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NOTE ON INTEGRAL CLOSURES OF SEMIGROUP RINGS

RYÛKI MATSUDA

Abstract. Let S be a subsemigroup which contains 0 of a torsion-free abelian (additive) group. Then S is called a grading monoid (or a g-monoid). The group $\{s-s'|s,s' \in S\}$ is called the quotient group of S, and is denored by $q(S)$. Let R be a commutative ring. The total quotient ring of R is denoted by $q(R)$. Throught the paper, we assume that a g-monoid properly contains $\{0\}$. A commutative ring is called a ring, and a non-zero-divisor of a ring is called a regular element of the ring.

We consider integral elements over the semigroup ring $R[X;S]$ of S over R. Let S be a g-monoid with quotient group G. If $n\alpha \in S$ for an element α of G and a natural number n implies $\alpha \in S$, then S is called an integrally closed semigroup. We know the following fact:

Theorem 1 ([G2, Corollary 12.11]). *Let D be an integral domain and S a g-monoid. Then $D[X;S]$ is integrally closed if and only if D is an integrally closed domain and S is an integrally closed semigroup.*

Let R be a ring. In this paper, we show that conditions for $R[X;S]$ to be integrally closed reduce to conditions for the polynomial ring of an indeterminate over a reduced total quotient ring to be integrally closed (Theorem 15). Clearly the quotient field of an integral domain is a von Neumann regular ring. Assume that $q(R)$ is a von Neumann regular ring. We show that $R[X;S]$ is integrally closed if and only if R is integrally closed and S is integrally closed (Theorem 20). Let G be a g-monoid which is a group. If R is a subring of the ring T which is integrally closed in T, we show that $R[X;G]$ is integrally closed in $T[X;S]$ (Theorem 13). Finally, let S be sub-g-monoid of a totally ordered abelian group. Let R be a subring of the ring T which is integrally closed in T. If g and h are elements of $T[X;S]$ with h monic and $gh \in R[X;S]$, we show that $g \in R[X;S]$ (Theorem 24).

1. General Rings

Let G be an abelian group. Then a maximal number n so that there exist a set of n-elements in G which is independent over $\mathbf{Z}$ is called the torsion-free rank of G, and is denoted by $t.f.r.(G)$.

Lemma 2. *Let G be a g-monoid which is finitely generated and with $t.f.r.(G) = n$. Then $R[X;G]$ is isomorphic to the ring $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ over R (where $X_1, \dots, X_n$ are indeterminates).*

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Proof. Since G is torsion-free, G is the direct sum $\mathbf{Z}u_1 \oplus \cdots \oplus \mathbf{Z}u_n$ for some elements u_i of G. Set $\sigma(X_i) = X^{u_i}$ for each i. Then we have an isomorphism σ of $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ onto $R[X; G]$.

Lemma 3. *Let f be an element of $R[X; S]$. Then f is a zero-divisor of $R[X; S]$ if and only if there exists a non-zero element a of R such that $aa_i = 0$ for every coefficient a_i of f.*

Proof. (1) Let f be a zero-divisor in $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$, then there exists a non-zero element a of R such that $af = 0$. For, there exists a natural number m such that $X^m f$ is a zero-divisor in $R[X_1, \ldots, X_n]$. By $[G1, (28.7)$ Proposition], there exists a non-zero element a of R such that $aX^m f = 0$, and hence $af = 0$.

(2) Let f be a zero-divisor in $R[X; S]$. Then f is a zero-divisor in $R[X; G]$, where $G = q(S)$. There exists a finitely generated subgroup H of G such that f is a zero-divisor in $R[X; H]$. $R[X; H]$ is isomorphic to the ring $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ for some n by Lemma 2. By (1), there exists a non-zero element a of R such that $af = 0$.

Let S be a g-monoid with quotient group G, and T a sub-g-monoid of S with quotient group $H \subset G$. Let R be a ring with total quotient ring K. Then all of R, $R[X; T]$, $K[X; T]$ and $q(R[X; T])$ are canonically regarded as subrings of $q(R[X; S])$ by Lemma 3.

Lemma 4([G1, (10.2) Proposition]). *Let R be a subring of the ring T, and A be the integral closure of R in T. If N is a multiplicative system in R, then the quotient ring A_N is the integral closure of R_N in T_N.*

Lemma 5. *If $R[X; S]$ is integrally closed, then R is integrally closed.*

Proof. Let x be an element of $K = q(R)$ which is integral over R. Since x is an element of $q(R[X; S])$ which is integral over $R[X; S]$, x belongs to $R[X; S]$, and hence $x \in R$. Therefore R is integrally closed.

Lemma 6. *If $R[X; S]$ is integrally closed, then S is integrally closed.*

Proof. Let α be an element of $q(S)$ which is integral over S. Since $n\alpha \in S$ for some natural number n, we have an integral equation of the element X^α of $q(R[X; S])$ over $R[X; S]$. It follows that $X^\alpha \in R[X; S]$, and hence $\alpha \in S$. Therefore S is integrally closed.

Lemma 7. *If $R[X; S]$ is integrally closed, then R is a reduced ring.*

Proof. Suppose that R has a non-zero nilpotent a. Take non-zero α of S. Then $1 + X^\alpha$ is a regular element of $R[X; S]$ by Lemma 3. Then $a/(1 + X^\alpha)$ is a nilpotent of $q(R[X; S])$, and $a/(1 + X^\alpha) \notin R[X; S]$; a contradiction. Hence R is reduced.

Lemma 8. *Let G be a g-monoid which is a group. Let $\{H_\lambda | \lambda\}$ be the set of finitely generated non-zero subgroups of G. Let A_λ be the integral closure of $R[X; H_\lambda]$. Then $\cup A_\lambda$ is the integral closure of $R[X; G]$.*

Proof. Let F be an element of $q(R[X;G])$ which is integral over $R[X;G]$. There exists a finitely generated subgroup H of G such that F is an element of $q(R[X;H])$ and F is integral over $R[X;H]$. Then we have $H = H_\lambda$ for some λ, and $F \in A_\lambda$. Hence the integral closure of $R[X;G]$ is $\cap A_\lambda$.

Lemma 9([BCL, Lemma 1]). *Let G be a g-monoid which is a group, and H a non-zero subgroup of G. Then $R[X;G]$ is a free $R[X;H]$-module. Let $\{\alpha_\lambda|\lambda\}$ be a system of complete representatives of G modulo H. Then $\{X^{\alpha_\lambda}|\lambda\}$ is a free basis of $R[X;G]$ over $R[X;H]$.*

Proposition 10. *Let G be a g-monoid which is a group. Then $R[X;G]$ is integrally closed if and only if, for every finitely generated non-zero subgroup H of G, $R[X;H]$ is integrally closed.*

Proof. The sufficiency follows from Lemma 8.

The necessity: Let F be an element of $q(R[X;H])$ which is integral over $R[X;H]$. Since $R[X;G]$ is integrally closed, we have $F \in R[X;G]$. Lemma 9 implies that $F \in R[X;H]$. Hence $R[X;H]$ is integrally closed.

Lemma 11. *Let $X_1, X_2, \dots$ be indeterminates. Let R_1 be the integral closure of $R[X_1, X_1^{-1}]$, let R_2 be the integral closure of $R_1[X_2, X_2^{-1}]$, R_3 be the integral closure of $R_2[X_3, X_3^{-1}], \dots$. Then R_n is the integral closure of $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$.*

Proof. We rely on the induction on n. Assume that R_{n-1} is the integral closure of $R[X_1, X_1^{-1}, \dots, X_{n-1}^{-1}, X_{n-1}^{-1}]$. Clearly $R_{n-1}[X_n, X_n^{-1}]$ is integral over $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. Hence R_n is integral over $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. Let F be an element of $q(R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1})$ which is integral over $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. Then F is integral over $R_{n-1}[X_n, X_n^{-1}]$. Hence $F \in R_n$. Therefore R_n is the integral closure of $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$.

Lemmas 8 and 11 show that to determine the integral closure of $R[X;G]$ reduces to determine the integral closures of $R'[X, X^{-1}]$ for some ring R'.

Lemma 12. *$R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ is integrally closed if and only if $R[X_1, \dots, X_n]$ is integrally closed.*

Proof. The sufficiency follows from Lemma 4.

The necessity: Let F be an element of $q(R[X_1, \dots, X_n])$ which is integral over $R[X_1, \dots, X_n]$. We have $F \in R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ by assumption. If $F \notin R[X_1, \dots, X_n]$, we may assume that $F = f_d X_n^d + f_{d+1} X_n^{d+1} + \cdots$, where each $f_i \in R[X_1, X_1^{-1}, \dots, X_{n-1}, X_{n-1}^{-1}]$, $f_d \neq 0$ and $d < 0$. R is reduced by Lemma 7. Hence there exists a prime ideal P of R such that $f_d \not\equiv 0 \bmod PR[X_1, X_1^{-1}, \dots, X_{n-1}, X_{n-1}^{-1}]$. Set $q(R/P) = k$. Then there arises an element of $k[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] - k[X_1, \dots, X_n]$ which is integral over $k[X_1, \dots, X_n]$; a contradiction. Therefore $R[X_1, \dots, X_n]$ is integrally closed.

Theorem 13. *Let G be a g-monoid which is a group. Let T be an extension ring of the ring R and let A be the integral closure of R in T. Then $A[X;G]$ is the integral closure of $R[X;G]$ in $T[X;G]$.*

Proof. (1) Let $X_1, \dots, X_n$ be a finite number of indeterminates. Then $A[X_1, \dots, X_n]$ is the integral closure of $R[X_1, \dots, X_n]$ in $T[X_1, \dots, X_n]$ (cf. [G1, (10.7) Theorem]).

(2) In (1), $A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ is the integral closure of $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ in $T[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. For, let N be the multiplicative system in $R[X_1, \dots, X_n]$ generated by $X_1, \dots, X_n$. Since $A[X_1, \dots, X_n]$ is the integral closure of $R[X_1, \dots, X_n]$ in $T[X_1, \dots, X_n]$ by (1), we see that the quotient ring $A[X_1, \dots, X_n]_N$ is the integral closure of $R[X_1, \dots, X_n]_N$ in $T[X_1, \dots, X_n]_N$ by Lemma 4. Hence $A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ is the integral closure of $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ in $T[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$.

(3) Assume that G is finitely generated. Then $A[X;G]$ is the integral closure of $R[X;G]$ in $T[X;G]$. For, $R[X;G]$ (resp. $A[X;G]$ and $T[X;G]$) is isomorphic to $R[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ (resp. $A[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ and $T[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$) for some n by Lemma 2. (2) implies that $A[X;G]$ is the integral closure of $R[X;G]$ in $T[X;G]$.

(4) Assume that G is a g-monoid which is a group. Let $F = \Sigma a_i X^{\alpha_i}$ be an element of $T[X;G]$ which is integral over $R[X;G]$. We have $F_m + F_{m-1} f_{m-1} + \cdots + f_0 = 0$ for the elements f_i of $R[X;G]$. There exists a finitely generated subgroup H of G such that F is integral over $R[X;H]$ and belongs to $T[X;H]$. (3) implies that $F \in A[X;H]$. Hence $A[X;G]$ is the integral closure of $R[X;G]$ in $T[X;G]$.

Proposition 14. *$R[X;S]$ is integrally closed if and only if S is integrally closed, R is integrally closed and $K[X;G]$ is integrally closed, where $K = q(R)$.*

Proof. The necessity: S is integrally closed by Lemma 6. R is integrally closed by Lemma 5. $K[X;G]$ is integrally closed by Lemma 4.

The sufficiency: Suppose that $R[X;S]$ is not integrally closed. There exists $F \in q(R[X;S]) - R[X;S]$ which is integral over $R[X;S]$. We have $F \in K[X;G]$ by assumption. Then we have $F \in R[X;G]$ by Theorem 13. Put $F = \Sigma a_i X^{\alpha_i}$. There exists k such that $a_k X^{\alpha_k} \notin R[X;S]$. R is reduced by assumption and by Lemma 7. Hence there exists a prime ideal P of R which does not contain a_k. Set $D = R/P$ and $\bar{a}_i = a_i + P \in D$ for each i. Then $\bar{F} = \Sigma \bar{a}_i X^{\alpha_i}$ is an element of $D[X;G] - D[X;S]$ which is integral over $D[X;S]$. Then $\bar{F}$ is an element of $k[X;G] - k[X;S]$ which is integral over $k[X;S]$, where $k = q(D)$. This contradicts to Theorem 1.

Let K be total quotient ring. We will denote the total quotient ring $q(K[X_1, \dots, X_n])$ of $K[X_1, \dots, X_n]$ by $K(X_1, \dots, X_n)$.

Theorem 15. *$R[X;S]$ is integrally closed if and only if S is integrally closed, R is integrally closed, $K[X_1]$ is integrally closed and $K(X_1, \dots, X_{n-1})[X_n]$ is integrally closed for every n with $n \leq t.f.r.(q(S))$, where $K = q(R)$.*

Proof. The necessity: S is integrally closed, R is integrally closed and $K[X;G]$ is integrally closed by Proposition 14. There exists a finitely generated subgroup H of G such that $t.f.r.(H) = n$. $K[X;H]$ is integrally closed by Proposition 10. Hence $(K[X_1, X_1^{-1},\ldots, X_{n-1}, X_{n-1}^{-1}])\ [X_n, X_n^{-1}]$ is integrally closed. Then $K(X_1,\ldots, X_{n-1})[X_n]$ is integrally closed by Lemma 12.

The sufficiency: We will show that $K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is integrally closed for every n with $n \leq t.f.r(G)$. Suppose that $K[X_1, X_1^{-1}, \ldots, X_k, X_k^{-1}]$ is integrally closed for $k < t.f.r.(G)$. Then $K[X_1^{-1}, \ldots, X_{k+1}, X_{k+1}^{-1}]$ is integrally closed by Proposition 14. Therefore $K[X;G]$ is integrally closed by Proposition 10. Then $R[X;S]$ is integrally closed by Proposition 14.

2. Von Neumann Regular Rings

Let R be a ring. If, for each element a of R, there exists an element b of R such that $a = a^2b$, then R is called a von Neumann regular ring. We confer [G1, §11] for von Neumann regular rings. Every field is clearly a von Neumann regular ring.

Lemma 16. *If R is a von Neumann regular ring, then $R[X]$ and $R[X, X^{-1}]$ are integrally closed.*

Proof. Let F be an element of $q(R[X])$ which is integral over $R[X]$. We have $F = f/g$, where f is an element of $R[X]$ and g is a regular element of $R[X]$. $R[X]$ is a Bezout ring, that is, every finitely generated ideal is principal by [GP, Corollary 3.1]. Hence there exist elements h, f', f_1, g_1 of $R[X]$ such that $f = hf'$, $g = hg'$, $h = ff_1+gg_1$, where h and g' are regular. Then we have $1/g' = Ff_1 + g_1$. Hence $1/g'$ is integral over $R[X]$. The integral equation of $1/g'$ over $R[X]$ shows that g' is a unit of $R[X]$. [G1, Corollary 11.4] implies that g' is a unit of R. Then we have $F = f/g = f'/g' \in R[X]$. Therefore $R[X]$ is integrally closed.

$R[X, X^{-1}]$ is integrally closed by Lemma 12.

Lemma 17. *If R is von Neumann regular ring, then $q(R[X;G])$ is a von Neumann regular ring.*

Proof. (1) Let $f = \Sigma_0^n a_i X^i$ be an element of $R[X]$. Then there exists $F \in q(R[X])$ such that $f = f^2F$. For the proof, we rely on the induction on n. Thus suppose that the assertion holds for fewer degrees. Assume that $a_n \neq 0$. There exists an idempotent e of R such that $Ra_n = Re$. Put $e' = 1 - e$. Since deg $(fe') < n$, there exist elements f_1 and g_1 of $R[X]$ with g_1e' regular in $Re'[X]$ such that $fe' = f^2f_1e'/(g_1e')$. Note that fe is regular in $Re[X]$. We see that $g_1e' + fe$ is a regular element of $R[X]$. Further we have $f = f^2(f_1e' + e)/(g_1e' + fe)$.

(2) $q(R[X_1, \ldots, X_n])$ is von Neumann regular. For the proof, we rely on the induction on n. (1) implies that $q(R[X_1])$ is von Neumann regular. Suppose that $q(R[X_1, \ldots, X_{n-1}])$ is von Neumann regular. It follows that $q(q(R[X_1, \ldots, X_{n-1}])[X_n])$ is von Neumann regular. That is, $q(R[X_1, \ldots, X_n])$ is von Neumann regular.

(3) Let f be an element of $q(R[X;G])$. There exists a finitely generated subgroup H of G such that $f \in q(R[X;H])$. Since $R[X;H]$ is isomorphic to the ring $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ for some n, $q(R[X;H])$ is von Neumann regular by (2). Hence there exists $F \in q(R[X;H])$ such that $f = f^2F$. Therefore $q(R[X;G])$ is von Neumann regular.

[G1, §11, Exercise 13] states that if R is its own total quotient ring and if R is reduced, then R is 0-dimensional. If this is the case, then $q(R[X;G])$ is a von Neumann regular ring for every reduced ring R. Now we have the following,

Example 18 (Gilmer and Matsuda). Let k be a field and let $X_1, X_2, \ldots$ be indeterminates.

(1) Set $R = k[[X_1, X_2, \ldots]]_1/(X_iX_j | i \neq j)$, where $k[[X_1, X_2, \ldots]]_1$ is the union of the ascending net of rings $k[[X_1, \ldots, X_n]]$ of all n. Then R is its own total quotient ring and reduced. But R is not 0-dimensional.

(2) Let $R = k[X_1, X_2, \ldots]/(X_iX_j | i \neq j)$ and $M = (X_1, X_2, \ldots)R$. Then R_M is its own total quotient ring and reduced. But R_M is not 0-dimensional.

Lemma 19. *If R is a von Neumann regular ring, then $R[X;G]$ is integrally closed.*

Proof. (1) $R[X, X^{-1}]$ is integrally closed by Lemma 16.

(2) $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ is integrally closed. For the proof, we rely on the induction on n. Suppose that $R[X_1, X_1^{-1}, \ldots, X_{n-1}, X_{n-1}^{-1}]$ is integrally closed. $q(R[X_1, \ldots, X_{n-1}])$ is von Neumann regular by Lemma 17. (1) implies that $q(R[X_1, \ldots, X_{n-1}])[X_n, X_n^{-1}]$ is integrally closed. Then Proposition 14 implies that $R[X_1, X_1^{-1}, \ldots, X_{n-1}, X_{n-1}^{-1}]$ is integrally closed.

(3) Let F by an element of $q(R[X;G])$ which is integral over $R[X;G]$. There exists a finitely generated subgroup H of G such that $F \in q(R[X;H])$ and F is integral over $R[X;H]$. Since $R[X;H]$ is isomorphic to the ring $R[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$ for some n, (2) implies that $F \in R[X;H]$. Therefore $R[X;G]$ is integrally closed.

Theorem 20. *Assume that $q(R)$ is a von Neumann regular ring. Then $R[X;S]$ is integrally closed if and only if S is integrally closed and R is integrally closed.*

Proof. The necessity is clear.

The sufficiency: Set $G = q(S)$ and $K = q(R)$. Then $K[X;G]$ is integrally closed by Lemma 19. Proposition 14 implies that $R[X;S]$ is integrally closed.

If R is a domain, then $q(R)$ is clearly a von Neumann regular ring.

3. A Theorem

Let G be a totally ordered abelian group. Let f be an element of $R[X;G]$. Put $f = a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n}$, where the a_i are non-zero elements of R and $\alpha_1 < \cdots < \alpha_n$. If $a_n = 1$, then f is called monic in $R[X;G]$.

Lemma 21. *Let $G = \mathbf{Z} \oplus \cdots \oplus \mathbf{Z}$ (the direct sum of n-copies of the additive group $\mathbf{Z}$) with the lexicographic order, and let $X_1 = X^{(1,0,\ldots,0)} < \cdots < X_n = X^{(0,\ldots,0,1)}$. Let T be a ring and R a subring of T which is integrally closed in T. Let g and h are elements of $T[X;G]$ with h monic and $gh \in R[X;G]$. Then $g \in R[X;G]$.*

Proof. We may assume that $g = g(X_1, \ldots, X_n)$ and $h = h(X_1, \ldots, X_n)$ belong to $T[X_1, \ldots, X_n]$. If $n = 1$, the assertion holds by [G1, (10.4) Theorem]. We rely on the induction on n. Suppose that the asertion holds for $n-1$. There exists a natural number m such that the coefficients of $g(X_1, \ldots, X_{n-1}, X_n)$ (resp. $h(X_1, \ldots, X_{n-1}, X_n)$) and $g(X_1, \ldots, X_{n-1}, X_{n-1}^m)$ (resp. $h(X_1, \ldots, X_{n-1}, X_{n-1}^m)$) are the same and $h(X_1, \ldots, X_{n-1}, X_{n-1}^m)$ is monic. Since $g(X_1, \ldots, X_{n-1}, X_{n-1}^m)h(X_1, \ldots, X_{n-1}X_{n-1}^m) \in R[X_1, \ldots, X_{n-1}]$, we have $g(X_1, \ldots, X_{n-1}, X_{n-1}^m) \in R[X_1, \ldots, X_{n-1}]$ by hypothesis. Hence $g \in R[X;G]$.

Lemma 22. *Let G be a finitely gnerated subgroup of the totally ordered abelian group $\mathbf{R}$. Let T be a ring and R a subring which is integrally closed in T. If, for elements g and h of $T[X;G]$ with h monic, gh belongs to $R[X;G]$, then g belongs to $R[X;G]$.*

Proof. There exist real numbers $\pi_1, \ldots, \pi_n$ so that $G = Z\pi_i + \cdots + Z\pi_n$ and $\{\pi_1, \ldots, \pi_n\}$ is independent over $\mathbf{Z}$. We may assume that $0 < \pi_1 < \cdots < \pi_n$. Set $X_i = X^{\pi_i}$ for each i. We may assume that $g = g(X_1, \ldots, X_n)$ and $h = h(X_1, \ldots, X_n)$ are elements of $T[X_1, \ldots, X_n]$. Arrange the powers of g and h as follows: $\Sigma k_{1i}\pi_i < \cdots < \Sigma k_{mi}\pi_i$. It follows that $\Sigma k_{1i}\pi_i/\pi_1 < \cdots < \Sigma k_{mi}\pi_i/\pi_1$.

Since all of $\pi_2/\pi_1, \ldots, \pi_n/\pi_1$ are irrational, there exist positive rational numbers $a_1 = 1, a_2, \ldots, a_n$ such that $\Sigma k_{1i}a_i < \cdots < \Sigma k_{mi}a_i$. Hence there exist positive integers $p_1, \ldots, p_n$ such that $\Sigma k_{1i}p_i < \cdots < \Sigma k_{mi}p_i$. Note that, if $(k_1, \ldots, k_n) \neq (l_1, \ldots, l_n)$, then $\Sigma k_i\pi_i \neq \Sigma l_i\pi_i$. Then the coefficients of $g(X_1, \ldots, X_n)$ (resp. $h(X_1, \ldots, X_n)$) and $g(Y^{p1}, \ldots, Y^{pn})$ (resp. $h(Y^{p1}, \ldots, Y^{pn})$) are the same and $h(Y^{p1}, \ldots, Y^{pn})$ is monic (where Y is an anothr indeterminate). Since $g(Y^{p1}, \ldots, Y^{pn})h(Y^{p1}, \ldots, Y^{pn}) \in R[Y]$, we have $g(Y^{p1}, \ldots, Y^{pn}) \in R[Y]$. Hence $g \in R[X;G]$.

Theorem 23. *Let S be a sub-g-monoid of a totally ordered abelian group. Let R be a subring of the ring T, and let A be the integral closure of R in T. If, for elements g and h of $T[X,S]$ with h monic, gh belongs to $R[X,S]$, then $g \in A[X;S]$.*

Proof. We may assume that $G = S$ is a finitely generated group and R is integrally closed in T. We may assume that $G = H_1 \oplus \cdots \oplus H_k$ with the lexicographic order, where the H_i are non-zero subgroups of the totally ordered abelian group $\mathbf{R}$. Since G is finitely generated, we may use a similar argument to [ZS, VI, (A)]. We rely on the induction on $t.f.r.(G)$. Assume that $t.f.r.(G) = r$, and suppose that the assertion holds for fewer torsion-free ranks. The powers of g and h are of the form $(h_1, \ldots, h_k)$ for the $h_i \in H_i$. Suppose that each k-component h_k is zero for every power of g and h. Then we have $g \in R[X;G]$ by induction. Suppose that h_k is non-zero for some power of g or h. Let $H_i = Z\pi_{i1} + \cdots + Z\pi_{in(i)}$, where the set $\{\pi_{i1}, \ldots, \pi_{in(i)}\}$ is independent over $\mathbf{Z}$ for each

i. We may assume that $0 < \pi_{i1} < \cdots < \pi_{in(i)}$ for each i. Put $X_{ij} = X^{(0,\ldots,\pi_{ij},0,\ldots)}$ for each i and j. We may assume that g and h belong to $T[X_{11},\ldots,X_{kn(k)}]$.

The case of $n(i) = 1$ for each i: Then G is order-isomorphic to $Z \oplus \cdots \oplus Z$ (the didrect sum of k-copies of $\mathbf{Z}$) with the lexicographic order. By Lemma 21, we have $g \in R[X; G]$.

The case of $n(i) > 1$ for some i: For each l, arrange the l-components of g and h as follows: $\Sigma k_{l1i}\pi_{li} < \Sigma k_{l2i}\pi_{li} < \cdots < \Sigma k_{lm(l)i}\pi_{li}$. Then, as in the proof of Lemma 22, there exist positive integers p_{ij} such that $\Sigma k_{l1i}p_{li} < \Sigma k_{l2i}p_{li} < \cdots < \Sigma k_{lm(l)i}p_{li}$. Let $Y_1,\ldots,Y_k$ be another indeterminates. Then the coefficients of $g = g(X_{11},\ldots,X_{kn(k)})$ (resp. $h = h(X_{11},\ldots,X_{kn(k)})$ and $g(Y_1^{p_{11}},\ldots,Y_1^{p_{1n}(1)},\ldots,Y_k^{p_{k1}},\ldots,Y_k^{p_{kn}(k)})$ (resp. $h(Y_1^{p_{11}},\ldots,Y_1^{p_{1n}(1)},\ldots, Y_k^{p_{k1}},\ldots,Y_k^{p_{kn}(k)})))$ are the same and $h(Y_1^{p_{11}},\ldots,Y_1^{p_{1n}(1)},\ldots, Y_k^{p_{k1}},\ldots,Y_k^{p_{kn}(k)})$ is monic. Since $g(Y_1^{p_{11}},\ldots,Y_1^{p_{1n}(1)},\ldots,Y_k^{p_{k1}},\ldots,Y_k^{p_{kn}(k)})h(Y_1^{p_{11}},\ldots, Y_1^{p_{1n}(1)},\ldots,Y_k^{p_{k1}},\ldots,Y_k^{p_{kn}(k)}) \in R[Y_1,\ldots,Y_k]$, we have $g(Y_1^{p_{11}},\ldots,Y_1^{p_{1n}(1)},\ldots,Y_k^{p_{k1}}, \ldots,Y_k^{p_{kn}(k)})h(Y_1^{p_{11}},\ldots,Y_1^{p_{1n}(1)},\ldots,Y_k^{p_{k1}},\ldots,Y_k^{p_{kn(k)}}) \in R[Y_1,\ldots,Y_k]$. Hence $g(Y_1^{p_{11}}, \ldots,Y_1^{p_{1n}(1)},\ldots,Y_k^{p_{k1}},\ldots,Y_k^{p_{kn}(k)}) \in R[Y_1,\ldots,Y_k]$. Therefore $g \in R[X; G]$.

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Department of Mathematics, Ibaraki University, Mito, Ibaraki 310, Japan.