ON PSEUDO GENERALIZED QUASI-EINSTEIN MANIFOLDS

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Abstract. The object of the present paper is to introduce a type of non-flat Riemannian manifold called pseudo generalized quasi-Einstein manifold and studied some properties of such a manifold with several non-trivial examples.

1. Introduction

In 2000 M. C. Chaki and R. K. Maity [1] introduced the notion of quasi-Einstein manifold. A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is said to be quasi-Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the following:

\[
S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y)
\]

where \(\alpha, \beta\) are scalars such that \(\beta \neq 0\) and \(A\) is a non-zero 1-form defined by \(g(X, U) = A(X)\) for all vector fields \(X\); \(U\) being a unit vector field, called the generator of the manifold. An \(n\)-dimensional manifold of this kind is denoted by \((QE)_n\). The scalars \(\alpha, \beta\) are known as the associated scalars.

Recently U. C. De and G. C. Ghosh [3] introduced the notion of generalized quasi-Einstein manifold. A non-flat Riemannian manifold \((M^n, g)\) \((n > 2)\) is said to be generalized quasi-Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the following condition:

\[
S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y)
\]

where \(\alpha, \beta, \gamma\) are non-zero scalars and \(A, B\) are non-zero 1-forms defined respectively by \(g(X, U) = A(X)\) and \(g(X, V) = B(X)\) for all vector fields \(X\). The unit vector fields \(U\) and \(V\) corresponding to the 1-forms \(A\) and \(B\) are orthogonal i.e., \(g(U, V) = 0\). Also \(U\) and \(V\) are known as the generators of the manifold. Such an \(n\)-dimensional manifold of this kind is denoted by \((GQE)_n\).

The present paper deals with a non-flat Riemannian manifold called pseudo generalized quasi-Einstein manifold.

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A Riemannian manifold \((M^n, g)\) \((n > 2)\) is called a \textit{pseudo generalized quasi-Einstein manifold} if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the following:

\[
S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta D(X, Y)
\]

where \(\alpha, \beta, \gamma\) and \(\delta\) are non-zero scalars and \(A, B\) are non-zero 1-forms such that

\[
g(X, U) = A(X), \quad g(X, V) = B(X)
\]

for all vector fields \(X, U, V\) being mutually orthogonal unit vector fields called the generators of the manifold, \(D\) is a symmetric \((0,2)\) tensor, with zero trace, which satisfies the condition

\[
D(X, U) = 0
\]

for all vector fields \(X\). Also \(\alpha, \beta, \gamma\) and \(\delta\) are called the associated scalars; \(A, B\) are the associated 1-forms of the manifold and \(D\) is called the structure tensor of the manifold. Such an \(n\)-dimensional manifold will be denoted by \(P\text{-}(GQE)_n\).

Section 2 is concerned with the preliminaries and it is shown that the scalars \(\alpha + \beta\) and \(\alpha + \gamma + \delta D(V, V)\) are the Ricci curvatures along the directions of the vector fields \(U\) and \(V\) respectively. After preliminaries, in section 3, we prove the existence theorem for a \(P\text{-}(GQE)_n\). Section 4 is devoted to the conformally flat \(P\text{-}(GQE)_n\) and introduced the notion of \textit{pseudo generalized quasi-constant curvature}. It is shown that a manifold of pseudo generalized quasi-constant curvature is a \(P\text{-}(GQE)_n\). But the converse is not true, in general. However a \(P\text{-}(GQE)_3\) is a manifold of pseudo generalized quasi-constant curvature. Section 5 deals with some geometric properties of \(P\text{-}(GQE)_n\). In section 6 we investigate the application of \(P\text{-}(GQE)_4\) to the general relativistic viscous fluid spacetime admitting heat flux and it is shown that such a spacetime obeying Einstein’s equation with a cosmological constant is a connected semi-Riemannian \(P\text{-}(GQE)_4\). In the last section we discover several non-trivial examples of the \(P\text{-}(GQE)_n\) which are neither \((QE)_n\) nor \(G(QE)_n\).

### 2. Preliminaries

We consider a \(P\text{-}(GQE)_n\) \((n > 2)\). Let \(\{e_i : i = 1, \ldots, n\}\) be an orthonormal basis of the tangent space at any point of the manifold. Then setting \(X = Y = e_i\) in (1.3) and taking summation over \(i, 1 \leq i \leq n\) we obtain

\[
r = n\alpha + \beta + \gamma,
\]

where \(r\) is the scalar curvature of the manifold. Also from (1.3) we have

\[
S(U, U) = \alpha + \beta,
\]

\[
S(V, V) = \alpha + \gamma + \delta D(V, V)
\]

and

\[
S(U, V) = 0.
\]

It is known that if \(P\) be a unit vector field, then \(S(P, P)\) is the Ricci curvature in the direction of \(P\). Hence from (2.2) and (2.3) we can state the following:
Theorem 2.1. In a $P(GQE)_n$ ($n > 2$), the scalars $\alpha + \beta$ and $\alpha + \gamma + \delta D(V, V)$ are the Ricci curvatures in the directions of the generators $U$ and $V$ respectively.

Let $Q$ and $L$ be the symmetric endomorphisms of the tangent space at any point of the manifold corresponding to the Ricci tensor $S$ and the structure tensor $D$ respectively i.e., $g(QX, Y) = S(X, Y)$ and $g(LX, Y) = D(X, Y)$. Further, let $s^2$ and $d^2$ denote the squares of the length of the Ricci tensor $S$ and the structure tensor $D$ respectively. Then $s^2 = \sum_{i=1}^{n} S(Qe_i, e_i)$ and $d^2 = \sum_{i=1}^{n} D(Le_i, e_i)$. Now from (1.3) we get

$$\sum_{i=1}^{n} S(Qe_i, e_i) = (n-2)\alpha^2 + (\alpha + \beta)^2 + (\alpha + \gamma)^2 + \gamma \delta D(V, V) + \delta \sum_{i=1}^{n} S(Le_i, e_i). \quad (2.5)$$

Also from (1.3) we obtain

$$\sum_{i=1}^{n} S(Le_i, e_i) = \gamma D(V, V) + \delta \sum_{i=1}^{n} D(Le_i, e_i). \quad (2.6)$$

Hence from (2.5) and (2.6) it follows that

$$s^2 = n\alpha^2 + \beta^2 - \gamma^2 + 2[\alpha \beta + \gamma S(V, V)] + \delta^2 d^2. \quad (2.7)$$

From (2.7) it follows that $\delta > \frac{n}{d}$ (resp. $<, =$) according as $n\alpha^2 + \beta^2 - \gamma^2 + 2[\alpha \beta + \gamma S(V, V)] < 0$ (resp. $>, =$). Hence we can state the following:

Theorem 2.2. In a $P(GQE)_n$ ($n > 2$) the associated scalar $\delta$ is less than or equal or greater than the ratio which the length of the Ricci tensor $S$ bears to the length of the structure tensor $D$ according as $n\alpha^2 + \beta^2 - \gamma^2 + 2[\alpha \beta + \gamma S(V, V)] > 0$ or, $= 0$ or, $< 0$ respectively.

3. Existence Theorem of $P(GQE)_n$ $(n > 2)$

To prove the existence theorem of $P(GQE)_n$ $(n > 2)$, we first state a well-known result ([5], [6]) as follows:

Proposition 3.1. For a connected orientable manifold $M^n$ the following assertions are equivalent:

1. There is a non-vanishing vector field $V$ on $M^n$.
2. Either $M^n$ is non-compact, or $M^n$ is compact and has Euler number $\chi(M^n) = 0$.

Theorem 3.1. Let $(M^n, g)$ be a connected orientable Riemannian manifold which is either non-compact or compact with vanishing Euler number. If the Ricci tensor $S$ of type $(0, 2)$ of a Riemannian manifold is non-vanishing and satisfies the following relation

$$S(Y, Z)S(X, W) - S(X, Z)S(Y, W) = p_1 [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] + p_2 g(TX, W)g(Y, Z) \quad (3.1)$$
where $p_1$, $p_2$ are non-zero scalars and $T$ is the symmetric endomorphism of the tangent space at any point of the manifold corresponding to a tensor field of type $(0,2)$, then the manifold is a pseudo generalized quasi-Einstein manifold.

**Proof.** From the Proposition 3.1, it follows that there is a non-vanishing vector field $V$ on the manifold $(M^n, g)$ under consideration such that $g(X, V) = B(X)$ for all vector fields $X$. Then setting $Y = Z = V$ in (3.1) yields

$$S(V, V)S(X, W) - S(X, V)S(W, V) = p_1 [g(V, V)g(X, W) - g(X, V)g(W, V)] + p_2 g(TX, W)g(V, V),$$

which can be written as

$$aS(X, W) - B(QX)B(QW) = p_1 ||V||^2 g(X, W) - p_1 B(X)B(W) + p_2 ||V||^2 g(TX, W)$$

(3.2)

where $a = S(V, V)$ and $B(QX) = g(QX, V) = S(X, V)$. Since $S(V, V)$ is the Ricci curvature in the direction of the generator $V$ and the Ricci tensor is non-vanishing, it follows that the scalar $a$ is non-vanishing. From (3.2) it follows that

$$S(X, W) = a g(X, W) + \beta A(X)A(W) + \gamma B(X)B(W) + \delta D(X, W)$$

where $\alpha = \frac{p_1 ||V||^2}{a}$, $\beta = \frac{1}{a}$, $\gamma = -\frac{p_1}{a}$, $\delta = \frac{p_2 ||V||^2}{a}$, $A(X) = B(QX)$ and $D(X, W) = g(TX, W)$. Since $V$ is non-null, $S \neq 0$, $p_1$ and $p_2$ are non-zero scalars, it follows that $\alpha$, $\beta$, $\gamma$, $\delta$ are non-zero scalars. Hence the manifold is a $P(GQE)_n$.

### 4. Conformally flat $P(GQE)_n$ ($n > 3$)

Let $R$ be the curvature tensor of type $(0, 4)$ of a conformally flat $P(GQE)_n$. Then we have

$$R(X, Y, Z, W) = \frac{1}{n-2} [g(Y, Z)S(X, W) - g(X, Z)S(Y, W) + g(X, W)S(Y, Z) - g(Y, W)S(X, Z)] - \frac{r}{(n-1)(n-2)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)],$$

(4.1)

where $r$ is the scalar curvature of the manifold. Using (1.3) and (2.1) in (4.1) we obtain

$$R(X, Y, Z, W) = \frac{\alpha(n-2) - \beta - \gamma}{(n-1)(n-2)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$+ \frac{\beta}{n-2} [g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)]$$

$$+ \frac{\gamma}{n-2} [g(Y, Z)B(X)B(W) - g(X, Z)B(Y)B(W)] + g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)]$$

$$+ \frac{\delta}{n-2} [g(Y, Z)D(X, W) - g(X, Z)D(Y, W)] + g(X, W)D(Y, Z) - g(Y, W)D(X, Z)].$$

(4.2)
According to Chen and Yano [2], a Riemannian manifold \((M^n, g)\) \((n \geq 3)\) is said to be of quasi-constant curvature if it is conformally flat and its curvature tensor \(R\) of type \((0, 4)\) has the form

\[
R(X, Y, Z, W) = a_1 [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
+ a_2 [g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \\
+ g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)],
\]

where \(A\) is a 1-form and \(a_1, a_2\) are scalars of which \(a_2 \neq 0\).

Also according to De and Ghosh [3], a Riemannian manifold \((M^n, g)\) \((n \geq 3)\) is said to be of generalized quasi-constant curvature if it is conformally flat and its curvature tensor \(R\) of type \((0, 4)\) has the form

\[
R(X, Y, Z, W) = b_1 [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
+ b_2 [g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \\
+ g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\
+ b_3 [g(Y, Z)B(X)B(W) - g(X, Z)B(Y)B(W)] \\
+ g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)],
\]

where \(A\) and \(B\) are 1-forms and \(b_1, b_2, b_3\) are non-zero scalars. Generalizing this notion we define the manifold of\ pseudo generalized quasi-constant curvature\ as follows:

A Riemannian manifold \((M^n, g)\) \((n \geq 3)\) is said to be of\ pseudo generalized quasi-constant curvature\ if it is conformally flat and its curvature tensor \(R\) of type \((0, 4)\) satisfies the condition

\[
R(X, Y, Z, W) = \alpha_1 [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
+ \alpha_2 [g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \\
+ g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\
+ \alpha_3 [g(Y, Z)B(X)B(W) - g(X, Z)B(Y)B(W)] \\
+ g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)] \\
+ \alpha_4 [g(Y, Z)D(X, W) - g(X, Z)D(Y, W)] \\
+ g(X, W)D(Y, Z) - g(Y, W)D(X, Z)],
\tag{4.3}
\]

where \(\alpha_1, \alpha_2, \ldots, \alpha_4\) are non-zero scalars and \(D\) is a symmetric tensor of type \((0, 2)\).

Now the relation (4.2) can be written as

\[
R(X, Y, Z, W) = \beta_1 [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\
+ \beta_2 [g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W)] \\
+ g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)] \\
+ \beta_3 [g(Y, Z)B(X)B(W) - g(X, Z)B(Y)B(W)] \\
+ g(X, W)B(Y)B(Z) - g(Y, W)B(X)B(Z)] \\
+ \beta_4 [g(Y, Z)D(X, W) - g(X, Z)D(Y, W)] \\
+ g(X, W)D(Y, Z) - g(Y, W)D(X, Z)],
\tag{4.4}
\]
where $\beta_1 = \frac{a(n-2) - \beta - \gamma}{n-1}$, $\beta_2 = \frac{\beta}{n-2}$, $\beta_3 = \frac{\beta}{n-2}$, and $\beta_4 = \frac{\beta}{n-2}$ are non-zero scalars. Comparing (4.3) and (4.4), it follows that the manifold is of pseudo generalized quasi-constant curvature. This leads to the following:

**Theorem 4.1.** A conformally flat $P(GQE)_n$ ($n > 3$) is a manifold of pseudo generalized quasi-constant curvature.

Let us now consider a manifold of pseudo generalized quasi-constant curvature. Then from (4.3) it follows that

$$S(X, Y) = \tilde{\alpha} g(X, Y) + \tilde{\beta} A(X) A(Y) + \tilde{\gamma} B(X) B(Y) + \tilde{\delta} D(X, Y),$$

where $\tilde{\alpha} = (n-1)\alpha_1 + \alpha_2 + \alpha_3$, $\tilde{\beta} = (n-2)\alpha_2$, $\tilde{\gamma} = (n-2)\alpha_3$, and $\tilde{\delta} = (n-2)\alpha_4$ are non-zero scalars. Thus we have the following:

**Theorem 4.2.** A manifold $(M^n, g)$ ($n > 2$) of pseudo generalized quasi-constant curvature is a $P(GQE)_n$.

Now a $P(GQE)_n$ is not a manifold of pseudo generalized quasi-constant curvature in general. However, since a 3-dimensional Riemannian manifold is conformally flat, it follows by virtue of Theorem 4.1 that a $P(GQE)_3$ is a manifold of pseudo generalized quasi-constant curvature. This leads to the following:

**Corollary 4.1.** A $P(GQE)_3$ is a manifold of pseudo generalized quasi-constant curvature.

5. Geometric Properties of $P(GQE)_n$ ($n > 2$)

This section deals with some geometric properties of $P(GQE)_n$ ($n > 2$). From (1.3) it follows that

$$S(X, Y) = (\alpha + \beta) g(X, Y) \quad \text{for all} \ X.$$

This leads to the following:

**Theorem 5.1.** In a $P(GQE)_n$ ($n > 2$) the generator $U$ is an eigenvector of the Ricci tensor $S$ corresponding to the eigen value $\alpha + \beta$.

Next we suppose that in a $P(GQE)_n$ ($n > 2$), the generator $U$ is a parallel vector field. Then we have $\nabla_X U = 0$ for all $X$, which implies that $R(X, Y) U = 0$ and hence $S(X, U) = 0$ for all $X$. Again from (1.3) we have

$$S(X, U) = (\alpha + \beta) A(X).$$

Therefore we must have $\alpha + \beta = 0$. Thus we have the following:

**Theorem 5.2.** If the generator $U$ of a $P(GQE)_n$ ($n > 2$) is a parallel vector field then the associated scalars $\alpha, \beta$ are related by $\alpha + \beta = 0$. 
Again since $U$ and $V$ are orthogonal unit vector fields, we have from (2.2) that

$$g(QU, U) = \alpha + \beta,$$

which implies that $QU$ is orthogonal to $U$ if and only if $\alpha + \beta = 0$. Hence we can state the following:

**Theorem 5.3.** In a $P(GQ E)_n$ $(n > 2)$, $QU$ is orthogonal to $U$ if and only if $\alpha + \beta = 0$.

Further from (2.3) we obtain

$$g(QV, V) = \alpha + \gamma + \delta D(V, V),$$

which implies that $QV$ is orthogonal to $V$ if and only if $\alpha + \gamma + \delta D(V, V) = 0$. Thus we have the following:

**Theorem 5.4.** In a $P(GQ E)_n$ $(n > 2)$, $QV$ is orthogonal to $V$ if and only if $\alpha + \gamma + \delta D(V, V) = 0$.

We now consider a compact orientable $P(GQ E)_n$ $(n > 2)$ without boundary. From (1.3) we have

$$S(X, X) = \alpha g(X, X) + \beta A(X) A(X) + \gamma B(X) B(X) + \delta D(X, X). \quad (5.1)$$

Let us assume that $\theta_U$ be the angle between $U$ and any vector field $X$; $\theta_V$ be the angle between $V$ and any vector field $X$. Therefore, $\cos \theta_U = \frac{g(X, U)}{\sqrt{g(X, X)}}$ and $\cos \theta_V = \frac{g(X, V)}{\sqrt{g(X, X)}}$. Further, we assume that $\theta_U \geq \theta_V$. Then we have $\cos \theta_V \geq \cos \theta_U$ and consequently $g(X, V) \geq g(X, U)$. Hence from (5.1) we have

$$S(X, X) \geq (\alpha + \beta + \gamma) |g(X, U)|^2 \quad \text{when} \quad \alpha, \beta, \gamma, \delta D(X, X) \quad \text{are positive.} \quad (5.2)$$

**Definition 5.1.** A vector field $H$ in a Riemannian manifold $(M^n, g)$ $(n > 2)$ is said to be harmonic [7] if

$$d\tau = 0 \quad \text{and} \quad \delta \tau = 0 \quad (5.3)$$

where $\tau(X) = g(X, H)$ for all $X$.

It is known from [7] that in a compact, orientable Riemannian manifold $(M^n, g)$ $(n > 2)$, the following relation holds

$$\int_{M = P(GQ E)_n} [S(X, X) - \frac{1}{2} |d\tau|^2 + |\nabla X|^2 - (\delta \tau)^2] d\nu = 0 \quad \text{for any vector field} \; X, \quad (5.4)$$

where $'d\nu'$ denotes the volume element of $M$. Now let $X \in \chi(M)$ be a harmonic vector field. Then (5.4) yields by virtue of (5.3) that

$$\int_{M} [S(X, X) + |\nabla X|^2] d\nu = 0 \quad \text{for any vector field} \; X. \quad (5.5)$$
Hence if each of $\alpha, \beta, \gamma, \delta D(X,X)$ of $P(GQE)_n$ are positive, then (5.2) and (5.5) together yields
\[
\int_M [(\alpha + \beta + \gamma) |g(X,U)|^2 + |\nabla X|^2] d\nu \leq 0,
\]
which implies by virtue of $\alpha + \beta + \gamma > 0$ that
\[
g(X,U) = 0 \text{ and } \nabla X = 0 \text{ for any vector field } X.
\]
(5.6)
Thus from (5.6), it follows that $X$ is orthogonal to $U$ and $X$ is a parallel vector field.
Similarly for the case, $\theta_U \leq \theta_V$, arguing as before it can be shown that $g(X,V) = 0$ and $\nabla X = 0$ for any vector field $X$. Thus we can state the following:

**Theorem 5.5.** In a compact, orientable $P(GQE)_n$ ($n > 2$) without boundary any harmonic vector field $X$ is parallel and orthogonal to one of the generators of the manifold which makes greatest angle with the vector $X$ provided that $\alpha, \beta, \gamma$ and $\delta D(X,X)$ are positive scalars.

6. General relativistic viscous fluid spacetime admitting heat flux

Let $(M^4, g)$ be a connected semi-Riemannian viscous fluid spacetime admitting heat flux and satisfying Einstein’s equation with a cosmological constant $\lambda$. Also let $U$ be the unit timelike velocity vector field of the fluid, $V$ be the unit heat flux vector field and $D$ be the anisotropic pressure tensor of the fluid. Then we have
\[
g(U,U) = -1, \quad g(V,V) = 1, \quad g(U,V) = 0,
\]
(6.1)
\[
D(X,Y) = D(Y,X), \quad \text{Tr}. D = 0, \quad D(X,U) = 0 \text{ for all vector fields } X.
\]
(6.2)
Let
\[
g(X,U) = A(X), \quad g(X,V) = B(X) \text{ for all vector field } X.
\]
(6.3)
Also let $T$ be the energy-momentum tensor of type $(0, 2)$ describing the matter distribution of such a fluid and it be of the following form [4]
\[
T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y) + B(X)B(Y) + D(X,Y)
\]
(6.4)
where $\sigma$ and $p$ are the energy density and isotropic pressure respectively. General relativity flows from Einstein’s equation given by
\[
S(X,Y) - \frac{r}{2} g(X,Y) + \lambda g(X,Y) = k T(X,Y)
\]
for all vector fields $X, Y$, where $S$ is the Ricci tensor of type $(0, 2)$, $r$ is the scalar curvature, $\lambda$ is a cosmological constant. Thus by virtue of (6.4), the above equation can be written as
\[
S(X,Y) - \frac{r}{2} g(X,Y) + \lambda g(X,Y) = k [pg(X,Y) + (\sigma + p)A(X)A(Y) + B(X)B(Y) + D(X,Y)].
\]
(6.5)
Let us now consider a $P(GQE)_4$ viscous fluid spacetime with the generator $U$ as the flow vector field of the fluid. Again from (6.5) we have

$$S(X, Y) = [kp + \frac{r}{2} - \lambda]g(X, Y) + k(\sigma + p)A(X)A(Y) + kB(X)B(Y) + kD(X, Y)$$

which shows that the spacetime under consideration is a $P(GQE)_4$ with $kp + \frac{r}{2} - \lambda, k(\sigma + p), k$ and $k$ as associated scalars; $A$ and $B$ as associated 1-forms; $U$ and $V$ as generators and $D$ as the structure tensor of type $(0, 2)$. Hence we can state the following:

**Theorem 6.1.** A viscous fluid spacetime admitting heat flux and satisfying Einstein’s equation with a cosmological constant is a 4-dimensional connected semi-Riemannian pseudo generalized quasi-Einstein manifold.

Using (1.3) and (2.1) in (6.5) we get

$$2kp - 2\lambda + 2\alpha + \beta + \gamma g(X, Y) = [k(\sigma + p)]A(X)A(Y) + (\gamma - k)B(X)B(Y) - kD(X, Y). \quad (6.6)$$

Setting $Y = U$ in (6.6) we obtain by virtue of (6.1)–(6.3) that

$$2kp - 2\lambda + 2\alpha + \beta + \gamma A(X) = [k\sigma + kp - \beta]A(X) \quad \text{for all vector field } X. \quad (6.7)$$

Again setting $X = U$ in (6.7) we obtain

$$\sigma = \frac{2\alpha + 3\beta + \gamma - 2\lambda}{2k}. \quad (6.8)$$

Now contracting (6.5) we get

$$r - 2r + 4\lambda = k(3p - \sigma + 1),$$

which yields by virtue of (2.1) and (6.8) that

$$p = \frac{6\lambda - 6\alpha + \beta - \gamma - 2k}{6k}. \quad (6.9)$$

Since $\alpha, \beta, \gamma$ are not constants, from (6.8) and (6.9) it follows that $\sigma$ and $p$ are not constants. Hence we can state the following:

**Theorem 6.2.** If a viscous fluid $P(GQE)_4$ spacetime admitting heat flux obeys Einstein’s equation with a cosmological constant then none of the energy density and isotropic pressure of the fluid can be a constant.

Now if the associated scalars $\alpha, \beta, \gamma$ are constants then it follows from (6.8) and (6.9) that $\sigma$ and $p$ are constants. Since $\sigma > 0$ and $p > 0$, we have from (6.8) and (6.9) that

$$\lambda < \frac{2\alpha + 3\beta + \gamma}{2} \quad \text{and} \quad \lambda > \frac{6\alpha - \beta + \gamma - 2k}{6}$$
and hence
\[ \frac{6\alpha - \beta + \gamma - 2k}{6} < \lambda < \frac{2\alpha + 3\beta + \gamma}{2}. \]  

(6.10)

Thus we have the following:

**Theorem 6.3.** If a viscous fluid \( P(GQE)_4 \) spacetime admitting heat flux obeys Einstein's equation with a cosmological constant \( \lambda \), then \( \lambda \) satisfies the relation (6.10).

7. Some examples of \( P(GQE)_n \)

This section deals with several examples of \( P(GQE)_n \). On the real number space \( \mathbb{R}^n \) (with coordinates \( x^1, x^2, \ldots, x^n \)) we define a suitable Riemannian metric \( g \) such that \( \mathbb{R}^n \) becomes a Riemannian manifold \( (M^n, g) \). We calculate the components of the Ricci tensor and then we verify the defining condition (1.3).

**Example 1.** We define a Riemannian metric on the 4-dimensional real number space \( \mathbb{R}^4 \) by the formula
\[
d s^2 = g_{ij} \, d x^i \, d x^j = (d x^1)^2 + (x^1)^2 (d x^2)^2 + (x^1 \sin x^2)^2 (d x^3)^2 + (d x^4)^2, \\
(i, j = 1, 2, \ldots, 4),
\]

(7.1)

where \( x^1 \neq 0 \) and \( 0 < x^2 < \frac{\pi}{2} \). Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are
\[
\Gamma_{122}^1 = -x^1, \Gamma_{33}^1 = -x^1 (\sin x^2)^2, \Gamma_{12}^2 = \frac{1}{x^1} = \Gamma_{13}^3, \Gamma_{23}^3 = \cot x^2, \Gamma_{23}^2 = -\sin x^2 \cos x^2, \\
R_{2332} = -(x^1 \sin x^2)^2
\]

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:
\[
S_{22} = -1, \\
S_{33} = -(\sin x^2)^2.
\]

Also it can be easily found that the scalar curvature of the manifold is non-zero and is given by \( r = \frac{2}{(x^1)^2} \neq 0 \). Therefore \( \mathbb{R}^4 \) with the considered metric is a Riemannian manifold \( (M^4, g) \) of non-vanishing scalar curvature. We shall now show that this \( M^4 \) is a \( P(GQE)_4 \) i.e., it satisfies the defining relation (1.3). Let us now consider the associated scalars, associated 1-forms and structure tensor as follows:
\[
\alpha = -\frac{1}{(x^1)^2}, \beta = \frac{1}{(x^1)^2}, \gamma = \frac{1}{(x^1)^2}, \delta = \frac{2}{(x^1)^2}, \\
A_i(x) = x^1 \text{ for } i = 2, \\
= 0 \text{ otherwise},
\]

(7.2)
\[ B_i(x) = \begin{cases} 
1 & \text{for } i = 2, \\
= x^1 & \text{for } i = 3, \\
= 0 & \text{otherwise}, 
\end{cases} \tag{7.4} \]

\[ D_{ij}(x) = \begin{cases} 
\frac{1}{2} & \text{for } i = j = 1, 4, \\
= -\frac{1}{2} & \text{for } i = j = 2, 3, \\
= -x^1 & \text{for } i = 2, j = 3, \\
= 0 & \text{otherwise} 
\end{cases} \tag{7.5} \]

at any point \( x \in M \). In our \( M^4 \), (1.3) reduces with these associated scalars, 1-forms and structure tensor to the following equations:

(i) \( S_{11} = a g_{11} + b A_1 A_1 + \gamma B_1 B_1 + \delta D_{11}, \)

(ii) \( S_{22} = a g_{22} + b A_2 A_2 + \gamma B_2 B_2 + \delta D_{22}, \)

(iii) \( S_{23} = a g_{23} + b A_2 A_3 + \gamma B_2 B_3 + \delta D_{23}, \)

(iv) \( S_{33} = a g_{33} + b A_3 A_3 + \gamma B_3 B_3 + \delta D_{33}, \)

(v) \( S_{44} = a g_{44} + b A_4 A_4 + \gamma B_4 B_4 + \delta D_{44} \)

since for the cases other than (i)–(v) the components of each term of (1.3) vanishes identically and the relation (1.3) holds trivially. Now from (7.2)–(7.5) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):

\[
\text{R.H.S. of (i)} = a g_{11} + b A_1 A_1 + \gamma B_1 B_1 + \delta D_{11} = 0 = S_{11} = \text{L.H.S. of (i)}. 
\]

Again

\[
\text{R.H.S. of (ii)} = a g_{22} + b A_2 A_2 + \gamma B_2 B_2 + \delta D_{22} = -1 = S_{22} = \text{L.H.S. of (iv)}.
\]

\[
\text{R.H.S. of (iv)} = a g_{33} + b A_3 A_3 + \gamma B_3 B_3 + \delta D_{33} = -(\sin x^2)^2 = S_{33} = \text{L.H.S. of (iv)}.
\]

By a similar argument as in (i), (ii) and (iv) it can be shown that the relations (iii) and (v) are true. Therefore, \( (M^4, g) \) is a \( P(GQE) \) which is neither \( (QE)_4 \) nor \( G(QE)_4 \). Hence we can state the following:

**Theorem 7.1.** Let \((M^4, g)\) be a Riemannian manifold endowed with the metric given by

\[
ds^2 = g_{ij} dx^i dx^j = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1 \sin x^2)^2 (dx^3)^2 + (dx^4)^2, 
\]

\[(i, j = 1, 2, \ldots, 4), \]

where \( x^1 \neq 0 \) and \( 0 < x^2 < \frac{\pi}{2} \). Then \((M^4, g)\) is a pseudo generalized quasi-Einstein manifold of non-vanishing scalar curvature which is neither quasi-Einstein nor generalized quasi-Einstein.

**Example 2.** We define a Riemannian metric on the 4-dimensional real number space \( R^4 \) by the formula

\[
ds^2 = g_{ij} dx^i dx^j = e^{x^1} (dx^1)^2 + e^{x^2} (dx^2)^2 + e^{x^3} (dx^3)^2 + (\sin x^3)^2 (dx^4)^2, 
\]

\[(i, j = 1, 2, \ldots, 4), \tag{7.6} \]
where $\frac{\pi}{4} < x^3 < \frac{\pi}{2}$. Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are

\[
\Gamma^1_{11} = -\frac{1}{2}, \quad \Gamma^2_{22} = \Gamma^3_{33}, \quad \Gamma^4_{34} = \cot x^3, \quad \Gamma^3_{44} = -e^{-x^3} \sin x^3 \cos x^3,
\]

\[
R_{3443} = 1 + \frac{3}{2} \sin x^3 \cos x^3 - 3(\cos x^3)^2
\]

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

\[
S^{33} = 1 + \frac{3}{2} \cot x^3 - 2(\cot x^3)^2,
\]

\[
S^{44} = e^{-x^3} [1 + \frac{3}{2} \sin x^3 \cos x^3 - 3(\cos x^3)^2].
\]

Also it can be easily shown that the scalar curvature of the manifold is non-vanishing. Therefore $R^4$ with the considered metric is a Riemannian manifold $(M^4, g)$ of non-vanishing scalar curvature. We shall now show that this $M^4$ is a $P(GQE)_4$ i.e., it satisfies the defining condition (1.3). Let us now consider the associated scalars, associated 1-forms and structure tensor as follows:

\[
\alpha = -e^{-x^3}(\cot x^3)^2, \quad \beta = 3e^{-x^3} \cot x^3, \quad \gamma = e^{-x^3}, \quad \delta = e^{-x^3}(\cot x^3)^2,
\]

(7.7)

\[
A_i(x) = \sqrt{\frac{3e^{x^3} + 2e^{x^1} \cot x^3}{6}} \quad \text{for } i = 3,
\]

\[
= \sqrt{\frac{3}{2}(\cot x^3)^2 + 1} \quad \text{for } i = 4,
\]

\[
= 0 \quad \text{otherwise}.
\]

(7.8)

\[
B_i(x) = \sqrt{e^{x^3} - (\cot x^3)^2} \quad \text{for } i = 3,
\]

\[
= 0 \quad \text{otherwise}.
\]

(7.9)

\[
D_{ij}(x) = e^{x^1} \quad \text{for } i = j = 1,
\]

\[
= e^{x^2} \quad \text{for } i = j = 2,
\]

\[
= e^{-x^1} \quad \text{for } i = j = 3,
\]

\[
= \sqrt{\frac{[e^{x^3} - (\cot x^3)^2][3e^{x^3} + 2e^{x^1} \cot x^3]}{6}} \quad \text{for } i = 3, j = 4,
\]

\[
= e^{-x^2} \quad \text{for } i = j = 4,
\]

\[
= 0 \quad \text{otherwise}
\]

(7.10)

at any point $x \in M$. In our $M^4$, (1.3) reduces with these associated scalars, 1-forms and structure tensor to the following equations
(i) $S_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta D_{11}$,
(ii) $S_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta D_{22}$,
(iii) $S_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta D_{33}$,
(iv) $S_{34} = \alpha g_{34} + \beta A_3 A_4 + \gamma B_3 B_4 + \delta D_{34}$,
(v) $S_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta D_{44}$,
since for the cases other than (i)–(v) the components of each term of (1.3) vanishes identically and the relation (1.3) holds trivially. By virtue of (7.7)–(7.10) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (iii):
\[ \text{R.H.S. of (iii)} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta D_{33} = 1 + \frac{3}{2} \cot x^3 - 2(\cot x^3)^2 \]
\[ = S_{33} = \text{L.H.S. of (iii)}. \]

By a similar argument as in (iii) it can be shown that the relations (i), (ii), (iv) and (v) are true. Therefore, $(M^4, g)$ is a $P(GQE)$ which is neither $(QE)_4$ nor $G(QE)_4$. Thus we can state the following:

**Theorem 7.2.** Let $(M^4, g)$ be a Riemannian manifold endowed with the metric given by
\[ ds^2 = g_{ij} dx^i dx^j = e^{x^1} (dx^1)^2 + e^{x^2} (dx^2)^2 + e^{x^3} (dx^3)^2 + (\sin x^3)^2 (dx^4)^2, \]
where $\frac{x}{3} < x^3 < \frac{x}{2}$. Then $(M^4, g)$ is a pseudo generalized quasi-Einstein manifold of non-vanishing scalar curvature which is neither quasi-Einstein nor generalized quasi-Einstein.

**Example 3.** We define a Riemannian metric on the 4-dimensional real number space $R^4$ by the formula
\[ ds^2 = g_{ij} dx^i dx^j = (1 + 2p)(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2, \]
where $p = \frac{\rho^2}{x^2}$ and $\rho$ is a non-zero constant. Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are
\[ \Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{p}{1 + 2p}, \quad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{p}{1 + 2p}, \]
\[ R_{1221} = R_{1331} = R_{1441} = \frac{p}{1 + 2p} \]
and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

\[ S_{11} = \frac{3p}{(1 + 2p)^2}, \]
\[ S_{22} = \frac{p}{(1 + 2p)^2}, \]
\[ S_{33} = \frac{p}{(1 + 2p)^2}, \]
\[ S_{44} = \frac{p}{(1 + 2p)^2}. \]
Also it can be easily shown that the scalar curvature of the manifold is non-vanishing. Therefore $R^4$ with the considered metric is a Riemannian manifold $(M^4, g)$ of non-vanishing scalar curvature. We shall now show that this $M^4$ is a $P(GQE_4)$ i.e., it satisfies the defining condition (1.3). Let us now consider the associated scalars, associated 1-forms and structure tensor as follows:

\[
\alpha = \frac{p}{(1 + 2p)^3}, \quad \beta = \frac{4p}{(1 + 2p)^3}, \quad \gamma = -\frac{2p}{(1 + 2p)^3}, \quad \delta = \frac{p}{(1 + 2p)^3}, \quad (7.12)
\]

\[
A_i(x) = \sqrt{p} \quad \text{for } i = 1,
\]
\[
= \frac{1}{2} \quad \text{for } i = 3,
\]
\[
= \frac{p}{2} \quad \text{for } i = 4,
\]
\[
= 0 \quad \text{otherwise,} \quad (7.13)
\]

\[
B_i(x) = \frac{p}{\sqrt{2}} \quad \text{for } i = 2,
\]
\[
= 0 \quad \text{otherwise,} \quad (7.14)
\]

\[
D_{ij}(x) = 1 \quad \text{for } i = j = 1,
\]
\[
= -2\sqrt{p} \quad \text{for } i = 1, j = 3,
\]
\[
= -2p^2 \quad \text{for } i = 1, j = 4,
\]
\[
= p^2 \quad \text{for } i = j = 2,
\]
\[
= -1 \quad \text{for } i = j = 3,
\]
\[
= -p \quad \text{for } i = 3, j = 4,
\]
\[
= -p^2 \quad \text{for } i = j = 4,
\]
\[
= 0 \quad \text{otherwise} \quad (7.15)
\]

at any point $x \in M$. In our $M^4$, scalars, 1-forms and structure tensor to the following equations

(i) $S_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta D_{11}$,
(ii) $S_{13} = \alpha g_{13} + \beta A_1 A_3 + \gamma B_1 B_3 + \delta D_{13}$,
(iii) $S_{14} = \alpha g_{14} + \beta A_1 A_4 + \gamma B_1 B_4 + \delta D_{14}$,
(iv) $S_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta D_{22}$,
(v) $S_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta D_{33}$,
(vi) $S_{34} = \alpha g_{34} + \beta A_3 A_4 + \gamma B_3 B_4 + \delta D_{34}$,
(vii) $S_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta D_{44}$,

since for the cases other than (i)–(vii) the components of each term of (1.3) vanishes identically and the relation (1.3) holds trivially. By virtue of (7.12)–(7.15) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):

\[
\text{R.H.S. of (i)} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta D_{11} = \frac{3p}{(1 + 2p)^2} = S_{11} = \text{L.H.S. of (i)}.
\]
By a similar argument as in (i) it can be shown that the relations (ii)–(vii) are true. Therefore, \((M^4, g)\) is a \(P(GQE)_4\) which is neither \((QE)_4\) nor \(G(QE)_4\). Thus we can state the following:

**Theorem 7.3.** Let \((M^4, g)\) be a Riemannian manifold endowed with the metric given by

\[
d s^2 = g_{ij} \, dx^i \, dx^j = (1 + 2p)(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2, \quad (i, j = 1, 2, \ldots, 4)
\]

where \(p = \frac{\epsilon_1}{\rho^2}\) and \(\rho\) is a constant. Then \((M^4, g)\) is a pseudo generalized quasi-Einstein manifold of non-vanishing scalar curvature which is neither quasi-Einstein nor generalized quasi-Einstein.

**References**


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