S \) is called a grading monoid (or a \( g \)-monoid). The group \( \{ s - s' | s, s' \in S \} \) is called the quotient group of \( S \), and is denoted by \( q(S) \). Let \( R \) be a commutative ring. The total quotient ring of \( R \) is denoted by \( q(R) \). Throughout the paper, we assume that a \( g \)-monoid properly contains \( \{ 0 \} \). A commutative ring is called a ring, and a non-zero-divisor of a ring is called a regular element of the ring.

We consider integral elements over the semigroup ring \( R[X; S] \) of \( S \) over \( R \). Let \( S \) be a \( g \)-monoid with quotient group \( G \). If \( n\alpha \in S \) for an element \( \alpha \) of \( G \) and a natural number \( n \) implies \( \alpha \in S \), then \( S \) is called an integrally closed semigroup. We know the following fact:

**Theorem 1** ([G2, Corollary 12.11]). Let \( D \) be an integral domain and \( S \) a \( g \)-monoid. Then \( D[X; S] \) is integrally closed if and only if \( D \) is an integrally closed domain and \( S \) is an integrally closed semigroup.

Let \( R \) be a ring. In this paper, we show that conditions for \( R[X; S] \) to be integrally closed reduce to conditions for the polynomial ring of an indeterminate over a reduced total quotient ring to be integrally closed (Theorem 15). Clearly the quotient field of an integral domain is a von Neumann regular ring. Assume that \( q(R) \) is a von Neumann regular ring. We show that \( R[X; S] \) is integrally closed if and only if \( R \) is integrally closed and \( S \) is integrally closed (Theorem 20). Let \( G \) be a \( g \)-monoid which is a group. If \( R \) is a subring of the ring \( T \) which is integrally closed in \( T \), we show that \( R[X; G] \) is integrally closed in \( T[X; S] \) (Theorem 13). Finally, let \( S \) be sub-\( g \)-monoid of a totally ordered abelian group. Let \( R \) be a subring of the ring \( T \) which is integrally closed in \( T \). If \( g \) and \( h \) are elements of \( T[X; S] \) with \( h \) monic and \( gh \in R[X; S] \), we show that \( g \in R[X; S] \) (Theorem 24).

1. General Rings

Let \( G \) be an abelian group. Then a maximal number \( n \) so that there exist a set of \( n \)-elements in \( G \) which is independent over \( \mathbb{Z} \) is called the torsion-free rank of \( G \), and is denoted by \( t.f.r.(G) \).

**Lemma 2.** Let \( G \) be a \( g \)-monoid which is finitely generated and with \( t.f.r.(G) = n \). Then \( R[X; G] \) is isomorphic to the ring \( R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \) over \( R \) (where \( X_1, \ldots, X_n \) are indeterminates).

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Proof. Since $G$ is torsion-free, $G$ is the direct sum $\mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_n$ for some elements $u_i$ of $G$. Set $\sigma(X_i) = X^{\alpha_i}$ for each $i$. Then we have an isomorphism $\sigma$ of $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ onto $R[X; G]$.

Lemma 3. Let $f$ be an element of $R[X; S]$. Then $f$ is a zero-divisor of $R[X; S]$ if and only if there exists a non-zero element $a$ of $R$ such that $aa_i = 0$ for every coefficient $a_i$ of $f$.

Proof. (1) Let $f$ be a zero-divisor in $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$, then there exists a non-zero element $a$ of $R$ such that $af = 0$. For, there exists a natural number $m$ such that $X^mf$ is a zero-divisor in $R[X_1, \ldots, X_n]$. By [G1, (28.7) Proposition], there exists a non-zero element $a$ of $R$ such that $aX^mf = 0$, and hence $af = 0$.

(2) Let $f$ be a zero-divisor in $R[X; S]$. Then $f$ is a zero-divisor in $R[X; G]$, where $G = q(S)$. There exists a finitely generated subgroup $H$ of $G$ such that $f$ is a zero-divisor in $R[X; H]$. $R[X; H]$ is isomorphic to the ring $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ for some $n$ by Lemma 2. By (1), there exists a non-zero element $a$ of $R$ such that $af = 0$.

Let $S$ be a $g$-monoid with quotient group $G$, and $T$ a sub-$g$-monoid of $S$ with quotient group $H \subset G$. Let $R$ be a ring with total quotient ring $K$. Then all of $R$, $R[X; T]$, $K[X; T]$ and $q(R[X; T])$ are canonically regarded as subrings of $q(R[X; S])$ by Lemma 3.

Lemma 4([G1, (10.2) Proposition]). Let $R$ be a subring of the ring $T$, and $A$ be the integral closure of $R$ in $T$. If $N$ is a multiplicative system in $R$, then the quotient ring $A_N$ is the integral closure of $R_N$ in $T_N$.

Lemma 5. If $R[X; S]$ is integrally closed, then $R$ is integrally closed.

Proof. Let $x$ be an element of $K = q(R)$ which is integral over $R$. Since $x$ is an element of $q(R[X; S])$ which is integral over $R[X; S]$, $x$ belongs to $R[X; S]$, and hence $x \in R$. Therefore $R$ is integrally closed.

Lemma 6. If $R[X; S]$ is integrally closed, then $S$ is integrally closed.

Proof. Let $\alpha$ be an element of $q(S)$ which is integral over $S$. Since $n\alpha \in S$ for some natural number $n$, we have an integral equation of the element $X^\alpha$ of $q(R[X; S])$ over $R[X; S]$. It follows that $X^\alpha \in R[X; S]$, and hence $\alpha \in S$. Therefore $S$ is integrally closed.

Lemma 7. If $R[X; S]$ is integrally closed, then $R$ is a reduced ring.

Proof. Suppose that $R$ has a non-zero nilpotent $a$. Take non-zero $\alpha$ of $S$. Then $1 + X^\alpha$ is a regular element of $R[X; S]$ by Lemma 3. Then $a/(1 + X^\alpha)$ is a nilpotent of $q(R[X; S])$, and $a/(1 + X^\alpha) \notin R[X; S]$; a contradiction. Hence $R$ is reduced.

Lemma 8. Let $G$ be a $g$-monoid which is a group. Let $\{H_\lambda | \lambda\}$ be the set of finitely generated non-zero subgroups of $G$. Let $A_\lambda$ be the integral closure of $R[X; H_\lambda]$. Then $\bigcup A_\lambda$ is the integral closure of $R[X; G]$. 
Proof. Let $F$ be an element of $q(R[X; G])$ which is integral over $R[X; G]$. There exists a finitely generated subgroup $H$ of $G$ such that $F$ is an element of $q(R[X; H])$ and $F$ is integral over $R[X; H]$. Then we have $H = H_{\lambda}$ for some $\lambda$, and $F \in A_{\lambda}$. Hence the integral closure of $R[X; G]$ is $\cap A_{\lambda}$.

Lemma 9([BCL, Lemma 1]). Let $G$ be a g-monoid which is a group, and $H$ a non-zero subgroup of $G$. Then $R[X; G]$ is a free $R[X; H]$-module. Let $\{\alpha_{\lambda}|\lambda\}$ be a system of complete representatives of $G$ modulo $H$. Then $\{X^{\alpha_{\lambda}}|\lambda\}$ is a free basis of $R[X; G]$ over $R[X; H]$.

Proposition 10. Let $G$ be a g-monoid which is a group. Then $R[X; G]$ is integrally closed if and only if, for every finitely generated non-zero subgroup $H$ of $G$, $R[X; H]$ is integrally closed.

Proof. The sufficiency follows from Lemma 8.

The necessity: Let $F$ be an element of $q(R[X; G])$ which is integral over $R[X; H]$. Since $R[X; G]$ is integrally closed, we have $F \in R[X; G]$. Lemma 9 implies that $F \in R[X; H]$. Hence $R[X; H]$ is integrally closed.

Lemma 11. Let $X_1, X_2, \ldots$ be indeterminates. Let $R_1$ be the integral closure of $R[X_1, X_1^{-1}], \ldots, X_n^{-1}$, then $R_2$ be the integral closure of $R_1[X_2, X_2^{-1}], R_3$ be the integral closure of $R_2[X_3, X_3^{-1}], \ldots$. Then $R_n$ is the integral closure of $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$.

Proof. We rely on the induction on $n$. Assume that $R_{n-1}$ is the integral closure of $R[X_1, X_1^{-1}, \ldots, X_{n-1}, X_{n-1}]$. Clearly $R_{n-1}[X_n, X_n^{-1}]$ is integral over $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. Hence $R_n$ is integral over $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. Let $F$ be an element of $q(R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}])$ which is integral over $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. Then $F$ is integral over $R_{n-1}[X_n, X_n^{-1}]$. Hence $F \in R_n$. Therefore $R_n$ is the integral closure of $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$.

Lemmas 8, 9, and 10 show that to determine the integral closure of $R[X; G]$ reduces to determine the integral closures of $R'[X, X^{-1}]$ for some ring $R'$.

Lemma 12. $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is integrally closed if and only if $R[X_1, \ldots, X_n]$ is integrally closed.

Proof. The sufficiency follows from Lemma 4.

The necessity: Let $F$ be an element of $q(R[X_1, \ldots, X_n])$. We have $F \in R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ by assumption. If $F \notin R[X_1, \ldots, X_n]$, we may assume that $F = f_0X_0 + f_1X_1^{d} + \ldots$, where each $f_i \in R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}], f_d \neq 0$ and $d < 0$. $R$ is reduced by Lemma 7. Hence there exists a prime ideal $P$ of $R$ such that $f_d \neq 0$ mod $PR[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. Set $q(R/P) = k$. Then there arises an element of $k[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] - k[X_1, \ldots, X_n]$ which is integral over $k[X_1, \ldots, X_n]$; a contradiction. Therefore $R[X_1, \ldots, X_n]$ is integrally closed.
Theorem 13. Let $G$ be a $g$-monoid which is a group. Let $T$ be an extension ring of the ring $R$ and let $A$ be the integral closure of $R$ in $T$. Then $A/X; G$ is the integral closure of $R[X; G]$ in $T[X; G]$.

Proof. (1) Let $X_1, \ldots, X_n$ be a finite number of indeterminates. Then $A[X_1, \ldots, X_n]$ is the integral closure of $R[X_1, \ldots, X_n]$ in $T[X_1, \ldots, X_n]$ (cf. [G1, (10.7) Theorem]).

(2) In (1), $A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is the integral closure of $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ in $T[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. For, let $N$ be the multiplicative system in $R[X_1, \ldots, X_n]$ generated by $X_1, \ldots, X_n$. Since $A[X_1, \ldots, X_n]$ is the integral closure of $R[X_1, \ldots, X_n]$ in $T[X_1, \ldots, X_n]$ by (1), we see that the quotient ring $A[X_1, \ldots, X_n]/N$ is the integral closure of $R[X_1, \ldots, X_n]/N$ by Lemma 4. Hence $A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is the integral closure of $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ in $T[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$.

(3) Assume that $G$ is finitely generated. Then $A/X; G$ is the integral closure of $R[X; G]$ in $T[X; G]$. For, $R[X; G]$ (resp. $A[X; G]$ and $T[X; G]$) is isomorphic to $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ (resp. $A[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ and $T[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$) for some $n$ by Lemma 2. (2) implies that $A[X; G]$ is the integral closure of $R[X; G]$ in $T[X; G]$.

(4) Assume that $G$ is a $g$-monoid which is a group. Let $F = \Sigma a_iX^{\alpha_i}$ be an element of $T[X; G]$ which is integral over $R[X; G]$. We have $F_m + F_{m-1}f_{m-1} + \cdots + F_0 = 0$ for the elements $f_i$ of $R[X; G]$. There exists a finitely generated subgroup $H$ of $G$ such that $F$ is integral over $R[X; H]$ and belongs to $T[X; H]$. (3) implies that $F \in A[X; H]$. Hence $A[X; G]$ is the integral closure of $R[X; G]$ in $T[X; G]$.

Proposition 14. $R[X; S]$ is integrally closed if and only if $S$ is integrally closed, $R$ is integrally closed and $K[X; G]$ is integrally closed, where $K = q(R)$.


The sufficiency: Suppose that $R[X; S]$ is not integrally closed. There exists $F \in q(R[X; S]) - R[X; S]$ which is integral over $R[X; S]$. We have $F \in K[X; G]$ by assumption. Then we have $F \in R[X; G]$ by Theorem 13. Put $F = \Sigma a_iX^{\alpha_i}$. There exists $k$ such that $a_kX^{\alpha_k} \notin R[X; S]$. $R$ is reduced by assumption and by Lemma 7. Hence there exists a prime ideal $P$ of $R$ which does not contain $a_k$. Set $D = R/P$ and $\bar{a}_i = a_i + P$ for each $i$. Then $\bar{F} = \Sigma \bar{a}_iX^{\alpha_i}$ is an element of $D[X; G] - D[X; S]$ which is integral over $D[X; S]$. Then $\bar{F}$ is an element of $k[X; G] - k[X; S]$ which is integral over $k[X; S]$, where $k = q(D)$. This contradicts to Theorem 1.

Let $K$ be total quotient ring. We will denote the total quotient ring $q(K[X_1, \ldots, X_n])$ of $K[X_1, \ldots, X_n]$ by $K(X_1, \ldots, X_n)$.

Theorem 15. $R[X; S]$ is integrally closed if and only if $S$ is integrally closed, $R$ is integrally closed, $K[X_1]$ is integrally closed and $K(X_1, \ldots, X_{n-1})[X_n]$ is integrally closed for every $n$ with $n \leq t.f.r. (q(S))$, where $K = q(R)$.
Proof. The necessity: $S$ is integrally closed, $R$ is integrally closed and $K[X;G]$ is integrally closed by Proposition 14. There exists a finitely generated subgroup $H$ of $G$ such that $t.f.r(H) = n$. $K[X;H]$ is integrally closed by Proposition 10. Hence $(K[X_1, X_1^{-1}, \ldots, X_{n-1}, X_{n-1}^{-1}]) [X_n, X_n^{-1}]$ is integrally closed. Then $K(X_1, \ldots, X_{n-1})[X_n]$ is integrally closed by Lemma 12.

The sufficiency: We will show that $K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is integrally closed for every $n$ with $n \leq t.f.r(G)$. Suppose that $K[X_1, X_1^{-1}, \ldots, X_k, X_k^{-1}]$ is integrally closed for $k < t.f.r(G)$. Then $K[X_1^{-1}, \ldots, X_{k+1}, X_{k+1}^{-1}]$ is integrally closed by Proposition 14. Therefore $K[X;G]$ is integrally closed by Proposition 10. Then $R[X;S]$ is integrally closed by Proposition 14.

2. Von Neumann Regular Rings

Let $R$ be a ring. If, for each element $a$ of $R$, there exists an element $b$ of $R$ such that $a = ab$, then $R$ is called a von Neumann regular ring. We confer [G1, §11] for von Neumann regular rings. Every field is clearly a von Neumann regular ring.

Lemma 16. If $R$ is a von Neumann regular ring, then $R[X]$ and $R[X, X^{-1}]$ are integrally closed.

Proof. Let $F$ be an element of $q(R[X])$ which is integral over $R[X]$. We have $F = f/g$, where $f$ is an element of $R[X]$ and $g$ is a regular element of $R[X]$. $R[X]$ is a Bezout ring, that is, every finitely generated ideal is principal by [GP, Corollary 3.1]. Hence there exist elements $h, f', f_1, g_1$ of $R[X]$ such that $f = hf'$, $g = hg'$, $h = ff_1 + gg_1$, where $h$ and $g'$ are regular. Then we have $1/g' = Ff_1 + g_1$. Hence $1/g'$ is integral over $R[X]$. The integral equation of $1/g'$ over $R[X]$ shows that $g'$ is a unit of $R[X]$. [G1, Corollary 11.4] implies that $g'$ is a unit of $R$. Then we have $F = f/g = f'/g' \in R[X]$. Therefore $R[X]$ is integrally closed.

$R[X, X^{-1}]$ is integrally closed by Lemma 12.

Lemma 17. If $R$ is von Neumann regular ring, then $q(R[X;G])$ is a von Neumann regular ring.

Proof. (1) Let $f = \sum a_i X^i$ be an element of $R[X]$. Then there exists $F \in q(R[X])$ such that $f = f^2 F$. For the proof, we rely on the induction on $n$. Thus suppose that the assertion holds for fewer degrees. Assume that $a_n \neq 0$. There exists an idempotent $e$ of $R$ such that $Ra_n = Re$. Put $e' = 1 - e$. Since deg $(f' e') < n$, there exist elements $f_1$ and $g_1$ of $R[X]$ with $g_1 e'$ regular in $R e'[X]$ such that $f e' = f f_1 e' / (g_1 e')$. Note that $f e$ is regular in $R e[X]$. We see that $g_1 e' + f e$ is a regular element of $R[X]$. Further we have $f = f^2 (f_1 e' + e) / (g_1 e' + f e)$.

(2) $q(R[X_1, \ldots, X_n])$ is von Neumann regular. For the proof, we rely on the induction on $n$. (1) implies that $q(R[X_1])$ is von Neumann regular. Suppose that $q(R[X_1, \ldots, X_{n-1}])$ is von Neumann regular. It follows that $q(q(R[X_1, \ldots, X_{n-1}][X_n])$ is von Neumann regular. That is, $q(R[X_1, \ldots, X_n])$ is von Neumann regular.
(3) Let \( f \) be an element of \( q(R[X;G]) \). There exists a finitely generated subgroup \( H \) of \( G \) such that \( f \in q(R[X;H]) \). Since \( R[X;H] \) is isomorphic to the ring \( R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \) for some \( n \), \( q(R[X;H]) \) is von Neumann regular by (2). Hence there exists \( F \in q(R[X;H]) \) such that \( f = f^2 F \). Therefore \( q(R[X;G]) \) is von Neumann regular.

[G1, §11, Exercise 13] states that if \( R \) is its own total quotient ring and if \( R \) is reduced, then \( R \) is 0-dimensional. If this is the case, then \( q(R[X;G]) \) is a von Neumann regular ring for every reduced ring \( R \). Now we have the following,

**Example 18** (Gilmer and Matsuda). Let \( k \) be a field and let \( X_1, X_2, \ldots \) be indeterminates.

1. Set \( R = k[[X_1, X_2, \ldots]]/(X_iX_j|i \neq j) \), where \( k[[X_1, X_2, \ldots]] \) is the union of the ascending net of rings \( k[[X_1, \ldots, X_n]] \) of all \( n \). Then \( R \) is its own total quotient ring and reduced. But \( R \) is not 0-dimensional.

2. Let \( R = k[[X_1, X_2, \ldots]]/(X_iX_j|i \neq j) \) and \( M = (X_1, X_2, \ldots)R \). Then \( R_M \) is its own total quotient ring and reduced. But \( R_M \) is not 0-dimensional.

**Lemma 19.** If \( R \) is a von Neumann regular ring, then \( R[X;G] \) is integrally closed.

**Proof.** (1) \( R[X, X^{-1}] \) is integrally closed by Lemma 16.

(2) \( R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \) is integrally closed. For the proof, we rely on the induction on \( n \). Suppose that \( R[X, X^{-1}], \ldots, X_{n-1}, X_{n-1}^{-1}] \) is integrally closed. \( q(R[X_1, \ldots, X_{n-1}]/[X_n, X_n^{-1}]) \) is von Neumann regular by Lemma 17. (1) implies that \( q(R[X_1, \ldots, X_{n-1}]/[X_n, X_n^{-1}]) \) is integrally closed. Then Proposition 14 implies that \( R[X_1, X_1^{-1}, \ldots, X_{n-1}, X_{n-1}^{-1}] \) is integrally closed.

(3) Let \( F \) be an element of \( q(R[X;G]) \) which is integral over \( R[X;G] \). There exists a finitely generated subgroup \( H \) of \( G \) such that \( F \in q(R[X;H]) \) and \( F \) is integral over \( R[X;H] \). Since \( R[X;H] \) is isomorphic to the ring \( R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}] \) for some \( n \), (2) implies that \( F \in R[X;H] \). Therefore \( R[X;G] \) is integrally closed.

**Theorem 20.** Assume that \( q(R) \) is a von Neumann regular ring. Then \( R[X;S] \) is integrally closed if and only if \( S \) is integrally closed and \( R \) is integrally closed.

**Proof.** The necessity is clear.

The sufficiency: Set \( G = q(S) \) and \( K = q(R) \). Then \( K[X;G] \) is integrally closed by Lemma 19. Proposition 14 implies that \( R[X;S] \) is integrally closed.

If \( R \) is a domain, then \( q(R) \) is clearly a von Neumann regular ring.

3. A Theorem

Let \( G \) be a totally ordered abelian group. Let \( f \) be an element of \( R[X;G] \). Put \( f = a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n} \), where the \( a_i \) are non-zero elements of \( R \) and \( \alpha_1 < \cdots < \alpha_n \).

If \( a_n = 1 \), then \( f \) is called monic in \( R[X;G] \).
Lemma 21. Let $G = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (the direct sum of $n$-copies of the additive group $\mathbb{Z}$) with the lexicographic order, and let $X_1 = X^{(1,0,\ldots,0)} < \cdots < X_n = X^{(0,\ldots,0,1)}$. Let $T$ be a ring and $R$ a subring of $T$ which is integrally closed in $T$. Let $g$ and $h$ are elements of $T[X;G]$ with $h$ monic and $gh \in R[X;G]$. Then $g \in R[X;G]$.

Proof. We may assume that $g = g(X_1, \ldots, X_n)$ and $h = h(X_1, \ldots, X_n)$ belong to $T[X_1, \ldots, X_n]$. If $n = 1$, the assertion holds by [G1, (10.4) Theorem]. We rely on the induction on $n$. Suppose that the assertion holds for $n - 1$. There exists a natural number $m$ such that the coefficients of $g(X_1, \ldots, X_{n-1}, X_n)$ (resp. $h(X_1, \ldots, X_{n-1}, X_n)$) and $g(X_1, \ldots, X_{n-1}, X_n^m)$ (resp. $h(X_1, \ldots, X_{n-1}, X_n^m)$) are the same and $h(X_1, \ldots, X_{n-1}, X_n^m)$ is monic. Since $g(X_1, \ldots, X_{n-1}, X_n^m)h(X_1, \ldots, X_{n-1}, X_n^m) \in R[X_1, \ldots, X_{n-1}]$, we have $g(X_1, \ldots, X_{n-1}, X_n^m) \in R[X_1, \ldots, X_{n-1}]$ by hypothesis. Hence $g \in R[X;G]$.

Lemma 22. Let $G$ be a finitely generated subgroup of the totally ordered abelian group $R$. Let $T$ be a ring and $R$ a subring which is integrally closed in $T$. If, for elements $g$ and $h$ of $T[X;G]$ with $h$ monic, $gh \in R[X;G]$, then $g$ belongs to $R[X;G]$.

Proof. There exist real numbers $\pi_1, \ldots, \pi_n$ so that $G = Z\pi_1 + \cdots + Z\pi_n$ and $\{\pi_1, \ldots, \pi_n\}$ is independent over $Z$. We may assume that $0 < \pi_1 < \cdots < \pi_n$. Set $X_i = X^{\pi_i}$ for each $i$. We may assume that $g = g(X_1, \ldots, X_n)$ and $h = h(X_1, \ldots, X_n)$ are elements of $T[X_1, \ldots, X_n]$. Arrange the powers of $g$ and $h$ as follows: $\Sigma k_i\pi_i < \cdots < \Sigma k_m\pi_i$. It follows that $\Sigma k_i\pi_i/\pi_1 < \cdots < \Sigma k_m\pi_i/\pi_i$.

Since all of $\pi_2/\pi_1, \ldots, \pi_n/\pi_1$ are irrational, there exist positive rational numbers $a_1 = 1, a_2, \ldots, a_n$ such that $\Sigma k_i a_i < \cdots < \Sigma k_m a_i$. Hence there exist positive integers $p_1, \ldots, p_n$ such that $\Sigma k_i p_i < \cdots < \Sigma k_m p_i$. Note that, if $(k_1, \ldots, k_n) \neq (l_1, \ldots, l_n)$, then $\Sigma k_i \pi_i \neq \Sigma l_i \pi_i$. Then the coefficients of $g(X_1, \ldots, X_n)$ (resp. $h(X_1, \ldots, X_n)$) and $g(Y^{p_1}, \ldots, Y^{p_n})$ (resp. $h(Y^{p_1}, \ldots, Y^{p_n})$) are the same and $h(Y^{p_1}, \ldots, Y^{p_n})$ is monic (where $Y$ is an another indeterminate). Since $g(Y^{p_1}, \ldots, Y^{p_n})h(Y^{p_1}, \ldots, Y^{p_n}) \in R[Y]$, we have $g(Y^{p_1}, \ldots, Y^{p_n}) \in R[Y]$. Hence $g \in R[X;G]$.

Theorem 23. Let $S$ be a sub-$g$-monoid of a totally ordered abelian group. Let $R$ be a subring of the ring $T$, and let $A$ be the integral closure of $R$ in $T$. If, for elements $g$ and $h$ of $T[X,S]$ with $h$ monic, $gh \in R[X,S]$, then $g \in A[X;S]$.

Proof. We may assume that $G = S$ is a finitely generated subring and $R$ is integrally closed in $T$. We may assume that $G = H_1 \oplus \cdots \oplus H_k$ with the lexicographic order, where the $H_i$ are non-zero subgroups of the totally ordered abelian group $R$. Since $G$ is finitely generated, we may use a similar argument to [ZS, VI, (A)]. We rely on the induction on $t.f.r(G)$. Assume that $t.f.r(G) = r$, and suppose that the assertion holds for fewer torsion-free ranks. The powers of $g$ and $h$ of the form $(h_1, \ldots, h_k)$ for the $h_i \in H_i$. Suppose that each $k$-component $h_k$ is zero for every power of $g$ and $h$. Then we have $g \in R[X;G]$ by induction. Suppose that $h_k$ is non-zero for some power of $g$ or $h$. Let $H_i = Z\pi_{i_1} + \cdots + Z\pi_{i_{m(i)}}$, where the set $\{\pi_{i_1}, \ldots, \pi_{i_{m(i)}}\}$ is independent over $Z$ for each
i. We may assume that $0 < \pi_{i1} < \cdots < \pi_{in(i)}$ for each $i$. Put $X_{ij} = X^{(0, \ldots, \pi_{ij}, 0, \ldots)}$ for each $i$ and $j$. We may assume that $g$ and $h$ belong to $T[X_{11}, \ldots, X_{kn(k)}]$.

The case of $n(i) = 1$ for each $i$: Then $G$ is order-isomorphic to $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (the direct sum of $k$-copies of $\mathbb{Z}$) with the lexicographic order. By Lemma 21, we have $g \in R[X; G]$.

The case of $n(i) > 1$ for some $i$: For each $l$, arrange the $l$-components of $g$ and $h$ as follows: $\Sigma k_{11} \pi_{i1} < \Sigma k_{12} \pi_{i1} < \cdots < \Sigma k_{lm(l)i} \pi_{i1}$. Then, as in the proof of Lemma 22, there exist positive integers $p_{ij}$ such that $\Sigma k_{11}p_{ii} < \Sigma k_{12}p_{ii} < \cdots < \Sigma k_{lm(l)i}p_{ii}$. Let $Y_1, \ldots, Y_k$ be another indeterminates. Then the coefficients of $g = g(X_{11}, \ldots, X_{kn(k)})$ (resp. $h = h(X_{11}, \ldots, X_{kn(k)})$) and $g(Y_1^{p_{i1}}, \ldots, Y_1^{p_{in(1)}}, \ldots, Y_k^{p_{k1}}, \ldots, Y_k^{p_{kn(k)}})$ (resp. $h(Y_1^{p_{i1}}, \ldots, Y_1^{p_{in(1)}}, \ldots, Y_k^{p_{k1}}, \ldots, Y_k^{p_{kn(k)}})$) are the same and $h(Y_1^{p_{i1}}, \ldots, Y_1^{p_{in(1)}}, \ldots, Y_k^{p_{k1}}, \ldots, Y_k^{p_{kn(k)}})$ is monic. Since $g(Y_1^{p_{i1}}, \ldots, Y_1^{p_{in(1)}}, \ldots, Y_k^{p_{k1}}, \ldots, Y_k^{p_{kn(k)}})h(Y_1^{p_{i1}}, \ldots, Y_1^{p_{in(1)}}, \ldots, Y_k^{p_{k1}}, \ldots, Y_k^{p_{kn(k)}}) \in R[Y_1, \ldots, Y_k]$, we have $g(Y_1^{p_{i1}}, \ldots, Y_1^{p_{in(1)}}, \ldots, Y_k^{p_{k1}}, \ldots, Y_k^{p_{kn(k)}}) \in R[Y_1, \ldots, Y_k]$. Hence $g(Y_1^{p_{i1}}, \ldots, Y_1^{p_{in(1)}}, \ldots, Y_k^{p_{k1}}, \ldots, Y_k^{p_{kn(k)}}) \in R[Y_1, \ldots, Y_k]$. Therefore $g \in R[X; G]$.

References


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