

MULTIVARIATE HARDY-TYPE INEQUALITIES

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Abstract. We present transparent proofs for some multivariate versions of both continuous and discrete Hardy-type inequalities. Our theorems subsume several known results and simplify the proofs of existing results.

1. Introduction

In 1920, Hardy [2] proved the following theorem.

Theorem A. *If $p > 1$, $f(x) \geq 0$ for $0 < x < \infty$ and $G(x) = x^{-1} \int_0^x f(t)dt$, then*

$$\int_0^\infty G^p(x)dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx$$

unless $f \equiv 0$. The multiplicative constant is best-possible.

In the same article, Hardy also established a discretized form of this result.

Theorem B. *If $p > 1$, $a_n \geq 0$ and $A_n = \sum_{i=1}^n a_i (n \geq 1)$, then*

$$\sum_{n=1}^\infty \left(\frac{A_n}{n}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^\infty a_n^p$$

unless each $a_n = 0$. The constant is best-possible.

In the case of Theorem B, he was unable to fix the multiplicative constant. This was rectified by Landau [6].

These relations have since been the object of sustained study and have undergone extensive generalization and found numerous applications. For comprehensive reviews of developments see the monograph of Opic and Kufner [9] and Chapter 4 of the monograph of Mitrinović, Pečarić and Fink [8]. An important early contribution was made by Copson [1], who established the following generalization of Theorem B.

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Theorem C. Suppose $p > 1$, $\lambda_n > 0$, $\Lambda_n = \sum_{i=1}^n \lambda_i$, $a_n > 0$ and $A_n = \sum_{i=1}^n \lambda_i a_i$ ($n \geq 1$). If $\sum_{n=1}^{\infty} \lambda_n a_n^p$ converges, then

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} \lambda_n a_n^p.$$

The constant is best-possible.

This reduces to Theorem B when the constants λ_n assume a common value. In 1928 Hardy [3] provided a generalization of Theorem A.

Theorem D. Suppose $p > 1$, $m \neq 1$, $f(x) \geq 0$ and $F(x)$ is defined by

$$F(x) = \begin{cases} \int_0^x f(t) dt, & m > 1, \\ \int_x^{\infty} f(t) dt, & m < 1, \end{cases}$$

Then

$$\int_0^{\infty} x^{-m} F^p(x) dx < \left\{ \frac{p}{|m-1|} \right\}^p \int_0^{\infty} x^{p-m} f^p(x) dx$$

unless $f \equiv 0$. The constant is best-possible.

This reduces to Theorem A for $m = p$.

A recent extension of Hardy's discrete inequality in the direction given by Copson in Theorem C is the following result of Hwang and Yang [5].

Theorem E. Suppose $p > 1$, $\beta_n > 0$, $\lambda_n > 0$, $a_n > 0$ ($n \geq 1$) and that $\sum_{n=1}^{\infty} \lambda_n a_n^p$ converges. Define $\Lambda_n = \sum_{i=1}^n \beta_i \lambda_i$ and $A_n = \sum_{i=1}^n \beta_i \lambda_i a_i$ ($n \geq 1$). If there exists $K > 0$ such that

$$p-1 + \frac{(\beta_{n+1} - \beta_n) \Lambda_n}{\beta_{n+1} \beta_n \lambda_n} \geq \frac{p}{K} \quad \text{for } n = 1, 2, 3, \dots,$$

then

$$\sum_{n=1}^{\infty} \lambda_n \left(\frac{A_n}{\Lambda_n} \right)^p \leq K^p \sum_{n=1}^{\infty} \lambda_n a_n^p.$$

This has led to generalizations by Hwang [4] to discrete inequalities of Hardy type involving multiple variables. For earlier work on discrete Hardy-type inequalities see Pachpatte [10]. The proofs involved are relatively complicated. Yang and Jean [14] and Pachpatte [13] have given several two-variable versions and Pachpatte [11-12] multivariate versions of the continuous Hardy inequality, again *via* fairly involved proofs.

Our aim in this paper is to provide transparent treatments of multivariate inequalities of Hardy type both for the discrete and for the continuous case. In Section 2 we show that several results for the continuous case may be derived directly from known univariate results obtained by Lee and Yang [4]. Our results subsume those of [11] and [14]. In section 3 we give a parallel analysis for the discrete case. Our results generalize [10, Theorem 1] and provide a simpler proof for the main result of [4].

2. Continuous Multivariate Inequalities

Our derivation is based on the following results derived by Lee and Yang.

Theorem 2.1. *Suppose $p > 1$ and $f(x) \geq 0$ and that $r(x) \geq 0$ is absolutely continuous for $x > 0$. Define $R(x) = \int_0^x r(t)dt$. If $m > 1$ and there is a $\lambda > 0$ such that*

$$p - 1 + \frac{R(x)r'(x)}{r^2(x)} - \frac{(p - m)R(x)}{xr(x)} \geq \frac{p}{\lambda} \tag{2.1}$$

for almost all $x > 0$, then

$$\int_0^\infty x^{p-m} \left(\frac{1}{R(x)} \int_0^x r(s)f(s)ds \right)^p dx \leq \lambda^p \int_0^\infty x^{p-m} f^p(x)dx. \tag{2.2}$$

Theorem 2.2. *Let p, f, r be as in Theorem 2.1 and define $G(x) = \frac{1}{xr(x)} \int_0^x r(t)f(t)dt$. If $m > 1$ and there is a $\lambda > 0$ such that*

$$\frac{m - 1}{p} + \frac{xr'(x)}{r(x)} \geq \frac{1}{\lambda}$$

for almost all $x > 0$, then

$$\int_0^\infty x^{p-m} G^p(x)dx \leq \lambda^p \int_0^\infty x^{p-m} f^p(x)dx.$$

Theorem 2.3. *Let p, f, r be as in Theorem 2.1, $\alpha > 1$ and $R(x) = \int_x^\infty r(t)dt$. If $m < 1$ and there is a $\lambda > 0$ such that*

$$p - 1 - \frac{(m + \alpha p - p)R(x)}{xr(x)} - \frac{R(x)r'(x)}{r^2(x)} \geq \frac{p}{\lambda}$$

for almost all $x \geq 0$, then

$$\int_0^\infty x^{p-\alpha p-m} \left(\frac{1}{R(x)} \int_x^\infty t^\alpha r(t)f(t)dt \right)^p dx \leq \lambda^p \int_0^\infty x^{p-m} f^p(x)dx.$$

Theorem 2.4. *Let p, α, f, r be as in Theorem 2.3 and $G(x) = \int_x^\infty t^\alpha r(t)dt / \{x^\alpha r(x)\}$. If $m < 1$ and there is a $\lambda > 0$ such that*

$$\frac{1 - m}{p} - \frac{xr'(x)}{r(x)} - \alpha \geq \frac{1}{\lambda}$$

for almost all $x > 0$, then

$$\int_0^\infty x^{-m} G^p(x)dx \leq \lambda^p \int_0^\infty x^{p-m} f^p(x)dx.$$

We now proceed to multivariate versions of these four theorems. First we consider a generalization of Theorem 2.1. This subsumes the main theorem of Pachpatte [11] under the case $m = p$.

Theorem 2.5. *Let $p > 1$ and f be a nonnegative integrable function defined on the positive orthant $\{(x_1, \dots, x_n) : x_i > 0, i = 1, \dots, n\}$. Suppose $r_i (i = 1, \dots, n)$ is positive and absolutely continuous on $(0, \infty)$ and define $R_i(x_i) = \int_0^{x_i} r_i(t) dt$. If $m_i > 1$ and there exist constants $\lambda_i > 0$ ($i = 1, \dots, n$) such that*

$$p - 1 + \frac{R_i(x_i)r'_i(x_i)}{r_i^2(x_i)} - \frac{(p - m_i)R_i(x_i)}{x_i r_i(x_i)} \geq \frac{p}{\lambda_i} \quad (2.3)$$

for all $x_i \in (0, \infty)$ ($i = 1, \dots, n$), then

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i^{-m_i} \right) H^p(x_1, \dots, x_n) dx_1 \cdots dx_n \\ & \leq \left(\prod_{i=1}^n \lambda_i \right)^p \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i^{p-m_i} \right) f^p(x_1, \dots, x_n) dx_1 \cdots dx_n, \end{aligned} \quad (2.4)$$

where

$$H(x_1, \dots, x_n) = \int_0^{x_1} \cdots \int_0^{x_n} \left\{ \prod_{i=1}^n \frac{r_i(s_i)x_i}{R_i(x_i)} \right\} f(s_1, \dots, s_n) ds_1 \cdots ds_n.$$

Proof. We proceed *via* mathematical induction. First note that the standing condition (2.3) for $n = k + 1$ subsumes that for $n = k$. A basis is provided by Theorem 2.1, which provides the result for $n = 1$. For the inductive step, suppose (2.4) holds for $n = k$ and conditions (2.3) for $n = k + 1$. Since the integrand is nonnegative, Fubini's theorem applies and we may arrange the left-hand side of (2.4) for $n = k + 1$ as

$$\begin{aligned} & \int_0^\infty \frac{x_{k+1}^{p-m_{k+1}}}{R_{k+1}^p(x_{k+1})} \left\{ \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k x_i^{-m_i} \right) \right. \\ & \times \left. \left[\int_0^{x_1} \cdots \int_0^{x_k} \left\{ \prod_{i=1}^k \frac{x_i r_i(s_i)}{R_i(x_i)} \right\} F(s_1, \dots, s_k) ds_1 \cdots ds_k \right]^p dx_1 \cdots dx_k \right\} dx_{k+1}, \end{aligned} \quad (2.5)$$

where x_{k+1} is regarded as a parameter in the function

$$F(s_1, \dots, s_k) := \int_0^{x_{k+1}} r_{k+1}(s_{k+1}) f(s_1, \dots, s_k, s_{k+1}) ds_{k+1}.$$

By the inductive hypothesis, the expression inside the outer braces in (2.5) is bounded above by

$$\left(\prod_{i=1}^k \lambda_i^p \right) \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k x_i^{p-m_i} \right) F^p(x_1, \dots, x_k) dx_1 \cdots dx_k$$

and so the left-hand side of (2.4) for $n = k + 1$ is bounded above by

$$\left(\prod_{i=1}^k \lambda_i^p \right) \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k x_i^{p-m_i} \right) \int_0^\infty x_{k+1}^{p-m_{k+1}} \left\{ \frac{1}{R_{k+1}(x_{k+1})} \right. \\ \left. \times \int_0^{x_{k+1}} r_{k+1}(s_{k+1}) f(x_1, \dots, x_k, s_{k+1}) ds_{k+1} \right\}^p dx_{k+1} dx_1 \cdots dx_k.$$

By Theorem 2.1, this in turn is bounded above by

$$\left(\prod_{i=1}^k \lambda_i \right)^p \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^k x_i^{p-m_i} \right) \lambda_{k+1}^p \int_0^\infty x_{k+1}^{p-m_{k+1}} f^p(x_1, \dots, x_k, x_{k+1}) dx_{k+1} dx_1 \cdots dx_k \\ = \left(\prod_{i=1}^{k+1} \lambda_i \right)^p \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^{k+1} x_i^{p-m_i} \right) f^p(x_1, \dots, x_{k+1}) dx_1 \cdots dx_{k+1},$$

and the inductive step is established.

Exactly similar reasoning may be used to establish the three following theorems from Theorems 2.2, 2.3 and 2.4.

Theorem 2.6. *Let p, f, r_i ($i = 1, \dots, n$) be as in Theorem 2.5. If $m_i > 1$ and there exist $\lambda_i > 0$ ($i = 1, \dots, n$) such that*

$$\frac{m_i - 1}{p} + \frac{x_i r_i'(x_i)}{r_i(x_i)} \geq \frac{1}{\lambda_i}, \text{ for all } x_i \in (0, \infty) \text{ (} i = 1, \dots, n \text{),}$$

then

$$\int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i^{-m_i} \right) H^p(x_1, \dots, x_n) dx_1 \cdots dx_n \\ \leq \left(\prod_{i=1}^n \lambda_i \right)^p \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i^{p-m_i} \right) f^p(x_1, \dots, x_n) dx_1 \cdots dx_n,$$

where

$$H(x_1, \dots, x_n) = \int_0^{x_1} \cdots \int_0^{x_n} \left(\prod_{i=1}^n \frac{r_i(s_i)}{r_i(x_i)} \right) f(s_1, \dots, s_n) ds_1 \cdots ds_n.$$

Theorem 2.7. *Let p, f, r_i ($i = 1, \dots, n$) be as in Theorem 2.5, $\alpha_i > 1$ and $R_i(x_i) = \int_{x_i}^\infty r_i(s_i) ds_i$ ($i = 1, \dots, n$). Suppose further that $m_i < 1$ and there exist constants λ_i ($i = 1, \dots, n$) such that*

$$p - 1 - \frac{(m_1 + \alpha_1 p - p) R_i(x_i)}{x_i r_i(x_i)} - \frac{R_i(x_i) r_i'(x_i)}{r_i^2(x_i)} \geq \frac{p}{\lambda_i}$$

for all $x_i \in (0, \infty)$ ($i = 1, \dots, n$). Then

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i^{-m_i} \right) H^p(x_1, \dots, x_n) dx_1 \cdots dx_n \\ & \leq \left(\prod_{i=1}^n \lambda_i \right)^p \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i^{p-m_i} \right) f^p(x_1, \dots, x_n) dx_1 \cdots dx_n, \end{aligned}$$

where

$$H(x_1, \dots, x_n) = \int_{x_1}^\infty \cdots \int_{x_n}^\infty \left(\prod_{i=1}^n \frac{x_i^{1-\alpha_i} s_i^{\alpha_i} r_i(s_i)}{R_i(x_i)} \right) f(s_1, \dots, s_n) ds_1 \cdots ds_n.$$

Theorem 2.8. Let p, α_i, f, r_i ($i = 1, \dots, n$) be as in Theorem 2.7. Suppose $m_i < 1$ and there exist λ_i ($i = 1, \dots, n$) such that

$$\frac{1 - m_i}{p} - \frac{x_i r_i'(x_i)}{r_i(x_i)} - \alpha_i \geq \frac{1}{\lambda_i}, \quad \text{for all } x_i \in (0, \infty) \text{ } (i = 1, \dots, n).$$

Then

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i^{-m_i} \right) H^p(x_1, \dots, x_n) dx_1 \cdots dx_n \\ & \leq \left(\prod_{i=1}^n \lambda_i \right)^p \int_0^\infty \cdots \int_0^\infty \left(\prod_{i=1}^n x_i^{p-m_i} \right) f^p(x_1, \dots, x_n) dx_1 \cdots dx_n, \end{aligned}$$

where

$$H(x_1, \dots, x_n) = \int_{x_1}^\infty \cdots \int_{x_n}^\infty \left(\prod_{i=1}^n \frac{s_i^{\alpha_i} r_i(s_i)}{x_i^{\alpha_i} r_i(x_i)} \right) f(s_1, \dots, s_n) ds_1 \cdots ds_n.$$

For $n = 2$, Theorems 2.5-2.8 extend the two-variable results of Yang and Jean [14].

3. Discrete Multivariate Inequalities

For use in the sequel we define $B_n \subset \mathbf{Z}_+^n$ by

$$B_n := \{x = (x_1, \dots, x_n) : x_i \in \mathbf{Z}_+\}.$$

Similarly for $x \in B_n$ we define

$$C_n(x) := \{y = (y_1, \dots, y_n) : y \in B_n, y_i \leq x_i (i = 1, \dots, n)\}.$$

For a function $u : B_n \rightarrow \mathbf{R}$ it is convenient to introduce the shorthand

$$\sum_{B_n} u(y) := \sum_{y_1=1}^\infty \cdots \sum_{y_n=1}^\infty u(y_1, y_2, \dots, y_n)$$

and

$$\sum_{C_n(x)} u(y) := \sum_{y_1=1}^{x_1} \cdots \sum_{y_n=1}^{x_n} u(y_1, y_2, \dots, y_n).$$

We are now in a position to state a discrete multivariate Hardy-type inequality given by Pachpatte [10].

Theorem 3.1. *Suppose $p > 0$ is a constant, $f(x) \geq 0$ for $x \in B_n$ and $A_n(x) = \sum_{C_n(x)} f(x)$ for $x \in B_n$. Then*

$$\sum_{B_n} \left\{ \frac{A_n(x)}{\prod_{i=1}^n x_i} \right\}^p \leq \left(\frac{p}{p-1} \right)^{np} \sum_{B_n} f^p(x).$$

Equality holds if $f(x) = 0$ for all x_i ($i = 1, 2, \dots, n$).

We provide a simple proof of the following extension of Theorem C. Theorem 3.1 occurs as the special case $\lambda_i(x_i) = 1$.

Theorem 3.2. *Let $p > 1$, $a(x) > 0$ and $\lambda_i(x_i) > 0$ for $x_i \geq 1$ ($i = 1, 2, \dots, n$). We define $\Lambda_i(x) = \sum_{y_i=1}^{x_i} \lambda_i(y_i)$ for $i = 1, 2, \dots, n$ and*

$$A_n(x) = \sum_{C_n(x)} \left\{ \prod_{i=1}^n \lambda_i(y_i) \right\} a(y) \quad x \in B_n.$$

Suppose $\sum_{B_n} \prod_{i=1}^n \lambda_i(x_i) a^p(x)$ converges for $x \in B_n$. Then

$$\sum_{B_n} \left(\prod_{i=1}^n \lambda_i(x_i) \right) \left\{ \frac{A_n(x)}{\prod_{i=1}^n \Lambda_i(x_i)} \right\}^p \leq \left(\frac{p}{p-1} \right)^{np} \sum_{B_n} \left(\prod_{i=1}^n \lambda_i(x_i) \right) a^p(x). \quad (3.1)$$

Proof. We employ induction. Theorem C gives the result for $n = 1$, supplying a basis. Let (3.1) be true for $n = k$, that is, let

$$\sum_{B_k} \left(\prod_{i=1}^k \lambda_i(x_i) \right) \left\{ \frac{A_k(x)}{\prod_{i=1}^k \Lambda_i(x_i)} \right\}^p \leq \left(\frac{p}{p-1} \right)^{kp} \sum_{B_k} \left(\prod_{i=1}^k \lambda_i(x_i) \right) a^p(x). \quad (3.2)$$

For $n = k + 1$, the left-hand side of (3.1) is

$$\sum_{B_k} \left(\prod_{i=1}^k \lambda_i(x_i) \right) \sum_{x_{k+1}=1}^{\infty} \lambda_{k+1}(x_{k+1}) \left\{ \frac{A_{k+1}(x)}{\Lambda_{k+1}(x_{k+1}) \prod_{i=1}^k \Lambda_i(x_i)} \right\}^p$$

$$= \sum_{B_k} \left(\prod_{i=1}^k \frac{\lambda_i(x_i)}{(\Lambda_i(x_i))^p} \right) \sum_{x_{k+1}=1}^{\infty} \lambda_{k+1}(x_{k+1}) \left\{ \frac{\sum_{y_{k+1}=1}^{x_{k+1}} \lambda_{k+1}(y_{k+1}) \sum_{C_k(x)} \left(\prod_{i=1}^k \lambda_i(y_i) \right) a(y)}{\Lambda_{k+1}(x_{k+1})} \right\}^p.$$

By Theorem C, the expression in braces is dominated by

$$\left(\frac{p}{p-1} \right)^p \left\{ \sum_{C_k(x)} \left(\prod_{i=1}^k \lambda_i(y_i) \right) a(y_1, \dots, y_k, x_{k+1}) \right\}^p.$$

Hence the left-hand side of (3.1) is dominated by

$$\left(\frac{p}{p-1} \right)^p \sum_{x_{k+1}=1}^{\infty} \lambda_{k+1}(x_{k+1}) \sum_{B_k} \left(\prod_{i=1}^k \lambda_i(x_i) \right) \left\{ \frac{\sum_{C_k(x)} \left(\prod_{i=1}^k \lambda_i(y_i) \right) a(y_1, \dots, y_k, x_{k+1})}{\prod_{i=1}^k \Lambda_i(x_i)} \right\}^p.$$

By the inductive hypothesis (3.2), this in turn is dominated by

$$\begin{aligned} & \left(\frac{p}{p-1} \right)^p \sum_{x_{k+1}=1}^{\infty} \lambda_{k+1}(x_{k+1}) \left(\frac{p}{p-1} \right)^{kp} \sum_{B_k} \left(\prod_{i=1}^k \lambda_i(x_i) \right) a^p(x_1, \dots, x_k, x_{k+1}) \\ &= \left(\frac{p}{p-1} \right)^{(k+1)p} \sum_{B_{k+1}} \left(\prod_{i=1}^{k+1} \lambda_i(x_i) \right) a^p(x_1, \dots, x_{k+1}). \end{aligned}$$

Hence we have the desired result for $n = k + 1$, as required.

To conclude, we give a simple proof of the following result of Hwang [4], which generalizes Theorem E.

Theorem 3.3. *Let $p > 1$, $a(x) > 0$, $\beta_i(x_i) > 0$, $\lambda_i(x_i) > 0$ for $x_i \geq 1$ ($i = 1, 2, \dots, n$). Define $\Lambda_i(x) = \sum_{y_i=1}^{x_i} \beta_i(y_i) \lambda_i(y_i)$ for $i = 1, 2, \dots, n$ and $A(x) = \sum_{C_n(x)} \left\{ \prod_{i=1}^n \beta_i(y_i) \lambda_i(y_i) \right\} a(y)$ for $x \in B_n$. Further, suppose $\sum_{B_n} \prod_{i=1}^n \lambda_i(x_i) a^p(x)$ converges for $x \in B_n$. If there exist constants $K_i > 0$ such that*

$$p-1 + \frac{[\beta_i(x_i+1) - \beta_i(x_i)] \Lambda_i(x_i)}{\beta_i(x_i+1) \beta_i(x_i) \lambda_i(x_i)} \geq \frac{p}{K_i} \quad (3.3)$$

for $x_i \geq 1$ ($i = 1, 2, \dots, n$), then

$$\sum_{B_n} \left(\prod_{i=1}^n \lambda_i(x_i) \right) \left\{ \frac{A(x)}{\prod_{i=1}^n \Lambda_i(x_i)} \right\}^p \leq \left(\prod_{i=1}^n K_i \right)^p \sum_{B_n} \left(\prod_{i=1}^n \lambda_i(x_i) \right) a^p(x). \quad (3.4)$$

Equality holds in (3.5) if $a(x) = 0$ for all $x_i (i = 1, 2, \dots, n)$.

Proof. We give a proof by induction. A basis is supplied for $n = 1$ by Theorem E. For the inductive step, suppose the theorem to be true for $n = k$ while the conditions hold for $n = k + 1$.

For $n = k + 1$, the left-hand side of (3.5) can be written as

$$\sum_{B_k} \left(\prod_{i=1}^k \frac{\lambda_i(x_i)}{\Lambda_i^p(x_i)} \right) \sum_{x_{k+1}=1}^{\infty} \lambda_{k+1}(x_{k+1}) \left\{ \frac{\sum_{y_{k+1}=1}^{x_{k+1}} \beta_{k+1}(y_{k+1}) \sum_{C_k(x)} \left(\prod_{i=1}^k \beta_i(y_i) \lambda_i(y_i) \right) a(y)}{\Lambda_{k+1}(x_{k+1})} \right\}^p.$$

By Theorem E, the expression in braces is less than or equal to

$$\sum_{B_k} K_{k+1}^p \left\{ \sum_{C_k(x)} \left(\prod_{i=1}^k \beta_i(y_i) \lambda_i(y_i) \right) a(y_1, \dots, y_k, x_{k+1}) \right\}^p.$$

Hence the left-hand side of (3.5) is less than or equal to

$$K_{k+1}^p \sum_{x_{k+1}=1}^{\infty} \lambda_{k+1}(x_{k+1}) \sum_{B_k} \left(\prod_{i=1}^k \lambda_i(x_i) \right) \left\{ \frac{\sum_{C_k(x)} \left(\prod_{i=1}^k \beta_i(y_i) \lambda_i(y_i) \right) a(y_1, \dots, y_k, x_{k+1})}{\prod_{i=1}^k \Lambda_i(x_i)} \right\}^p.$$

By the inductive hypothesis, this is less than or equal to

$$\begin{aligned} & K_{k+1}^p \sum_{x_{k+1}=1}^{\infty} \lambda_{k+1}(x_{k+1}) \left(\prod_{i=1}^k K_i \right)^p \sum_{B_k} \left(\prod_{i=1}^k \lambda_i(x_i) \right) a^p(x_1, \dots, x_k, x_{k+1}) \\ &= \left(\prod_{i=1}^{k+1} K_i \right)^p \sum_{B_{k+1}} \left(\prod_{i=1}^{k+1} \lambda_i(x_i) \right) a^p(x_1, \dots, x_{k+1}) \end{aligned}$$

and the proof is complete.

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