TAMKANG JOURNAL OF MATHEMATICS
Volume 54, Number 2, 155-173, 2023
DOI:10.5556/j.tkjm.54.2023.4061

Refinements of some numerical radius inequalities for Hilbert space operators

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Abstract. Some power inequalities for the numerical radius based on the recent Dragomir extension of Furuta's inequality are established. Some particular cases are also provided. Moreover, we get an improvement of the Hölder-McCarthy operator inequality in the case when $r \geq 1$ and refine generalized inequalities involving powers of the numerical radius for sums and products of Hilbert space operators.

Keywords. Numerical radius, convex function operator, Mixed Schwarz inequality, Furuta inequality, Young inequality

1 Introduction

In what follows $\mathcal{H}$ denotes a complex Hilbert space equipped with the inner product $\langle .,. \rangle$ and corresponding norm $\|x\| = \langle x, x\rangle^{1/2}$ and $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ with identity I. An operator $A \in \mathcal{B}(\mathcal{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we then write $A \geq 0$. In addition, we write $A > 0$ if A is a positive invertible operator. For $A, B \in \mathcal{B}(\mathcal{H})$ we say $B \geq A$ if $B - A \geq 0$. For $T \in B(\mathcal{H})$, set $|T| = (T^*T)^{\frac{1}{2}}$ as usual. By taking $U|T|x = Tx$ for $x \in \mathcal{H}$ and $Ux = 0$ for $x \in \ker|T|$, T has a unique polar decomposition $T = U|T|$ with $\ker U = \ker|T|$.
If T is a positive self-adjoint on $\mathcal{H}$, then the following inequality is a generalization of the Schwarz inequality on $\mathcal{H}$

$$|\langle Tx, y\rangle|^2 \leq \langle Tx, x\rangle \langle Ty, y\rangle \tag{1.1}$$

for any $x, y \in \mathcal{H}$. In 1952, Kato [7] proved the following celebrated generalization of Schwarz inequality for any operator $T \in \mathcal{B}(\mathcal{H})$:

$$|\langle Tx, y\rangle|^2 \leq \left\langle |T|^{2\alpha}x, x\right\rangle \left\langle |T^*|^{2(1-\alpha)}y, y\right\rangle \tag{1.2}$$

for any $x, y \in \mathcal{H}$ and $0 \leq \alpha \leq 1$.
In order to generalize this result, in 1994 Furuta [5] obtained the following result:

$$|\left\langle T|T|^{\alpha+\beta-1}x, y\right\rangle|^2 \leq \left\langle |T|^{2\alpha}x, x\right\rangle \left\langle |T^*|^{2\beta}y, y\right\rangle \tag{1.3}$$

Received date: December 21, 2020; Published online: October 14, 2022.
2010 *Mathematics Subject Classification*. 47A12, 47A30, 47B15, 47A63.

for any $x, y \in \mathcal{H}$ and $0 \leq \alpha, \beta \leq 1$ with $\alpha + \beta \geq 1$.
The inequality (1.3) was generalized for any $\alpha, \beta \geq$ with $\alpha+\beta \geq 1$ by Dragomir in [4]. Indeed, as noted by Dragomir the condition $0 \leq \alpha, \beta \leq 1$ was assumed by Furuta to fit with the Heinz-Kato inequality, which reads:

$$|\langle Tx, y\rangle| \leq \|A^{\alpha}\| \left\|B^{1-\alpha}\right\| \tag{1.4}$$

for any $x, y \in \mathcal{H}$ and $0 \leq \alpha \leq 1$, where A and B are positive operators such that $\|Tx\| \leq \|Ax\|$ and $\|T^*y\| \leq \|By\|$ for any $x, y \in \mathcal{H}$. In the same work [4], Dragomir provides a useful extension of Furuta's inequality, as follows:

$$|\langle DCBAx, y\rangle|^2 \leq \left\langle A^*|B|^2 Ax, x\right\rangle \left\langle D|C^*|^2 D^* y, y\right\rangle \tag{1.5}$$

for any $D, C, B, A \in \mathcal{B}(\mathcal{H})$ and any vectors $x, y \in \mathcal{H}$. The equality in (1.5) holds if and only if the vectors BAx and $(DC)^*y$ are linearly dependent in $\mathcal{H}$.
For a bounded linear operator $A \in \mathcal{B}(\mathcal{H})$ we use the operator norm $\|A\|$ and denote by $W(A)$ its numerical range :

$$\begin{aligned} \|A\| &= \sup\{|\langle Ax, y\rangle| : x, y \in \mathcal{H}, \|x\| = \|y\| = 1\} \\ W(A) &= \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}. \end{aligned}$$

Recall that the numerical range is a convex subset of $\mathbb{C}$ and that its closure contains the spectrum of A.

The numerical radius of A, denoted by $w(A)$, is given by

$$w(A) = \sup\{|\langle Ax, x\rangle| : x \in \mathcal{H}, \|x\| = 1\}.$$

It is well-known that $w(A)$ defines a norm on $\mathcal{B}(\mathcal{H})$ which is equivalent to the usual operator norm $\|A\|$. In fact for $A \in \mathcal{B}(\mathcal{H})$, we have

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.6}$$

Also if $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then $w(A) = \|A\|$.
An important inequality for $w(A)$ is the power inequality stating that

$$w(A^n) \leq w^n(A) \text{ for } \quad n = 1, 2, \cdots.$$

Several numerical radius inequalities improving the inequalities in (1.6) have been recently given in [1, 2, 9, 18]. An interesting numerical radius inequality has been established by Rashid [18], it has been shown that if $T, S, X \in \mathcal{B}(\mathcal{H})$ such that T and S are positive, then for any $r \geq 1$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$w^r\left(T^{\frac{1}{p}} X S^{\frac{1}{q}}\right) \leq \|X\|^r \left\|\frac{1}{p}T^r + \frac{1}{q}S^r\right\|. \tag{1.7}$$

Recently, Shebrawi and Albadawi [19] proved that if $A_i, B_i \in \mathcal{B}(\mathcal{H})$ $(i = 1, \cdots, n)$ and $r \geq 1$, then

$$w^r\left(\sum_{i=1}^n A_i^* B_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^n \left(|A|^{2r} + |B|^{2r}\right)\right\|. \tag{1.8}$$

Very recently, Dragomir [4] showed that if $A, B, C, D \in \mathcal{B}(\mathcal{H})$, then for any $r \geq 1$ we have

$$w^r(DCBA) \leq \frac{1}{2}\left\||BA|^{2r} + |(DC)^*|^{2r}\right\|. \tag{1.9}$$

Motivated by the above results, we establish in this paper some power inequalities for the numerical radius based on the recent Dragomir extension of Furuta's inequality are obtained. Also, we get an improvement of the Hölder-McCarthy operator inequality in the case when $r \geq 1$ In addition, we establish some improvements of norm and numerical radius inequalities for sums and powers of operators acting on a Hilbert space.

2 Preliminaries

To prove our generalized numerical radius inequalities, we need several well-known lemmas. The first lemma follows from the spectral theorem for positive operators and Jensen's inequality.

Lemma 2.1. *(Hölder Mc-Carty inequality). Let $A \in \mathcal{B}(\mathcal{H})$, $A \geq 0$ and let $x \in \mathcal{H}$ be any unit vector. Then we have*

(i) $\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ *for* $r \geq 1$.

(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for* $0 < r \leq 1$.

The second lemma is a simple consequence of the classical Jensen's inequality concerning the convexity or concavity of certain power functions. It is a special case of Schlömilch's inequality for weighted means of nonnegative real numbers.

Lemma 2.2. *Let $a, b > 0$ and $0 \leq \alpha \leq 1$. Then*

$$a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \leq (\alpha a^r + (1-\alpha) b^r)^{\frac{1}{r}} \quad \text{for} \quad r \geq 1. \tag{2.1}$$

The next lemma is an immediate consequence of the spectral theorem for self-adjoint operators concerning the convexity or concavity of certain function (see e.g., [9]).

Lemma 2.3. *Let f be a convex function defined on a real interval I. Then for every self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$ whose spectrum $\sigma(A) \subset I$, we have*

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle \quad \text{for all } x \in \mathcal{H}. \tag{2.2}$$

Moreover, if f is concave, then inequality (2.2) is reversed.

Kittaneh and Manasrah [13] obtained the following result which is a refinement of the scalar Young inequality.

Lemma 2.4. *Let $a, b > 0$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab + r_0(a^{\frac{p}{2}} - b^{\frac{q}{2}})^2 \leq \frac{a^p}{p} + \frac{a^q}{q}, \tag{2.3}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$.

Manasrah and Kittaneh have generalized (2.3) in [14], as follows:

Lemma 2.5. *Let $a, b > 0$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $m = 1, 2, \cdots$, we have*

$$(a^{\frac{1}{p}} b^{\frac{1}{q}})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq \left(\frac{a^r}{p} + \frac{b^r}{q}\right)^{\frac{m}{r}}, \quad r \geq 1 \tag{2.4}$$

where $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$. *In particular, if* $p = q = 2$, *then*

$$(\sqrt{ab})^m + \frac{1}{2^m}(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \le 2^{-\frac{m}{r}}(a^r + b^r)^{\frac{m}{r}}. \tag{2.5}$$

For $m = 1$, *and* $p = q = 2$, *we have*

$$\sqrt{ab} + \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \le 2^{-\frac{1}{r}}(a^r + b^r)^{\frac{1}{r}}. \tag{2.6}$$

The following fact concerning convexifiable functions plays an important role in our discussion (see [21, Corollary 2.8]): If f is twice continuously differentiable, then $\lambda = \min_{t\in I} f''(t)$ is a convexifier of f.

Lemma 2.6 ([16]). *Let* f *be a twice differentiable on* $[a, b]$. *If* f *is convex such that* $\lambda := \inf_{x\in[a,b]} f''(x) > 0$. *Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{8}\lambda(b-a)^2. \tag{2.7}$$

3 Extensions of the Dragomir and Furuta inequality

In this section we provide some key lemmas which are play the main role in the proof of our main results. First, we begain with the following result provides a simple however useful extension for four operators of the Schwarz inequality was established by Dragomir [4].

Lemma 3.1. *Let* $A, B, C, D \in \mathcal{B}(\mathcal{H})$. *Then for* $x, y \in \mathcal{B}(\mathcal{H})$, *we have the inequality*

$$|\langle DCBAx, y\rangle|^2 \le \langle A^*|B|^2Ax, x\rangle \langle D|C^*|^2D^*y, y\rangle \tag{3.1}$$

The equality case holds in (3.1) if and only if the vectors BAx *and* C^*D^*y *are linearly dependent in* $\mathcal{H}$.

The following results are an easy consequence of Lemma 3.1 and Cauchy-Schwarz inequality.

Corollary 3.1. *Let* $S, T, X \in \mathcal{B}(\mathcal{H})$ *such that* S *and* T *are positive. Then for all* $x, y \in \mathcal{H}$ *and* $0 \le \alpha, \beta \le 1$, *we have*

$$\begin{aligned} |\langle S^\alpha XT^\beta x, y\rangle|^2 &\le \langle T^\beta|X|^2T^\beta x, x\rangle \langle S^{2\alpha}y, y\rangle \\ &\le \|X\|^2 \langle T^{2\beta}x, x\rangle \langle S^{2\alpha}y, y\rangle. \end{aligned} \tag{3.2}$$

with equality in the first inequality if and only if the vectors $XT^\beta x$ *and* $S^\alpha y$ *are linearly dependent in* $\mathcal{H}$.

Proof. Letting $A = T^\beta, B = X, C = S^\alpha$ and $D = I$ in Lemma 3.1,

$$\begin{aligned} |\langle S^\alpha XT^\beta x, y\rangle|^2 &\le \langle T^\beta|X|^2T^\beta x, x\rangle \langle S^{2\alpha}y, y\rangle = \langle XT^\beta x, XT^\beta x\rangle \langle S^{2\alpha}y, y\rangle \\ &\le \left\|XT^\beta x\right\|^2 \langle S^{2\alpha}y, y\rangle \le \|X\|^2 \left\|T^\beta x\right\|^2 \langle S^{2\alpha}y, y\rangle \\ &= \|X\|^2 \langle T^{2\beta}x, x\rangle \langle S^{2\alpha}y, y\rangle. \end{aligned}$$

□

Corollary 3.2 ([4]). *For any operator $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$, we have the inequality*

$$|\left\langle T|T|^{\alpha-1}T|T|^{\beta-1}x, y\right\rangle|^2 \leq \left\langle |T|^{2\alpha}x, x\right\rangle \left\langle |T^*|^{2\beta}y, y\right\rangle \tag{3.3}$$

for any $\alpha, \beta \geq 1$.

Corollary 3.3 ([4]). *For any operator $T \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$, we have the inequality*

$$|\left\langle T^*|T^*|^{\alpha+\beta-2}Tx, y\right\rangle|^2 \leq \left\langle |T|^{2\alpha}x, x\right\rangle \left\langle |T|^{2\beta}y, y\right\rangle \tag{3.4}$$

for any $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 2$.

Lemma 3.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and let f be a positive, increasing and convex functions on an interval $I \subset \mathbb{R}^+$. If f is twice differentiable such that $\lambda = \inf_{t\in I} f''(x)$, then*

$$\begin{aligned} f\left(\left|\left\langle T|T|^{\alpha-1}T|T|^{\beta-1}x, y\right\rangle\right|\right) &\leq \frac{1}{2}\left(\left\langle f(|T|^{2\alpha})x, x\right\rangle + \left\langle f(|T^*|^{2\beta})y, y\right\rangle\right) \\ &- \frac{1}{8}\lambda\left(\left\langle |T|^{2\alpha}x, x\right\rangle - \left\langle |T^*|^{2\beta}y, y\right\rangle\right)^2 \end{aligned} \tag{3.5}$$

for every $x, y \in \mathcal{H}$ and any $\alpha, \beta \geq 1$.

Proof. Employing the monotonicity and convexity of f for the inequality (3.3), we have

$$\begin{aligned} f\left(\left|\left\langle T|T|^{\alpha-1}T|T|^{\beta-1}x, y\right\rangle\right|\right) &\leq f\left(\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} \left\langle |T^*|^{2\beta}y, y\right\rangle^{\frac{1}{2}}\right) \quad (f \text{ increasing}) \\ &\leq f\left(\frac{\left\langle |T|^{2\alpha}x, x\right\rangle + \left\langle |T^*|^{2\beta}y, y\right\rangle}{2}\right) \quad (\text{by AM-GM}) \\ &\leq \frac{f(\left\langle |T|^{2\alpha}x, x\right\rangle) + f(\left\langle |T^*|^{2\beta}y, y\right\rangle)}{2} \\ &- \frac{1}{8}\lambda\left(\left\langle |T|^{2\alpha}x, x\right\rangle - \left\langle |T^*|^{2\beta}y, y\right\rangle\right)^2 \quad (\text{by Lemma 2.6}) \\ &\leq \frac{1}{2}\left(\left\langle f(|T|^{2\alpha})x, x\right\rangle + \left\langle f(|T^*|^{2\beta})y, y\right\rangle\right) \\ &- \frac{1}{8}\lambda\left(\left\langle |T|^{2\alpha}x, x\right\rangle - \left\langle |T^*|^{2\beta}y, y\right\rangle\right)^2 \quad (\text{by Lemma 2.3}) \end{aligned}$$

for all $x, y \in \mathcal{H}$ and any $\alpha, \beta \geq 1$. □

In a similar way, we can prove the following result:

Lemma 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and let f be a positive, increasing and convex functions on an interval I. If f is twice differentiable such that $\lambda = \inf_{t\in I} f''(x)$, then*

$$\begin{aligned} f(|\left\langle T^*|T^*|^{\alpha+\beta-2}Tx, y\right\rangle|) &\leq \frac{1}{2}\left(\left\langle f(|T|^{2\alpha})x, x\right\rangle + \left\langle f(|T|^{2\beta})y, y\right\rangle\right) \\ &- \frac{1}{8}\lambda\left(\left\langle |T|^{2\alpha}x, x\right\rangle - \left\langle |T|^{2\beta}y, y\right\rangle\right)^2 \end{aligned} \tag{3.6}$$

for all $x, y \in \mathcal{H}$ and any $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 2$.

Recall that a function $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$, where $\mathbb{R}^+=[0,\infty)$ is said to be sub-multiplicative if it satisfies the inequality

$$f(ts)\le f(t)f(s)\quad\text{for all }\ t,s\in\mathbb{R}^+. \tag{3.7}$$

Lemma 3.4. *Let $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a positive, increasing, convex and sub-multiplicative i.e., $f(ts)\le f(t)f(s)$ for all $t,s\in\mathbb{R}$ and let $T\in\mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} f(|\langle T|T|^{\alpha-1}T|T|^{\beta-1}x,y\rangle|^2) &\le \frac{1}{p}\langle f^p(|T|^{2\alpha})x,x\rangle+\frac{1}{q}\langle f^q(|T^*|^{2\beta})y,y\rangle \\ &- r_0\left(\langle f(|T|^{2\alpha})x,x\rangle^{\frac{p}{2}}-\langle f(|T^*|^{2\beta})y,y\rangle^{\frac{q}{2}}\right)^2 \end{aligned} \tag{3.8}$$

for all $x,y\in\mathcal{H}$, $\alpha,\beta\ge0$ with $\alpha+\beta\ge2$ and $\frac{1}{p}+\frac{1}{q}=1$.

Lemma 3.5. *Let $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a positive, increasing, convex and sub-multiplicative i.e., $f(ts)\le f(t)f(s)$ for all $t,s\in\mathbb{R}$ and let $T,S,X\in\mathcal{B}(\mathcal{H})$ such that T and S are positive. Then*

$$\begin{aligned} f\left(|\langle T^\alpha XS^\beta x,y\rangle|^2\right) &\le 2^{-\frac{2}{r}}f(\|X\|)\left(\langle f^r(S^{2\beta})x,x\rangle+\langle f^r(T^{2\alpha})y,y\rangle\right)^{\frac{2}{r}} \\ &- \frac{1}{4}f(\|X\|)\left(\langle f(S^{2\beta})x,x\rangle-\langle f(T^{2\alpha})y,y\rangle\right)^2 \end{aligned} \tag{3.9}$$

for all $x,y\in\mathcal{H}$, $0\le\alpha,\beta\le1$, and $r\ge1$. In particular case if $r=1$ and $m=2$, we have

$$\begin{aligned} f\left(|\langle T^\alpha XS^\beta x,y\rangle|^2\right) &\le \frac{1}{4}f(\|X\|)\left(\langle f(S^{2\beta})x,x\rangle+\langle f(T^{2\alpha})y,y\rangle\right)^2 \\ &- \frac{1}{4}f(\|X\|)\left(\langle f(S^{2\beta})x,x\rangle-\langle f(T^{2\alpha})y,y\rangle\right)^2 \end{aligned} \tag{3.10}$$

Proof. Since f is increasing and convex, then by applying Corollary 3.1, with $p=q=2$, we get

$$\begin{aligned} f(|\langle T^\alpha XS^\beta x,y\rangle|^2) &\le f\left(\|X\|^2\langle S^{2\beta}x,x\rangle\langle T^{2\alpha}y,y\rangle\right) \qquad (f\text{ increasing}) \\ &\le f(\|X\|^2)\left(f(\langle S^{2\beta}x,x\rangle)f(\langle T^{2\alpha}y,y\rangle)\right) \qquad (f\text{ sub-multiplicative}) \\ &\le f(\|X\|^2)\left(\langle f(S^{2\beta})x,x\rangle\langle f(T^{2\alpha})y,y\rangle\right) \qquad (\text{by Lemma 2.3}) \\ &\le 2^{-\frac{2}{r}}f(\|X\|^2)\left(\langle f(S^{2\beta})x,x\rangle^r+\langle f(T^{2\alpha})y,y\rangle^r\right)^{\frac{2}{r}} \\ &- \frac{1}{4}f(\|X\|^2)\left(\langle f(S^{2\beta})x,x\rangle-\langle f(T^{2\alpha})y,y\rangle\right)^2 \qquad (\text{by Lemma 2.5}) \\ &\le 2^{-\frac{2}{r}}f(\|X\|^2)\left(\langle f^r(S^{2\beta})x,x\rangle+\langle f^r(T^{2\alpha})y,y\rangle\right)^{\frac{2}{r}} \\ &- \frac{1}{4}f(\|X\|^2)\left(\langle f(S^{2\beta})x,x\rangle-\langle f(T^{2\alpha})y,y\rangle\right)^2 \end{aligned}$$

□

Lemma 3.6. *Let $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a positive, increasing, convex and sub-multiplicative i.e., $f(ts)\le f(t)f(s)$ for all $t,s\in\mathbb{R}$ and let $T\in\mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned} f\left(|\langle T|T|^{\alpha-1}T|T|^{\beta-1}x,y\rangle|^2\right) &\le 2^{-\frac{2}{r}}\left(\langle f^r(|T|^{2\alpha})x,x\rangle+\langle f^r(|T^*|^{2\beta})y,y\rangle\right)^{\frac{2}{r}} \\ &- \frac{1}{4}\left(\langle f(|T|^{2\alpha})x,x\rangle-\langle f(|T^*|^{2\beta})y,y\rangle\right)^2 \end{aligned} \tag{3.11}$$

for all $x, y \in \mathcal{H}$, $\alpha, \beta \geq 1$, and $r \geq 1$. In particular case if $r = 1$, we have

$$\begin{aligned} f\left(|\left\langle T|T|^{\alpha-1}T|T|^{\beta-1}x, y\right\rangle|^2\right) &\leq \frac{1}{4}\left(\left\langle f(|T|^{2\alpha})x, x\right\rangle + \left\langle f(T^{2\beta})y, y\right\rangle\right)^2 \\ &- \frac{1}{4}\left(\left\langle f(|T|^{2\alpha})x, x\right\rangle - \left\langle f(|T^*|^{2\beta})y, y\right\rangle\right)^2 \end{aligned} \tag{3.12}$$

4 Numerical radius inequalities

In this section we provide some numerical radius inequalities. Let us begin with the following result. The following theorem gives us a new bound for powers of the numerical radius.

Theorem 4.1. *Let $T, S, X \in \mathcal{B}(\mathcal{H})$ such that T and S are positive and let $f : I \subseteq \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be an increasing, convex and sub-multiplicative, i.e., $f(ts) \leq f(t)f(s)$ for all $s, t \in \mathbb{R}$. If f is twice differentiable such that $\lambda = \inf\limits_{t \in I} f''(x)$, then for every $\alpha, \beta \in [0, 1]$, we have*

$$f(w(S^\alpha X T^\beta)) \leq \frac{1}{2} f(\|X\|) w\left(f(T^{2\beta}) + f(S^{2\alpha})\right) - \inf_{\|x\|=1} \psi(x), \tag{4.1}$$

where $\psi(x) = \frac{1}{16} f(\|X\|)\lambda(\langle [T^{2\beta} - S^{2\alpha}]x, x\rangle)^2$.

Proof. Employing the monotonicity and convexity of f for the inequality (3.2), we have

$$\begin{aligned} f\left(\left|\left\langle S^\alpha X T^\beta x, y\right\rangle\right|\right) &\leq f\left(\|X\| \left\langle T^{2\beta}x, x\right\rangle^{\frac{1}{2}} \left\langle S^{2\alpha}y, y\right\rangle^{\frac{1}{2}}\right) \quad (f \text{ is increasing}) \\ &\leq f(\|X\|) f\left(\left\langle T^{2\beta}x, x\right\rangle^{\frac{1}{2}} \left\langle S^{2\alpha}y, y\right\rangle^{\frac{1}{2}}\right) \quad (\text{ by sub-multiplicativity of } f) \\ &\leq f(\|X\|) f\left(\frac{\left\langle T^{2\beta}x, x\right\rangle + \left\langle S^{2\alpha}y, y\right\rangle}{2}\right) \quad (\text{by AM-GM}) \\ &\leq f(\|X\|)\left[\frac{f\left(\left\langle T^{2\beta}x, x\right\rangle\right) + f\left(\left\langle S^{2\alpha}y, y\right\rangle\right)}{2}\right. \\ &- \left.\frac{1}{16}\lambda(\left\langle T^{2\beta}x, x\right\rangle - \left\langle S^{2\alpha}y, y\right\rangle)^2\right] \quad (\text{by Lemma 2.6}) \\ &\leq f(\|X\|)\left[\frac{1}{2}\left[\left\langle f(T^{2\beta})x, x\right\rangle + \left\langle f(S^{2\alpha})y, y\right\rangle\right]\right. \\ &- \left.\frac{1}{16}\lambda(\left\langle T^{2\beta}x, x\right\rangle - \left\langle S^{2\alpha}y, y\right\rangle)^2\right] \quad (\text{by Lemma 2.3}) \end{aligned}$$

for all $x, y \in \mathcal{H}$. Letting $y = x$, we have

$$\begin{aligned} & f(|\left\langle S^\alpha X T^\beta x, x\right\rangle|) \\ \leq\ & \frac{1}{2} f(\|X\|)\left\{\left[\left\langle f(T^{2\beta})x, x\right\rangle + \left\langle f(S^{2\alpha})x, x\right\rangle\right] - \frac{1}{8}\lambda\left(\left\langle T^{2\beta}x, x\right\rangle - \left\langle S^{2\alpha}x, x\right\rangle\right)^2\right\} \\ =\ & \frac{1}{2} f(\|X\|)\left\{\left[\left\langle (f(T^{2\beta}) + f(S^{2\alpha}))x, x\right\rangle\right] - \frac{1}{8}\lambda(\left\langle [T^{2\beta} - S^{2\alpha}]x, x\right\rangle)^2\right\}. \end{aligned}$$

Taking the supremum over all unit vector $x \in \mathcal{H}$, we get the required result. □

Note that our inequality in the previous theorem is a generalization and refinement of an inequality shown in [20, Theorem 3.3].

Theorem 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and let f be a positive, increasing and convex functions on an interval I. If f is twice differentiable such that $\lambda = \inf_{t\in I} f''(x)$, then for every $\alpha, \beta \geq 1$, we have*

$$f(w(T|T|^{\alpha-1}T|T|^{\beta-1})) \leq \frac{1}{2}\left\|f(|T|^{2\alpha}) + f(|T^*|^{2\beta})\right\| - \inf_{\|x\|=1} \xi(x), \tag{4.2}$$

where $\xi(x) = \frac{1}{8}\lambda(\langle[|T|^{2\alpha} - |T^*|^{2\beta}]x, x\rangle)^2$. In particular case, if $\alpha = \beta = 1$, then

$$f(w(T^2)) \leq \frac{1}{2}\left\|f(|T|^2) + f(|T^*|^2)\right\| - \inf_{\|x\|=1} \xi(x),$$

where $\xi(x) = \frac{1}{8}\lambda(\langle[|T|^2 - |T^*|^2]x, x\rangle)^2$.

Proof. The proof is similar to proof of Theorem 4.1, so we omit it. □

Theorem 4.3. *Let $T, S, X \in \mathcal{B}(\mathcal{H})$ such that T and S are positive. Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be an increasing, convex and sub-multiplicative, i.e., $f(ts) \leq f(t)f(s)$ for all $s, t \in \mathbb{R}$. Then for all $x \in \mathcal{H}$, $0 \leq \alpha, \beta \leq 1, p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$f(w^2(T^\alpha X S^\beta)) \quad = \quad f(\|X\|^2) w\left(\frac{1}{p} f^p(T^{2\beta}) + \frac{1}{q} f^q(S^{2\alpha})\right) - \inf_{\|x\|=1} \gamma(x),$$

where $\gamma(x) = r_0 f(\|X\|^2)\left(\langle f(S^{2\beta})x, x\rangle^{\frac{p}{2}} - \langle f(T^{2\alpha})x, x\rangle^{\frac{q}{2}}\right)^2$ and $r_0 = \min\{\frac{1}{p}, \frac{1}{q}\}$.

Proof. From (3.2), we have

$$\begin{aligned}
f\left(\left|\langle T^\alpha X S^\beta x, y\rangle\right|^2\right) &\leq f\left((\|X\|^2)\langle S^{2\beta}x, x\rangle\langle T^{2\alpha}y, y\rangle\right) \quad (f \text{ increasing}) \\
&\leq f\left(\|X\|^2\right) f(\langle S^{2\beta}x, x\rangle) f(\langle T^{2\alpha}y, y\rangle) \quad (f \text{ super-multiplicative}) \\
&\leq f(\|X\|^2)\langle f(S^{2\beta})x, x\rangle\langle f(T^{2\alpha})y, y\rangle \quad (\text{by Lemma 2.3}) \\
&\leq f(\|X\|^2)\left\{\frac{1}{p}\langle f(S^{2\beta})x, x\rangle^p + \frac{1}{q}\langle f(T^{2\alpha})y, y\rangle^q\right. \\
&- \left. r_0\left(\langle f(S^{2\beta})x, x\rangle^{\frac{p}{2}} - \langle f(T^{2\alpha})y, y\rangle^{\frac{q}{2}}\right)^2\right\} \quad (\text{by Lemma 2.4}) \\
&\leq f(\|X\|^2)\left\{\frac{1}{p}\langle f^p(S^{2\beta})x, x\rangle + \frac{1}{q}\langle f^q(T^{2\alpha})y, y\rangle\right. \\
&- \left. r_0\left(\langle f(S^{2\beta})x, x\rangle^{\frac{p}{2}} - \langle f(T^{2\alpha})y, y\rangle^{\frac{q}{2}}\right)^2\right\} \quad (\text{by Lemma 2.3}).
\end{aligned}$$

Letting $x = y$, we get

$$\begin{aligned}
& f(|\langle T^\alpha X S^\beta x, x\rangle|^2) \\
\leq \; & f(\|X\|^2)\left\{\frac{1}{p}\langle f^p(S^{2\beta})x, x\rangle + \frac{1}{q}\langle f^q(T^{2\alpha})x, x\rangle - r_0\left(\langle f(S^{2\beta})x, x\rangle^{\frac{p}{2}} - \langle f(T^{2\alpha})x, x\rangle^{\frac{q}{2}}\right)^2\right\} \\
= \; & f(\|X\|^2)\left\{\left\langle\left[\frac{1}{p} f^p(S^{2\beta}) + \frac{1}{q} f^q(T^{2\alpha})\right]x, x\right\rangle - r_0\left(\langle f(S^{2\beta})x, x\rangle^{\frac{p}{2}} - \langle f(T^{2\alpha})x, x\rangle^{\frac{q}{2}}\right)^2\right\}.
\end{aligned}$$

Taking the supremum over all unit vector $x \in \mathcal{H}$, we get the required result. □

Theorem 4.4. *Let $f:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ be a positive, increasing, convex and sub-multiplicative i.e., $f(ts)\le f(t)f(s)$ for all $t,s\in\mathbb{R}$ and let $T\in\mathcal{B}(\mathcal{H})$. Then for every $\alpha,\beta\ge 1$, we have*

$$f(w^2(T|T|^{\alpha-1}T|T|^{\beta-1}))\le 2^{-\frac{2}{r}}\left\|f^r(|T|^{2\alpha})+f^r(|T^*|^{2\beta})\right\|^{\frac{2}{r}}-\inf_{\|x\|=1}\rho(x),$$

where $\rho(x):=\frac{1}{4}\left\langle\left(f(|T|^{2\alpha})-f(|T^*|^{2\beta})\right)x,x\right\rangle$. In particular, if $\alpha=\beta=1$ and $r=1$, we have

$$f(w^2(T^2))\le\frac{1}{4}\left\|f(|T|^2)+f(|T^*|^2)\right\|^2-\inf_{\|x\|=1}\rho_1(x),$$

where $\rho_1(x):=\frac{1}{4}\left\langle\left(f(|T|^2)-f(|T^*|^2)\right)x,x\right\rangle$.

Theorem 4.5. *Let $T\in\mathcal{B}(\mathcal{H})$. Then for all $x,y\in\mathcal{H}$ and every $\alpha,\beta\ge1$, $r\ge1$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m=1,2,\cdots$, we have*

$$w^{2m}\left(T|T|^{\alpha-1}T|T|^{\beta-1}\right)\le w^{\frac{m}{r}}\left[\frac{1}{p}|T|^{2r\alpha}+\frac{1}{q}|T^*|^{2r\beta}\right]-r_0^m\inf_{\|x\|=1}\eta(x),$$

where $\eta(x):=\left[\left\langle|T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}}-\left\langle|T^*|^{2\beta}x,x\right\rangle^{\frac{m}{2}}\right]^2$, and $r_0=\min\{\frac{1}{p},\frac{1}{q}\}$.

Proof. From inequality (3.3), we have

$$\begin{aligned}
&|\left\langle T|T|^{\alpha-1}T|T|^{\beta-1}x,y\right\rangle|^{2m}\\
\le\ &\left\langle|T|^{2\alpha}x,x\right\rangle^{\frac{m}{p}}\left\langle|T^*|^{2\beta}y,y\right\rangle^{\frac{m}{q}}=\left[\left\langle|T|^{2\alpha}x,x\right\rangle^{\frac{1}{p}}\left\langle|T^*|^{2\beta}y,y\right\rangle^{\frac{1}{q}}\right]^m\\
\le\ &\left[\frac{1}{p}\left\langle|T|^{2\alpha}x,x\right\rangle^r+\frac{1}{q}\left\langle|T^*|^{2\beta}y,y\right\rangle^r\right]^{\frac{m}{r}}-r_0^m\left[\left\langle|T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}}-\left\langle|T^*|^{2\beta}y,y\right\rangle^{\frac{m}{2}}\right]^2\\
&\text{(by Lemma 2.5)}\\
\le\ &\left[\frac{1}{p}\left\langle|T|^{2r\alpha}x,x\right\rangle+\frac{1}{q}\left\langle|T^*|^{2r\beta}y,y\right\rangle\right]^{\frac{m}{r}}-r_0^m\left[\left\langle|T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}}-\left\langle|T^*|^{2\beta}y,y\right\rangle^{\frac{m}{2}}\right]^2\\
&\text{(by Lemma 2.3).}
\end{aligned}$$

Let $x=y$, we get

$$\begin{aligned}
&|\left\langle T|T|^{\alpha-1}T|T|^{\beta-1}x,x\right\rangle|^{2m}\\
\le\ &\left[\frac{1}{p}\left\langle|T|^{2r\alpha}x,x\right\rangle+\frac{1}{q}\left\langle|T^*|^{2r\beta}x,x\right\rangle\right]^{\frac{m}{r}}-r_0^m\left[\left\langle|T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}}-\left\langle|T^*|^{2\beta}x,x\right\rangle^{\frac{m}{2}}\right]^2\\
=\ &\left[\left\langle\left(\frac{1}{p}|T|^{2r\alpha}+\frac{1}{q}|T^*|^{2r\beta}\right)x,x\right\rangle\right]^{\frac{m}{r}}-r_0^m\left[\left\langle|T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}}-\left\langle|T^*|^{2\beta}x,x\right\rangle^{\frac{m}{2}}\right]^2
\end{aligned}$$

Taking the supremum over all unit vector $x\in\mathcal{H}$, we get the required result. □

Theorem 4.6. *Let $T\in\mathcal{B}(\mathcal{H})$. Then for all $x,y\in\mathcal{H}$ and every $\alpha,\beta\ge0$ with $\alpha+\beta\ge2$, $r\ge1$, $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m=1,2,\cdots$, we have*

$$\begin{aligned}
w^{2m}\left(T^*|T^*|^{\alpha+\beta-2}T\right)&\le w^{\frac{m}{r}}\left[\frac{1}{p}|T|^{2r\alpha}+\frac{1}{q}|T|^{2r\beta}\right]-r_0^m\inf_{\|x\|=1}\mu(x),\quad\textit{where}\\
\mu(x):&=\left[\left\langle|T|^{2\alpha}x,x\right\rangle^{\frac{m}{2}}-\left\langle|T|^{2\beta}x,x\right\rangle^{\frac{m}{2}}\right]^2\ \textit{and}\ r_0=\min\{\frac{1}{p},\frac{1}{q}\}.
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 4.5, so we omit it. □

Theorem 4.7. *Let $T, S \in \mathcal{B}(\mathcal{H})$, $\alpha, \beta, \gamma, \delta \geq 1$. Then for all $r \geq 1$, we have*

$$\begin{aligned} & w\left(T|T|^{\alpha-1}T|T|^{\beta-1} + S|S|^{\gamma-1}S|S|^{\delta-1}\right) \\ \leq\ & 2^{-\frac{1}{r}} w^{\frac{1}{r}} \left[|T|^{2r\alpha} + |T^*|^{2r\beta}\right] + 2^{-\frac{1}{r}} w^{\frac{1}{r}} \left[|S|^{2r\gamma} + |S^*|^{2r\delta}\right] \\ -\ & \frac{1}{2} \inf_{\|x\|=1} \left[\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}x, x\right\rangle^{\frac{1}{2}}\right]^2 - \frac{1}{2} \inf_{\|x\|=1} \left[\left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x, x\right\rangle^{\frac{1}{2}}\right]^2 \end{aligned} \tag{4.3}$$

Proof. For all $x, y \in \mathcal{H}$, we have

$$\begin{aligned} & \left|\left(\left\langle T|T|^{\alpha-1}T|T|^{\beta-1} + S|S|^{\gamma-1}S|S|^{\delta-1}\right) x, y\right\rangle\right| \\ \leq\ & \left|\left\langle T|T|^{\alpha-1}T|T|^{\beta-1}x, y\right\rangle\right| + \left|\left\langle S|S|^{\gamma-1}S|S|^{\delta-1}x, y\right\rangle\right| \quad \text{(by Cauchy-Swartz inequality)} \\ \leq\ & \left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} \left\langle |T^*|^{2\beta}y, y\right\rangle^{\frac{1}{2}} + \left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} \left\langle |S^*|^{2\delta}y, y\right\rangle^{\frac{1}{2}} \quad \text{(by inequality (3.3))} \\ \leq\ & 2^{-\frac{1}{r}} \left[\left\langle |T|^{2\alpha}x, x\right\rangle^r + \left\langle |T^*|^{2\beta}y, y\right\rangle^r\right]^{\frac{1}{r}} - \frac{1}{2}\left[\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}y, y\right\rangle^{\frac{1}{2}}\right]^2 \\ +\ & 2^{-\frac{1}{r}} \left[\left\langle |S|^{2\gamma}x, x\right\rangle^r + \left\langle |S^*|^{2\delta}y, y\right\rangle^r\right]^{\frac{1}{r}} \\ -\ & \frac{1}{2}\left[\left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x, x\right\rangle^{\frac{1}{2}}\right]^2 \quad \text{(by Lemma 2.5)} \\ \leq\ & 2^{-\frac{1}{r}} \left[\left\langle |T|^{2r\alpha}x, x\right\rangle + \left\langle |T^*|^{2r\beta}y, y\right\rangle\right]^{\frac{1}{r}} - \frac{1}{2}\left[\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}x, x\right\rangle^{\frac{1}{2}}\right]^2 \\ +\ & 2^{-\frac{1}{r}} \left[\left\langle |S|^{2r\gamma}x, x\right\rangle + \left\langle |S^*|^{2r\delta}y, y\right\rangle\right]^{\frac{1}{r}} \\ -\ & \frac{1}{2}\left[\left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x, x\right\rangle^{\frac{1}{2}}\right]^2 \quad \text{(by Lemma 2.3)} \end{aligned}$$

Let $x = y$, we get

$$\begin{aligned} & \left|\left(\left\langle T|T|^{\alpha-1}T|T|^{\beta-1} + S|S|^{\gamma-1}S|S|^{\delta-1}\right) x, x\right\rangle\right| \\ \leq\ & 2^{-\frac{1}{r}} \left[\left\langle |T|^{2r\alpha}x, x\right\rangle + \left\langle |T^*|^{2r\beta}x, x\right\rangle\right]^{\frac{1}{r}} - \frac{1}{2}\left[\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}x, x\right\rangle^{\frac{1}{2}}\right]^2 \\ +\ & 2^{-\frac{1}{r}} \left[\left\langle |S|^{2r\gamma}x, x\right\rangle + \left\langle |S^*|^{2r\delta}y, y\right\rangle\right]^{\frac{1}{r}} - \frac{1}{2}\left[\left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x, x\right\rangle^{\frac{1}{2}}\right]^2 \\ \leq\ & 2^{-\frac{1}{r}} \left\langle (|T|^{2r\alpha} + |T^*|^{2r\beta})x, x\right\rangle^{\frac{1}{r}} + 2^{-\frac{1}{r}} \left\langle (|S|^{2r\gamma} + |S^*|^{2r\delta})x, x\right\rangle^{\frac{1}{r}} \\ -\ & \frac{1}{2}\left[\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}x, x\right\rangle^{\frac{1}{2}}\right]^2 - \frac{1}{2}\left[\left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x, x\right\rangle^{\frac{1}{2}}\right]^2 \end{aligned}$$

Taking the supremum over all unit vector $x \in \mathcal{H}$, we get the required result. □

If we take $r = 1$, we have

Corollary 4.8. *Let $T, S \in \mathcal{B}(\mathcal{H})$, $\alpha, \beta, \gamma, \delta \geq 1$. Then*

$$\begin{aligned} & w\left(T|T|^{\alpha-1}T|T|^{\beta-1} + S|S|^{\gamma-1}S|S|^{\delta-1}\right) \\ \leq\ & \frac{1}{2} w\left[|T|^{2\alpha} + |T^*|^{2\beta} + |S|^{2\gamma} + |S^*|^{2\delta}\right] \\ -\ & \frac{1}{2} \inf_{\|x\|=1} \left[\left\langle |T|^{2\alpha}x, x\right\rangle^{\frac{1}{2}} - \left\langle |T^*|^{2\beta}x, x\right\rangle^{\frac{1}{2}}\right]^2 - \frac{1}{2} \inf_{\|x\|=1} \left[\left\langle |S|^{2\gamma}x, x\right\rangle^{\frac{1}{2}} - \left\langle |S^*|^{2\delta}x, x\right\rangle^{\frac{1}{2}}\right]^2 \end{aligned} \tag{4.4}$$

Remark 1. In Corollary 4.8, if we choose

(i) $\alpha = \beta = \gamma = \delta = 1$, we have

$$\begin{aligned} w(T^2 + S^2) &\leq \frac{1}{2} w\left[|T|^2 + |T^*|^2 + |S|^2 + |S^*|^2\right] \\ &- \frac{1}{2} \inf_{\|x\|=1} [\langle |T|^2 x, x\rangle^{\frac{1}{2}} - \langle |T^*|^2 x, x\rangle^{\frac{1}{2}}]^2 \\ &- \frac{1}{2} \inf_{\|x\|=1} [\langle |S|^2 x, x\rangle^{\frac{1}{2}} - \langle |S^*|^2 x, x\rangle^{\frac{1}{2}}]^2 \end{aligned}$$

(ii) $\alpha = \beta = \gamma = \delta = 2$, we have

$$\begin{aligned} w\left((T|T|)^2 + (S|S|)^2\right) &\leq \frac{1}{2} w\left[|T|^4 + |T^*|^4 + |S|^4 + |S^*|^4\right] \\ &- \frac{1}{2} \inf_{\|x\|=1} \left[\langle |T|^4 x, x\rangle^{\frac{1}{2}} - \langle |T^*|^4 x, x\rangle^{\frac{1}{2}}\right]^2 \\ &- \frac{1}{2} \inf_{\|x\|=1} \left[\langle |S|^4 x, x\rangle^{\frac{1}{2}} - \langle |S^*|^4 x, x\rangle^{\frac{1}{2}}\right]^2 \end{aligned}$$

By the same of the proof of Theorem 4.7, we can prove the following result.

Theorem 4.9. *Let $T, S \in \mathcal{B}(\mathcal{H})$, $\alpha, \beta, \gamma, \delta \geq 0$ such that $\alpha + \beta \geq 2$ and $\gamma + \delta \geq 2$. Then for all $r \geq 1$, we have*

$$\begin{aligned} & w\left(T^*|T^*|^{\alpha+\beta-2}T + S^*|T^*|^{\gamma+\delta-2}S\right) \qquad (4.5) \\ \leq\ & 2^{-\frac{1}{r}} w^{\frac{1}{r}}\left[|T|^{2r\alpha} + |T|^{2r\beta}\right] + 2^{-\frac{1}{r}} w^{\frac{1}{r}}\left[|S|^{2r\gamma} + |S|^{2r\delta}\right] \\ -\ & \frac{1}{2} \inf_{\|x\|=1} \left[\langle |T|^{2\alpha} x, x\rangle^{\frac{1}{2}} - \langle |T|^{2\beta} x, x\rangle^{\frac{1}{2}}\right]^2 - \frac{1}{2} \inf_{\|x\|=1} \left[\langle |S|^{2\gamma} x, x\rangle^{\frac{1}{2}} - \langle |S|^{2\delta} x, x\rangle^{\frac{1}{2}}\right]^2 \end{aligned}$$

5 Refinements of the Hölder-McCarthy Operator Inequality

In this section, we give some new refinements of the mixed Schwarz inequality and its generalization based on a new refinement of Hölder-McCarty inequality. The next lemma plays a main role in our main results.

Lemma 5.1 ([8]). *Let A be a positive operator on $\mathcal{H}$. If $x \in \mathcal{H}$ is a unit vector, then*

$$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle - \langle |A - \langle Ax, x\rangle|^r x, x\rangle \quad \text{for} \quad r \geq 2. \qquad (5.1)$$

The following result is a generalization of refinements of [6, Theorem 3.1] and [3, Theorem 2]

Theorem 5.1. *Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$ and let $0 < \alpha < 1$ and $r \geq 1$. Then*

$$w^{2r}(DCBA) \leq \left\| \alpha |BA|^{\frac{2r}{\alpha}} + (1-\alpha)|(DC)^*|^{\frac{2r}{1-\alpha}} \right\| - \inf_{\|x\|=1} \varsigma(x), \qquad (5.2)$$

where

$$\varsigma(x) := \alpha \left\langle \left| |BA|^{\frac{2}{\alpha}} - \left\langle |BA|^{\frac{2}{\alpha}} x, x\right\rangle \right|^r x, x \right\rangle + (1-\alpha) \left\langle \left| |(DC)^*|^{\frac{2}{1-\alpha}} - \left\langle |(DC)^*|^{\frac{2}{1-\alpha}} x, x\right\rangle \right|^r x, x \right\rangle$$

Proof. For every unit vectors $x, y \in \mathcal{H}$, we have

$$\begin{aligned}
|\langle DCBAx, y\rangle|^2 &\leq \langle |BA|^2 x, x\rangle \langle |(DC)^*|^2 y, y\rangle \quad \text{(by Lemma 3.1)} \\
&= \left\langle |BA|^{\alpha \cdot \frac{2}{\alpha}} x, x\right\rangle \left\langle |(DC)^*|^{(1-\alpha)\cdot \frac{2}{1-\alpha}} y, y\right\rangle \\
&\leq \left\langle |BA|^{\frac{2}{\alpha}} x, x\right\rangle^{\alpha} \left\langle |(DC)^*|^{\frac{2}{1-\alpha}} y, y\right\rangle^{(1-\alpha)} \quad \text{(by Lemma 2.1)} \\
&\leq \left[\alpha \left\langle |BA|^{\frac{2}{\alpha}} x, x\right\rangle^{r} + (1-\alpha) \left\langle |(DC)^*|^{\frac{2}{1-\alpha}} y, y\right\rangle^{r}\right]^{\frac{1}{r}} \quad \text{(by Lemma 2.2)}
\end{aligned}$$

Now for every unit vector $x \in \mathcal{H}$ and and $r \geq 1$, we have

$$\begin{aligned}
& |\langle DCBAx, x\rangle|^{2r} \\
\leq\; & \alpha \left\langle |BA|^{\frac{2}{\alpha}} x, x\right\rangle^{r} + (1-\alpha) \left\langle |(DC)^*|^{\frac{2}{1-\alpha}} x, x\right\rangle^{r} \\
\leq\; & \alpha \left[\left\langle |BA|^{\frac{2r}{\alpha}} x, x\right\rangle - \left\langle \left| |BA|^{\frac{2}{\alpha}} - \left\langle |BA|^{\frac{2}{\alpha}} x, x\right\rangle \right|^{r} x, x\right\rangle\right] \\
+\; & (1-\alpha) \left[\left\langle |(DC)^*|^{\frac{2r}{1-\alpha}} x, x\right\rangle - \left\langle \left| |(DC)^*|^{\frac{2}{1-\alpha}} - \left\langle |(DC)^*|^{\frac{2}{1-\alpha}} x, x\right\rangle \right|^{r} x, x\right\rangle\right] \\
& \text{(by Lemma 5.1)}
\end{aligned}$$

Taking the supremum over all unit vector $x \in \mathcal{H}$, we get the required result. □

Corollary 5.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $0 < \nu < 1$, $\alpha + \beta \geq 1$ and $r \geq 2$. Then*

$$w^{2r}(T|T|^{\alpha-1}T|T|^{\beta-1}) \leq \left\| \nu |T|^{\frac{2r\alpha}{\nu}} + (1-\nu)|T^*|^{\frac{2r\beta}{1-\nu}} \right\| - \inf_{\|x\|=1} \varsigma(x), \tag{5.3}$$

where

$$\varsigma(x) := \nu \left\langle \left| |T|^{\frac{2\alpha}{\nu}} - \left\langle |T|^{\frac{2\alpha}{\nu}} x, x\right\rangle \right|^{r} x, x\right\rangle + (1-\nu) \left\langle \left| |T^*|^{\frac{2\beta}{1-\nu}} - \left\langle |T^*|^{\frac{2\beta}{1-\nu}} x, x\right\rangle \right|^{r} x, x\right\rangle$$

Proof. In Theorem 5.1, choose $A = |T|^{\beta-1}, B = T, C = |T|^{\alpha-1}$ and $D = T$, we get the desired result. □

Corollary 5.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and let $0 < \nu < 1$, $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 2$ and $r \geq 2$. Then*

$$w^{2r}(T^*|T^*|^{\alpha+\beta-2}T) \leq \left\| \nu |T|^{\frac{2r\alpha}{\nu}} + (1-\nu)|T|^{\frac{2r\beta}{1-\nu}} \right\| - \inf_{\|x\|=1} \varsigma(x), \tag{5.4}$$

where

$$\varsigma(x) := \nu \left\langle \left| |T|^{\frac{2\alpha}{\nu}} - \left\langle |T|^{\frac{2\alpha}{\nu}} x, x\right\rangle \right|^{r} x, x\right\rangle + (1-\nu) \left\langle \left| |T|^{\frac{2\beta}{1-\nu}} - \left\langle |T|^{\frac{2\beta}{1-\nu}} x, x\right\rangle \right|^{r} x, x\right\rangle$$

Proof. In Theorem 5.1, choose $A = T, B = |T^*|^{\beta-1}, C = |T^*|^{\alpha-1}$ and $D = T^*$, we get the desired result. □

Remark 2. If we choose $D = T^*, C = I, B = I, A = S$ in Theorem 5.1, we have

$$w^{2r}(T^*S) \leq \left\| \alpha|S|^{\frac{2r}{\alpha}} + (1-\alpha)|T|^{\frac{2r}{1-\alpha}} \right\| - \inf_{\|x\|=1} \tau(x),$$

where

$$\tau(x) := \left\langle \left[\alpha \left| |S|^{\frac{2}{\alpha}} - \left\langle |S|^{\frac{2}{\alpha}} x, x\right\rangle \right|^{r} + (1-\alpha) \left| |T|^{\frac{2}{1-\beta}} - \left\langle |T|^{\frac{2}{1-\alpha}} x, x\right\rangle \right|^{r}\right] x, x\right\rangle$$

The following lemma is very useful in the sequel.

Lemma 5.2. *If $a, b \geq 0$, then*

(i) $a^\nu + b^\nu \leq (a+b)^\nu \leq 2^{\nu-1}(a^\nu + b^\nu)$, *for* $\nu \geq 1$, *and*

(ii) $2^{\nu-1}(a^\nu + b^\nu) \leq (a+b)^\nu \leq a^\nu + b^\nu$, *for* $\nu \leq 1$.

Our next result is a generalization and refinements of inequality (1.9).

Theorem 5.4. *Let $A, B, C, D, A_1, B_1, C_1, D_1 \in \mathcal{B}(\mathcal{H})$ and $r \geq 2$. Then*

$$\begin{aligned} w^r(DCBA + D_1C_1B_1A_1) &\leq 2^{r-2}\left\| |BA|^{2r} + |(DC)^*|^{2r} + |B_1A_1|^{2r} + |(D_1C_1)^*|^{2r} \right\| \\ &- \inf_{\|x\|=1} \psi(x), \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} \psi(x) : &= 2^{r-2}\Big[\Big\langle \left| |BA|^2 - \langle |BA|^2 x, x\rangle \right|^r x, x \Big\rangle + \Big\langle \left| |(DC)^*|^2 - \langle |(DC)^*|^2 x, x\rangle \right|^r x, x \Big\rangle \\ &+ \Big\langle \left| |B_1A_1|^2 - \langle |B_1A_1|^2 x, x\rangle \right|^r x, x \Big\rangle + \Big\langle \left| |(D_1C_1)^*|^2 - \langle |(D_1C_1)^*|^2 y, y\rangle \right|^r x, x \Big\rangle\Big] \end{aligned}$$

Proof. By Cauchy-Schwartz inequality, Lemma 3.1, Lemma 2.2 and Lemma 5.2. Then for every unit vectors $x, y \in \mathcal{H}$, we have

$$\begin{aligned} & |\langle (DCBA + D_1C_1B_1A_1)x, y\rangle| \\ \leq\ & |\langle DCBAx, y\rangle| + |\langle D_1C_1B_1A_1x, y\rangle| \\ \leq\ & \langle |BA|^2x, x\rangle^{\frac{1}{2}} \langle |(DC)^*|^2y, y\rangle^{\frac{1}{2}} + \langle |B_1A_1|^2x, x\rangle^{\frac{1}{2}} \langle |(D_1C_1)^*|^2y, y\rangle^{\frac{1}{2}} \\ \leq\ & \frac{1}{2}\langle |BA|^2x, x\rangle + \frac{1}{2}\langle |(DC)^*|^2y, y\rangle + \frac{1}{2}\langle |B_1A_1|^2x, x\rangle + \frac{1}{2}\langle |(D_1C_1)^*|^2y, y\rangle \\ \leq\ & \left[\frac{1}{2}\langle |BA|^2x, x\rangle^r + \frac{1}{2}\langle |(DC)^*|^2y, y\rangle^r\right]^{\frac{1}{r}} + \left[\frac{1}{2}\langle |B_1A_1|^2x, x\rangle^r + \frac{1}{2}\langle |(D_1C_1)^*|^2y, y\rangle^r\right]^{\frac{1}{r}} \\ \leq\ & 2^{1-\frac{1}{r}}\left[\frac{1}{2}\langle |BA|^2x, x\rangle^r + \frac{1}{2}\langle |(DC)^*|^2y, y\rangle^r + \frac{1}{2}\langle |B_1A_1|^2x, x\rangle^r + \frac{1}{2}\langle |(D_1C_1)^*|^2y, y\rangle^r\right]^{\frac{1}{r}}. \end{aligned}$$

Hence

$$\begin{aligned} & |\langle (DCBA + D_1C_1B_1A_1)x, y\rangle|^r \\ =\ & 2^{r-1}\left[\frac{1}{2}\langle |BA|^2x, x\rangle^r + \frac{1}{2}\langle |(DC)^*|^2y, y\rangle^r + \frac{1}{2}\langle |B_1A_1|^2x, x\rangle^r + \frac{1}{2}\langle |(D_1C_1)^*|^2y, y\rangle^r\right] \\ =\ & 2^{r-2}\left[\langle |BA|^2x, x\rangle^r + \langle |(DC)^*|^2y, y\rangle^r + \langle |B_1A_1|^2x, x\rangle^r + \langle |(D_1C_1)^*|^2y, y\rangle^r\right] \\ \leq\ & 2^{r-2}\Big[\langle |BA|^{2r}x, x\rangle - \Big\langle \left| |BA|^2 - \langle |BA|^2x, x\rangle \right|^r x, x \Big\rangle \\ +\ & \langle |(DC)^*|^{2r}y, y\rangle - \Big\langle \left| |(DC)^*|^2 - \langle |(DC)^*|^2y, y\rangle \right|^r y, y \Big\rangle\Big] \\ +\ & 2^{r-2}\Big[\langle |B_1A_1|^{2r}x, x\rangle - \Big\langle \left| |B_1A_1|^2 - \langle |B_1A_1|^2x, x\rangle \right|^r x, x \Big\rangle \\ +\ & \langle |(D_1C_1)^*|^{2r}y, y\rangle - \Big\langle \left| |(D_1C_1)^*|^2 - \langle |(D_1C_1)^*|^2y, y\rangle \right|^r y, y \Big\rangle\Big] \quad \text{(by Lemma 5.1)} \end{aligned}$$

Let $x = y$, we have

$$\begin{aligned}
& |\langle (DCBA + D_1C_1B_1A_1)\,x, x\rangle|^r \\
\le\ & 2^{r-2}\left\langle \left[|BA|^{2r} + |(DC)^*|^{2r} + |B_1A_1|^{2r} + |(D_1C_1)^*|^{2r}\right] x, x\right\rangle \\
-\ & 2^{r-2}\left[\left\langle \left||BA|^2 - \langle |BA|^2x, x\rangle\right|^r x, x\right\rangle + \left\langle \left||(DC)^*|^2 - \langle |(DC)^*|^2x, x\rangle\right|^r x, x\right\rangle\right. \\
+\ & \left.\left\langle \left||B_1A_1|^2 - \langle |B_1A_1|^2x, x\rangle\right|^r x, x\right\rangle + \left\langle \left||(D_1C_1)^*|^2 - \langle |(D_1C_1)^*|^2x, x\rangle\right|^r x, x\right\rangle\right]
\end{aligned}$$

Taking the supremum over all unit vector $x \in \mathcal{H}$, we get the required result. □

Remark 3. In Theorem, if we choose $D = T^*, C = B = I, A = T$ and $D_1 = T, C_1 = B_1 = I, A_1 = T^*$, then

$$w^r(T^*T + TT^*) \le 2^{r-1}\left\||T|^{2r} + |T^*|^{2r}\right\| - \inf_{\|x\|=1}\psi(x),$$

where

$$\psi(x) := 2^{r-1}\left[\left\langle \left||T|^2 - \langle |T|^2x, x\rangle\right|^r x, x\right\rangle + \left\langle \left||T^*|^2 - \langle |T^*|^2x, x\rangle\right|^r x, x\right\rangle\right].$$

Many mathematicians improved the Young inequality and its reverse. Kober [15], proved that for $a, b > 0$

$$(1-\lambda)a + \lambda b \le a^{1-\lambda}b^\lambda + (1-\lambda)(\sqrt{a} - \sqrt{b})^2, \quad \lambda \ge 1. \tag{5.6}$$

By using (5.6), we obtain a refinement of the Hölder-McCarthy inequality.

Lemma 5.3 ([6])**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator. Then*

$$\langle Ax, x\rangle^\lambda \left(1 + 2(\lambda-1)\left(1 - \frac{\left\langle A^{\frac{1}{2}}x, x\right\rangle}{\langle Ax, x\rangle^{\frac{1}{2}}}\right)\right) \le \left\langle A^\lambda x, x\right\rangle \tag{5.7}$$

for all $\lambda \ge 1$ and $x \in \mathcal{H}$ with $\|x\| = 1$.

Remark 4. If we denote $\rho := \left(1 + 2(\lambda-1)\inf_{\|x\|=1}\left(1 - \frac{\left\langle A^{\frac{1}{2}}x,x\right\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}\right)\right)$, we have

$$\langle Ax, x\rangle^\lambda \le \frac{1}{\rho}\left\langle A^\lambda x, x\right\rangle, \quad \text{for } \lambda \ge 1. \tag{5.8}$$

Theorem 5.5. *Let A, B, C, D be invertible operators, $0 < \lambda < 1$ and $r > 1$. If for each unit vector $x \in \mathcal{H}$*

$$\xi(x) = \left(1 + 2(r-1)\left(1 - \frac{\left\langle |BA|^\lambda x, x\right\rangle}{\langle |BA|^{2\lambda}x, x\rangle^{\frac{1}{2}}}\right)\right)$$

and

$$\eta(x) = \left(1 + 2(r-1)\left(1 - \frac{\left\langle |(DC)^*|^{1-\lambda}x, x\right\rangle}{\left\langle |(DC)^*|^{2(1-\lambda)}x, x\right\rangle^{\frac{1}{2}}}\right)\right)$$

then

$$w^r(DCBA) \le \frac{1}{2\rho}\left\||BA|^{2r\lambda} + |(DC)^*|^{2r(1-\lambda)}\right\|,$$

where

$$\xi = \inf_{\|x\|=1}\xi(x), \eta = \inf_{\|x\|=1}\eta(x) \quad \textit{and } \rho = \min\{\xi, \eta\}.$$

Proof. Let $x \in \mathcal{H}$ be a unit vector. Then

$$\begin{aligned} |\langle DCBAx, x\rangle| &\leq \left\langle |BA|^{2\lambda}x, x\right\rangle^{\frac{1}{2}} \left\langle |(DC)^*|^{2(1-\lambda)}x, x\right\rangle^{\frac{1}{2}} \quad \text{(by Lemma 3.1)} \\ &\leq \left(\frac{\left\langle |BA|^{2\lambda}x, x\right\rangle^r + \left\langle |(DC)^*|^{2(1-\lambda)}x, x\right\rangle^r}{2}\right)^{\frac{1}{r}} \end{aligned}$$

Applying Lemma 5.3 to the positive operators $|BA|^{2\lambda}$ and $|(DC)^*|^{2(1-\lambda)}$, we have

$$\begin{aligned} \left\langle |BA|^{2r\lambda}x, x\right\rangle &\geq \left\langle |BA|^{2\lambda}x, x\right\rangle^r \left(1 + 2(r-1)\left(1 - \frac{\left\langle |BA|^{\lambda}x, x\right\rangle}{\langle |BA|^{2\lambda}x, x\rangle^{\frac{1}{2}}}\right)\right) \\ &= \xi(x)\left\langle |BA|^{2\lambda}x, x\right\rangle^r \\ &\Rightarrow \left\langle |BA|^{2\lambda}x, x\right\rangle^r \leq \frac{1}{\xi}\left\langle |BA|^{2r\lambda}x, x\right\rangle \quad \text{(by inequality (5.8)),} \end{aligned}$$

and

$$\begin{aligned} \left\langle |(DC)^*|^{2r\lambda}x, x\right\rangle &\geq \left\langle |(DC)^*|^{2\lambda}x, x\right\rangle^r \left(1 + 2(r-1)\left(1 - \frac{\left\langle |(DC)^*|^{\lambda}x, x\right\rangle}{\langle |(DC)^*|^{2\lambda}x, x\rangle^{\frac{1}{2}}}\right)\right). \\ &= \eta(x)\left\langle |(DC)^*|^{2\lambda}x, x\right\rangle^r \\ &\Rightarrow \left\langle |(DC)^*|^{2\lambda}x, x\right\rangle^r \leq \frac{1}{\xi}\left\langle |(DC)^*|^{2r\lambda}x, x\right\rangle \quad \text{(by inequality (5.8).} \end{aligned}$$

Hence

$$\begin{aligned} |\langle DCBAx, x\rangle| &\leq 2^{-\frac{1}{r}}\left(\frac{1}{\xi}\left\langle |BA|^{2r\lambda}x, x\right\rangle + \frac{1}{\eta}\left\langle |(DC)^*|^{2r(1-\lambda)}x, x\right\rangle\right)^{\frac{1}{r}} \\ |\langle DCBAx, x\rangle|^r &\leq \frac{1}{2\rho}\left\langle \left(|BA|^{2r\lambda} + |(DC)^*|^{2r(1-\lambda)}\right)x, x\right\rangle \end{aligned}$$

By taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired relation. □

Corollary 5.6. *Let T be an invertible operators, $\alpha, \beta \geq 1$, $0 < \lambda < 1$ and $r > 1$. If for each unit vector $x \in \mathcal{H}$*

$$\xi(x) = \left(1 + 2(r-1)\left(1 - \frac{\left\langle |T|^{\beta\lambda}x, x\right\rangle}{\langle |T|^{2\beta\lambda}x, x\rangle^{\frac{1}{2}}}\right)\right),$$

and

$$\eta(x) = \left(1 + 2(r-1)\left(1 - \frac{\left\langle |T^*|^{(1-\lambda)\alpha}x, x\right\rangle}{\left\langle |T^*|^{2(1-\lambda)\alpha}x, x\right\rangle^{\frac{1}{2}}}\right)\right),$$

then

$$w^r(T|T|^{\alpha-1}T|T|^{\beta-1}) \leq \frac{1}{2\rho}\left\| |T|^{2r\alpha\lambda} + |T^*|^{2r\beta(1-\lambda)}\right\|,$$

where

$$\xi = \inf_{\|x\|=1}\xi(x), \eta = \inf_{\|x\|=1}\eta(x) \quad \textit{and } \rho = \min\{\xi, \eta\}.$$

Proof. In Theorem 5.5, choose $A = |T|^{\beta-1}, B = T, C = |T|^{\alpha-1}$ and $D = T$, we get the desired result. □

Corollary 5.7. *Let T be an invertible operators, $\alpha, \beta \geq 0$ with $\alpha+\beta \geq 2$, $0<\lambda<1$ and $r>1$. If for each unit vector $x \in \mathcal{H}$*

$$\xi(x)=\left(1+2(r-1)\left(1-\frac{\langle |T|^{\beta\lambda}x,x\rangle}{\langle |T|^{2\beta\lambda}x,x\rangle^{\frac{1}{2}}}\right)\right),$$

and

$$\eta(x)=\left(1+2(r-1)\left(1-\frac{\left\langle |T|^{(1-\lambda)\alpha}x,x\right\rangle}{\left\langle |T|^{2(1-\lambda)\alpha}x,x\right\rangle^{\frac{1}{2}}}\right)\right),$$

then

$$w^r(T^*|T^*|^{\alpha+\beta-2}T) \leq \frac{1}{2\rho}\left\| |T|^{2r\alpha\lambda}+|T|^{2r\beta(1-\lambda)}\right\|,$$

where

$$\xi=\inf_{\|x\|=1}\xi(x), \eta=\inf_{\|x\|=1}\eta(x) \quad \text{and } \rho=\min\{\xi,\eta\}.$$

Proof. In Theorem 5.5, choose $A=T, B=|T^*|^{\beta-1}, C=|T^*|^{\alpha-1}$ and $D=T^*$, we get the desired result. □

We are going to establish a refinement of a numerical inequality for Hilbert space operators. We need the following lemma.

Lemma 5.4 ([19]). *Let $a_i, i=1,\cdots,n$ be positive real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1}\sum_{i=1}^{n} a_i^r \quad \text{for } r\geq 1. \tag{5.9}$$

The next result is a generalization and refinements of inequality (1.8).

Theorem 5.8. *Let $A_i, B_i, C_i, D_i \in \mathcal{B}(\mathcal{H})$, $i=1,\cdots,n$ be invertible operators. Then for all $r>1$*

$$w^r\left(\sum_{i=1}^{n} D_iC_iB_iA_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n}\left[|B_iA_i|^{2r}+|(D_iC_i)^*|^{2r}\right]\right\|, \tag{5.10}$$

where $\mu=\min\{\xi,\eta\}$,

$$\xi=\inf_{\|x\|=1}\left(1+2(r-1)\left(1-\frac{\langle |B_iA_i|x,x\rangle}{\langle |B_iA_i|^2x,x\rangle^{\frac{1}{2}}}\right)\right)$$

and

$$\eta=\inf_{\|x\|=1}\left(1+2(r-1)\left(1-\frac{\langle |(D_iC_i)^*|x,x\rangle}{\langle |(D_iC_i)^*|x,x\rangle^{\frac{1}{2}}}\right)\right).$$

Proof. For every unit vector $x\in\mathcal{H}$, we have

$$\left|\left\langle\left(\sum_{i=1}^{n} D_iC_iB_iA_i\right)x,x\right\rangle\right|^r$$

$$
\begin{aligned}
&= \left|\sum_{i=1}^{n} \langle D_iC_iB_iA_i x, x\rangle\right|^r \le \left(\sum_{i=1}^{n} |\langle D_iC_iB_iA_i x, x\rangle|\right)^r \\
&\le \left(\sum_{i=1}^{n} \left|\left\langle |B_iA_i|^2 x, x\right\rangle^{\frac{1}{2}} \left\langle |(D_iC_i)^*|^2 x, x\right\rangle^{\frac{1}{2}}\right|\right)^r \quad \text{(by Lemma 3.1)} \\
&\le n^{r-1} \sum_{i=1}^{n} \left\langle |B_iA_i|^2 x, x\right\rangle^{\frac{r}{2}} \left\langle |(D_iC_i)^*|^2 x, x\right\rangle^{\frac{r}{2}} \quad \text{(by Lemma 5.4)} \\
&\le \frac{n^{r-1}}{2} \left(\sum_{i=1}^{n} \left[\left\langle |B_iA_i|^2 x, x\right\rangle^r + \left\langle |(D_iC_i)^*|^2 x, x\right\rangle^r\right]\right) \quad \text{(AM-GM)} \\
&\le \frac{n^{r-1}}{2} \left(\sum_{i=1}^{n} \left[\frac{1}{\xi} \left\langle |B_iA_i|^{2r} x, x\right\rangle + \frac{1}{\eta} \left\langle |(D_iC_i)^*|^{2r} x, x\right\rangle\right]\right) \quad \text{(by inequality 5.8)} \\
&\le \frac{n^{r-1}}{2\mu} \sum_{i=1}^{n} \left\langle \left(|B_iA_i|^{2r} + |(D_iC_i)^*|^{2r}\right) x, x\right\rangle .
\end{aligned}
$$

Taking the supremum over all unit vector $x \in \mathcal{H}$, we obtain the desired result. □

Corollary 5.9. *Let $T_i \in \mathcal{B}(\mathcal{H})$, $i = 1, \cdots, n$ be invertible operators, $\alpha + \beta \ge 1$. Then for each $r > 1$*

$$
w^r \left(\sum_{i=1}^{n} T_i|T_i|^{\alpha-1}T_i|T_i|^{\beta-1}\right) \le \frac{n^{r-1}}{2\mu} \left\|\sum_{i=1}^{n} \left[|T_i|^{2r\beta} + |T_i^*|^{2r\alpha}\right]\right\|, \tag{5.11}
$$

where $\mu = \min\{\xi, \eta\}$,

$$
\xi = \inf_{\|x\|=1} \left(1 + 2(r-1)\left(1 - \frac{\langle |T_i|^{\beta} x, x\rangle}{\langle |T_i|^{2\beta} x, x\rangle^{\frac{1}{2}}}\right)\right)
$$

and

$$
\eta = \inf_{\|x\|=1} \left(1 + 2(r-1)\left(1 - \frac{\langle |T_i^*|^{\alpha} x, x\rangle}{\langle |T_i^*|^{2\alpha} x, x\rangle^{\frac{1}{2}}}\right)\right).
$$

Proof. In Theorem 5.8, choose $A_i = |T_i|^{\beta-1}, B_i = T_i, C_i = |T_i|^{\alpha-1}$ and $D_i = T_i$, we get the desired result. □

Acknowledgement. The author thank anonymous referees for giving valuable comments and suggestions to improve our manuscript.

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