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ON PSEUDO GENERALIZED QUASI-EINSTEIN MANIFOLDS

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Abstract. The object of the present paper is to introduce a type of non-flat Riemannian manifold called *pseudo generalized quasi-Einstein manifold* and studied some properties of such a manifold with several non-trivial examples.

1. Introduction

In 2000 M. C. Chaki and R. K. Maity [1] introduced the notion of quasi-Einstein manifold. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is said to be quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) \tag{1.1}$$

where α, β are scalars such that $\beta \neq 0$ and A is a non-zero 1-form defined by $g(X,U) = A(X)$ for all vector fields X; U being a unit vector field, called the generator of the manifold. An n-dimensional manifold of this kind is denoted by $(QE)_n$. The scalars α, β are known as the associated scalars.

Recently U. C. De and G. C. Ghosh [3] introduced the notion of generalized quasi-Einstein manifold. A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is said to be generalized quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following condition:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) \tag{1.2}$$

where α, β and γ are non-zero scalars and A, B are non-zero 1-forms defined respectively by $g(X,U) = A(X)$ and $g(X,V) = B(X)$ for all vector fields X. The unit vector fields U and V corresponding to the 1-forms A and B are orthogonal i.e., $g(U,V) = 0$. Also U and V are known as the generators of the manifold. Such an n-dimensional manifold of this kind is denoted by $G(QE)_n$.

The present paper deals with a non-flat Riemannian manifold called *pseudo generalized quasi-Einstein manifold*.

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A Riemannian manifold (M^n, g) $(n > 2)$ is called a *pseudo generalized quasi-Einstein manifold* if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following:

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta D(X,Y) \tag{1.3}$$

where α, β, γ and δ are non-zero scalars and A, B are non-zero 1-forms such that

$$g(X,U) = A(X), \quad g(X,V) = B(X) \tag{1.4}$$

for all vector fields X; U, V being mutually orthogonal unit vector fields called the generators of the manifold, D is a symmetric (0,2) tensor, with zero trace, which satisfies the condition

$$D(X,U) = 0 \tag{1.5}$$

for all vector fields X. Also α, β, γ and δ are called the associated scalars; A, B are the associated 1-forms of the manifold and D is called the structure tensor of the manifold. Such an n-dimensional manifold will be denoted by $P(GQE)_n$.

Section 2 is concerned with the preliminaries and it is shown that the scalars $\alpha+\beta$ and $\alpha+\gamma+\delta D(V,V)$ are the Ricci curvatures along the directions of the vector fields U and V respectively. After preliminaries, in section 3, we prove the existence theorem for a $P(GQE)_n$. Section 4 is devoted to the conformally flat $P(GQE)_n$ and introduced the notion of *pseudo generalized quasi-constant curvature*. It is shown that a manifold of pseudo generalized quasi-constant curvature is a $P(GQE)_n$. But the converse is not true, in general. However a $P(GQE)_3$ is a manifold of pseudo generalized quasi-constant curvature. Section 5 deals with some geometric properties of $P(GQE)_n$. In section 6 we investigate the application of $P(GQE)_4$ to the general relativistic viscous fluid spacetime admitting heat flux and it is shown that such a spacetime obeying Einstein's equation with a cosmological constant is a connected semi-Riemannian $P(GQE)_4$. In the last section we discover several non-trivial examples of the $P(GQE)_n$ which are neither $(QE)_n$ nor $G(QE)_n$.

2. Preliminaries

We consider a $P(GQE)_n$ $(n > 2)$. Let $\{e_i : i = 1, \ldots, n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then setting $X = Y = e_i$ in (1.3) and taking summation over $i, 1 \le i \le n$ we obtain

$$r = n\alpha + \beta + \gamma, \tag{2.1}$$

where r is the scalar curvature of the manifold. Also from (1.3) we have

$$S(U,U) = \alpha + \beta, \tag{2.2}$$

$$S(V,V) = \alpha + \gamma + \delta D(V,V) \text{ and} \tag{2.3}$$

$$S(U,V) = 0. \tag{2.4}$$

It is known that if P be a unit vector field, then $S(P,P)$ is the Ricci curvature in the direction of P. Hence from (2.2) and (2.3) we can state the following:

Theorem 2.1. *In a* $P(GQE)_n$ $(n>2)$*, the scalars* $\alpha+\beta$ *and* $\alpha+\gamma+\delta D(V,V)$ *are the Ricci curvatures in the directions of the generators* U *and* V *respectively.*

Let Q and L be the symmetric endomorphisms of the tangent space at any point of the manifold corresponding to the Ricci tensor S and the structure tensor D respectively i.e., $g(QX,Y)=S(X,Y)$ and $g(LX,Y)=D(X,Y)$. Further, let s^2 and d^2 denote the squares of the length of the Ricci tensor S and the structure tensor D respectively. Then $s^2=\sum_{i=1}^{n} S(Qe_i,e_i)$ and $d^2=\sum_{i=1}^{n} D(Le_i,e_i)$. Now from (1.3) we get

$$\sum_{i=1}^{n} S(Qe_i,e_i)=(n-2)\alpha^2+(\alpha+\beta)^2+(\alpha+\gamma)^2+\gamma\delta D(V,V)+\delta\sum_{i=1}^{n} S(Le_i,e_i). \tag{2.5}$$

Also from (1.3) we obtain

$$\sum_{i=1}^{n} S(Le_i,e_i)=\gamma D(V,V)+\delta\sum_{i=1}^{n} D(Le_i,e_i). \tag{2.6}$$

Hence from (2.5) and (2.6) it follows that

$$s^2=n\alpha^2+\beta^2-\gamma^2+2[\alpha\beta+\gamma S(V,V)]+\delta^2 d^2. \tag{2.7}$$

From (2.7) it follows that $\delta>\frac{s}{d}$ (resp. $<,=$) according as $n\alpha^2+\beta^2-\gamma^2+2[\alpha\beta+\gamma S(V,V)]<0$ (resp. $>,=$). Hence we can state the following:

Theorem 2.2. *In a* $P(GQE)_n$ $(n>2)$ *the associated scalar* δ *is less than or equal or greater than the ratio which the length of the Ricci tensor* S *bears to the length of the structure tensor* D *according as* $n\alpha^2+\beta^2-\gamma^2+2[\alpha\beta+\gamma S(V,V)]>0$ *or,* $=0$ *or,* <0 *respectively.*

3. Existence Theorem of $P(GQE)_n$ $(n>2)$

To prove the existence theorem of $P(GQE)_n$ $(n>2)$, we first state a well-known result ([5], [6]) as follows:

Proposition 3.1. *For a connected orientable manifold* M^n *the following assertions are equivalent:*

1. *There is a non-vanishing vector field* V *on* M^n.
2. *Either* M^n *is non-compact, or* M^n *is compact and has Euler number* $\chi(M^n)=0$.

Theorem 3.1. *Let* (M^n,g) *be a connected orientable Riemannian manifold which is either non-compact or compact with vanishing Euler number. If the Ricci tensor* S *of type* $(0,2)$ *of a Riemannian manifold is non-vanishing and satisfies the following relation*

$$\begin{aligned} S(Y,Z)S(X,W)-S(X,Z)S(Y,W) = {} & p_1[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\ & +p_2 g(TX,W)g(Y,Z) \end{aligned} \tag{3.1}$$

where p_1, p_2 are non-zero scalars and T is the symmetric endomorphism of the tangent space at any point of the manifold corresponding to a tensor field of type $(0,2)$*, then the manifold is a pseudo generalized quasi-Einstein manifold.*

Proof. From the Proposition 3.1, it follows that there is a non-vanishing vector field V on the manifold (M^n, g) under consideration such that $g(X,V) = B(X)$ for all vector fields X. Then setting $Y = Z = V$ in (3.1) yields

$$\begin{aligned} S(V,V)S(X,W) - S(X,V)S(W,V) &= p_1[g(V,V)g(X,W) - g(X,V)g(W,V)] \\ &\quad + p_2 g(TX,W)g(V,V), \end{aligned}$$

which can be written as

$$\begin{aligned} aS(X,W) - B(QX)B(QW) &= p_1||V||^2 g(X,W) - p_1 B(X)B(W) \\ &\quad + p_2||V||^2 g(TX,W) \end{aligned} \tag{3.2}$$

where $a = S(V,V)$ and $B(QX) = g(QX,V) = S(X,V)$. Since $S(V,V)$ is the Ricci curvature in the direction of the generator V and the Ricci tensor is non-vanishing, it follows that the scalar a is non-vanishing. From (3.2) it follows that

$$S(X,W) = \alpha g(X,W) + \beta A(X)A(W) + \gamma B(X)B(W) + \delta D(X,W)$$

where $\alpha = \frac{p_1||V||^2}{a}, \beta = \frac{1}{a}, \gamma = -\frac{p_1}{a}, \delta = \frac{p_2||V||^2}{a}, A(X) = B(QX)$ and $D(X,W) = g(TX,W)$. Since V is non-null, $S \neq 0$, p_1 and p_2 are non-zero scalars, it follows that α, β, γ, δ are non-zero scalars. Hence the manifold is a $P(GQE)_n$.

4. Conformally flat $P(GQE)_n$ ($n > 3$)

Let R be the curvature tensor of type (0, 4) of a conformally flat $P(GQE)_n$. Then we have

$$\begin{aligned} R(X,Y,Z,W) &= \frac{1}{n-2}[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &\quad + g(X,W)S(Y,Z) - g(Y,W)S(X,Z)] \\ &\quad - \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned} \tag{4.1}$$

where r is the scalar curvature of the manifold. Using (1.3) and (2.1) in (4.1) we obtain

$$\begin{aligned} R(X,Y,Z,W) &= \frac{\alpha(n-2) - \beta - \gamma}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + \frac{\beta}{n-2}[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &\quad + g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\ &\quad + \frac{\gamma}{n-2}[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) \\ &\quad + g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)] \\ &\quad + \frac{\delta}{n-2}[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ &\quad + g(X,W)D(Y,Z) - g(Y,W)D(X,Z)]. \end{aligned} \tag{4.2}$$

According to Chen and Yano [2], a Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be of quasi-constant curvature if it is conformally flat and its curvature tensor R of type (0, 4) has the form

$$\begin{aligned} R(X,Y,Z,W) = &\; a_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ a_2[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)], \end{aligned}$$

where A is a 1-form and a_1, a_2 are scalars of which $a_2 \neq 0$.
Also according to De and Ghosh [3], a Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be of generalized quasi-constant curvature if it is conformally flat and its curvature tensor R of type (0, 4) has the form

$$\begin{aligned} R(X,Y,Z,W) = &\; b_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ b_2[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\ &+ b_3[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) \\ &+ g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)], \end{aligned}$$

where A and B are 1-forms and b_1, b_2, b_3 are non-zero scalars. Generalizing this notion we define the manifold of *pseudo generalized quasi-constant curvature* as follows:
A Riemannian manifold (M^n, g) $(n \geq 3)$ is said to be of *pseudo generalized quasi-constant curvature* if it is conformally flat and its curvature tensor R of type (0, 4) satisfies the condition

$$\begin{aligned} R(X,Y,Z,W) = &\; \alpha_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ \alpha_2[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\ &+ \alpha_3[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) \\ &+ g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)] \\ &+ \alpha_4[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ &+ g(X,W)D(Y,Z) - g(Y,W)D(X,Z)], \end{aligned} \tag{4.3}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_4$ are non-zero scalars and D is a symmetric tensor of type (0, 2).
Now the relation (4.2) can be written as

$$\begin{aligned} R(X,Y,Z,W) = &\; \beta_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &+ \beta_2[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)] \\ &+ \beta_3[g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) \\ &+ g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)] \\ &+ \beta_4[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ &+ g(X,W)D(Y,Z) - g(Y,W)D(X,Z)], \end{aligned} \tag{4.4}$$

where $\beta_1 = \frac{\alpha(n-2)-\beta-\gamma}{(n-1)(n-2)}$, $\beta_2 = \frac{\beta}{n-2}$, $\beta_3 = \frac{\gamma}{n-2}$ and $\beta_4 = \frac{\delta}{n-2}$ are non-zero scalars. Comparing (4.3) and (4.4), it follows that the manifold is of pseudo generalized quasi-constant curvature. This leads to the following:

Theorem 4.1. *A conformally flat $P(GQE)_n$ $(n > 3)$ is a manifold of pseudo generalized quasi-constant curvature.*

Let us now consider a manifold of pseudo generalized quasi-constant curvature. Then from (4.3) it follows that

$$S(X,Y) = \bar{\alpha} g(X,Y) + \bar{\beta} A(X)A(Y) + \bar{\gamma} B(X)B(Y) + \bar{\delta} D(X,Y),$$

where $\bar{\alpha} = (n-1)\alpha_1 + \alpha_2 + \alpha_3$, $\bar{\beta} = (n-2)\alpha_2$, $\bar{\gamma} = (n-2)\alpha_3$ and $\bar{\delta} = (n-2)\alpha_4$ are non-zero scalars. Thus we have the following:

Theorem 4.2. *A manifold (M^n, g) $(n > 2)$ of pseudo generalized quasi-constant curvature is a $P(GQE)_n$.*

Now a $P(GQE)_n$ is not a manifold of pseudo generalized quasi-constant curvature in general. However, since a 3-dimensional Riemannian manifold is conformally flat, it follows by virtue of Theorem 4.1 that a $P(GQE)_3$ is a manifold of pseudo generalized quasi-constant curvature. This leads to the following:

Corollary 4.1. *A $P(GQE)_3$ is a manifold of pseudo generalized quasi-constant curvature.*

5. Geometric Properties of $P(GQE)_n$ $(n > 2)$

This section deals with some geometric properties of $P(GQE)_n$ $(n > 2)$. From (1.3) it follows that

$$S(X,U) = (\alpha + \beta) g(X,U) \quad \text{for all } X.$$

This leads to the following:

Theorem 5.1. *In a $P(GQE)_n$ $(n > 2)$ the generator U is an eigenvector of the Ricci tensor S corresponding to the eigen value $\alpha + \beta$.*

Next we suppose that in a $P(GQE)_n$ $(n > 2)$, the generator U is a parallel vector field. Then we have $\nabla_X U = 0$ for all X, which implies that $R(X,Y)U = 0$ and hence $S(X,U) = 0$ for all X. Again from (1.3) we have

$$S(X,U) = (\alpha + \beta) A(X).$$

Therefore we must have $\alpha + \beta = 0$. Thus we have the following:

Theorem 5.2. *If the generator U of a $P(GQE)_n$ $(n > 2)$ is a parallel vector field then the associated scalars α, β are related by $\alpha + \beta = 0$.*

Again since U and V are orthogonal unit vector fields, we have from (2.2) that

$$g(QU,U)=\alpha+\beta,$$

which implies that QU is orthogonal to U if and only if $\alpha+\beta=0$. Hence we can state the following:

Theorem 5.3. *In a $P(GQE)_n$ $(n>2)$, QU is orthogonal to U if and only if $\alpha+\beta=0$.*

Further from (2.3) we obtain

$$g(QV,V)=\alpha+\gamma+\delta D(V,V),$$

which implies that QV is orthogonal to V if and only if $\alpha+\gamma+\delta D(V,V)=0$. Thus we have the following:

Theorem 5.4. *In a $P(GQE)_n$ $(n>2)$, QV is orthogonal to V if and only if $\alpha+\gamma+\delta D(V,V)=0$.*

We now consider a compact orientable $P(GQE)_n$ $(n>2)$ without boundary. From (1.3) we have

$$S(X,X)=\alpha g(X,X)+\beta A(X)A(X)+\gamma B(X)B(X)+\delta D(X,X). \tag{5.1}$$

Let us assume that θ_U be the angle between U and any vector field X; θ_V be the angle between V and any vector field X. Therefore, $\cos\theta_U=\frac{g(X,U)}{\sqrt{g(X,X)}}$ and $\cos\theta_V=\frac{g(X,V)}{\sqrt{g(X,X)}}$. Further, we assume that $\theta_U\geq\theta_V$. Then we have $\cos\theta_V\geq\cos\theta_U$ and consequently $g(X,V)\geq g(X,U)$. Hence from (5.1) we have

$$S(X,X)\geq(\alpha+\beta+\gamma)\{g(X,U)\}^2 \text{ when } \alpha,\beta,\gamma,\delta D(X,X) \text{ are positive.} \tag{5.2}$$

Definition 5.1. A vector field H in a Riemannian manifold (M^n,g) $(n>2)$ is said to be harmonic [7] if

$$d\tau=0 \quad\text{and}\quad \delta\tau=0 \tag{5.3}$$

where $\tau(X)=g(X,H)$ for all X.
It is known from [7] that in a compact, orientable Riemannian manifold (M^n,g) $(n>2)$, the following relation holds

$$\int_{M=P(GQE)_n}[S(X,X)-\frac{1}{2}|d\tau|^2+|\nabla X|^2-(\delta\tau)^2]dv=0 \quad \text{for any vector field} X, \tag{5.4}$$

where 'dv' denotes the volume element of M. Now let $X\in\chi(M)$ be a harmonic vector field. Then (5.4) yields by virtue of (5.3) that

$$\int_M[S(X,X)+|\nabla X|^2]dv=0 \quad \text{for any vector field} X. \tag{5.5}$$

Hence if each of α, β, γ, $\delta D(X,X)$ of $P(GQE)_n$ are positive, then (5.2) and (5.5) together yields

$$\int_M [(\alpha+\beta+\gamma)|g(X,U)|^2 + |\nabla X|^2]\,dv \le 0,$$

which implies by virtue of $\alpha+\beta+\gamma > 0$ that

$$g(X,U) = 0 \text{ and } \nabla X = 0 \quad \text{for any vector field } X. \tag{5.6}$$

Thus from (5.6), it follows that X is orthogonal to U and X is a parallel vector field.
Similarly for the case, $\theta_U \le \theta_V$, arguing as before it can be shown that $g(X,V) = 0$ and $\nabla X = 0$ for any vector field X. Thus we can state the following:

Theorem 5.5. *In a compact, orientable $P(GQE)_n$ $(n > 2)$ without boundary any harmonic vector field X is parallel and orthogonal to one of the generators of the manifold which makes greatest angle with the vector X provided that α, β, γ and $\delta D(X,X)$ are positive scalars.*

6. General relativistic viscous fluid spacetime admitting heat flux

Let (M^4, g) be a connected semi-Riemannian viscous fluid spacetime admitting heat flux and satisfying Einstein's equation with a cosmological constant λ. Also let U be the unit timelike velocity vector field of the fluid, V be the unit heat flux vector field and D be the anisotropic pressure tensor of the fluid. Then we have

$$g(U,U) = -1, \quad g(V,V) = 1, \quad g(U,V) = 0, \tag{6.1}$$

$$D(X,Y) = D(Y,X), \quad Tr.D = 0, \quad D(X,U) = 0 \text{ for all vector fields } X. \tag{6.2}$$

Let

$$g(X,U) = A(X), \quad g(X,V) = B(X) \text{ for all vector field } X. \tag{6.3}$$

Also let T be the energy-momentum tensor of type $(0,2)$ describing the matter distribution of such a fluid and it be of the following form [4]

$$T(X,Y) = pg(X,Y) + (\sigma+p)A(X)A(Y) + B(X)B(Y) + D(X,Y) \tag{6.4}$$

where σ and p are the energy density and isotropic pressure respectively. General relativity flows from Einstein's equation given by

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y)$$

for all vector fields X, Y, where S is the Ricci tensor of type $(0,2)$, r is the scalar curvature, λ is a cosmological constant. Thus by virtue of (6.4), the above equation can be written as

$$\begin{aligned} S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = {} & k[pg(X,Y) + (\sigma+p)A(X)A(Y) \\ & +B(X)B(Y) + D(X,Y)]. \end{aligned} \tag{6.5}$$

Let us now consider a $P(GQE)_4$ viscous fluid spacetime with the generator U as the flow vector field of the fluid.
Again from (6.5) we have

$$S(X,Y) = [kp + \frac{r}{2} - \lambda]g(X,Y) + k(\sigma + p)A(X)A(Y) + kB(X)B(Y) + kD(X,Y)$$

which shows that the spacetime under consideration is a $P(GQE)_4$ with $kp + \frac{r}{2} - \lambda$, $k(\sigma + p)$, k and k as associated scalars; A and B as associated 1-forms; U and V as generators and D as the structure tensor of type $(0,2)$. Hence we can state the following:

Theorem 6.1. *A viscous fluid spacetime admitting heat flux and satisfying Einstein's equation with a cosmological constant is a 4-dimensional connected semi-Riemannian pseudo generalized quasi-Einstein manifold.*

Using (1.3) and (2.1) in (6.5) we get

$$\frac{2kp - 2\lambda + 2\alpha + \beta + \gamma}{2} g(X,Y) = [\beta - k(\sigma + p)]A(X)A(Y) + (\gamma - k)B(X)B(Y) - kD(X,Y). \tag{6.6}$$

Setting $Y = U$ in (6.6) we obtain by virtue of (6.1)–(6.3) that

$$\frac{2kp - 2\lambda + 2\alpha + \beta + \gamma}{2} A(X) = [k\sigma + kp - \beta]A(X) \text{ for all vector field } X. \tag{6.7}$$

Again setting $X = U$ in (6.7) we obtain

$$\sigma = \frac{2\alpha + 3\beta + \gamma - 2\lambda}{2k}. \tag{6.8}$$

Now contracting (6.5) we get

$$r - 2r + 4\lambda = k(3p - \sigma + 1),$$

which yields by virtue of (2.1) and (6.8) that

$$p = \frac{6\lambda - 6\alpha + \beta - \gamma - 2k}{6k}. \tag{6.9}$$

Since α, β, γ are not constants, from (6.8) and (6.9) it follows that σ and p are not constants. Hence we can state the following:

Theorem 6.2. *If a viscous fluid $P(GQE)_4$ spacetime admitting heat flux obeys Einstein's equation with a cosmological constant then none of the energy density and isotropic pressure of the fluid can be a constant.*

Now if the associated scalars α, β, γ are constants then it follows from (6.8) and (6.9) that σ and p are constants. Since $\sigma > 0$ and $p > 0$, we have from (6.8) and (6.9) that

$$\lambda < \frac{2\alpha + 3\beta + \gamma}{2} \text{ and } \lambda > \frac{6\alpha - \beta + \gamma - 2k}{6}$$

and hence

$$\frac{6\alpha-\beta+\gamma-2k}{6} < \lambda < \frac{2\alpha+3\beta+\gamma}{2}. \tag{6.10}$$

Thus we have the following:

Theorem 6.3. *If a viscous fluid $P(GQE)_4$ spacetime admitting heat flux obeys Einstein's equation with a cosmological constant λ, then λ satisfies the relation (6.10).*

7. Some examples of $P(GQE)_n$

This section deals with several examples of $P(GQE)_n$. On the real number space R^n (with coordinates $x^1, x^2, \dots, x^n$) we define a suitable Riemannian metric g such that R^n becomes a Riemannian manifold (M^n, g). We calculate the components of the Ricci tensor and then we verify the defining condition (1.3).

Example 1. We define a Riemannian metric on the 4-dimensional real number space R^4 by the formula

$$\begin{aligned} ds^2 = g_{ij}dx^i dx^j &= (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1 \sin x^2)^2(dx^3)^2 + (dx^4)^2, \\ &(i, j = 1, 2, \dots, 4), \end{aligned} \tag{7.1}$$

where $x^1 \neq 0$ and $0 < x^2 < \frac{\pi}{2}$. Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are

$$\Gamma^1_{22} = -x^1, \Gamma^1_{33} = -x^1(\sin x^2)^2, \Gamma^2_{12} = \frac{1}{x^1} = \Gamma^3_{13}, \Gamma^3_{23} = \cot x^2, \Gamma^2_{33} = -\sin x^2 \cos x^2,$$
$$R_{2332} = -(x^1 \sin x^2)^2$$

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

$$S_{22} = -1,$$
$$S_{33} = -(\sin x^2)^2.$$

Also it can be easily found that the scalar curvature of the manifold is non-zero and is given by $r = -\frac{2}{(x^1)^2} \neq 0$. Therefore R^4 with the considered metric is a Riemannian manifold (M^4, g) of non-vanishing scalar curvature. We shall now show that this M^4 is a $P(GQE)_4$ i.e., it satisfies the defining relation (1.3). Let us now consider the associated scalars, associated 1-forms and structure tensor as follows:

$$\alpha = -\frac{1}{(x^1)^2}, \ \beta = \frac{1}{(x^1)^2}, \ \gamma = \frac{1}{(x^1)^2}, \ \delta = \frac{2}{(x^1)^2}, \tag{7.2}$$

$$\begin{aligned} A_i(x) &= x^1 \quad \text{for } i = 2, \\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{7.3}$$

$$B_i(x) = 1 \quad \text{for } i=2,$$
$$= x^1 \quad \text{for } i=3,$$
$$= 0 \quad \text{otherwise}, \tag{7.4}$$

$$D_{ij}(x) = \frac{1}{2} \quad \text{for } i=j=1,4,$$
$$= -\frac{1}{2} \quad \text{for } i=j=2,3,$$
$$= -x^1 \quad \text{for } i=2, j=3,$$
$$= 0 \quad \text{otherwise} \tag{7.5}$$

at any point $x \in M$. In our M^4, (1.3) reduces with these associated scalars, 1-forms and structure tensor to the following equations:

(i) $S_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta D_{11}$,
(ii) $S_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta D_{22}$,
(iii) $S_{23} = \alpha g_{23} + \beta A_2 A_3 + \gamma B_2 B_3 + \delta D_{23}$,
(iv) $S_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta D_{33}$,
(v) $S_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta D_{44}$

since for the cases other than (i)−(v) the components of each term of (1.3) vanishes identically and the relation (1.3) holds trivially. Now from (7.2)−(7.5) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):

$$\text{R.H.S. of (i)} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta D_{11} = 0 = S_{11} = \text{ L.H.S. of (i)}.$$

Again

$$\text{R.H.S. of (ii)} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta D_{22} = -1 = S_{22} = \text{L.H.S. of (v)},$$
$$\text{R.H.S. of (iv)} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta D_{33} = -(\sin x^2)^2 = S_{33} = \text{L.H.S. of (iv)}.$$

By a similar argument as in (i), (ii) and (iv) it can be shown that the relations (iii) and (v) are true. Therefore, (M^4, g) is a $P(GQE)_4$ which is neither $(QE)_4$ nor $G(QE)_4$. Hence we can state the following:

Theorem 7.1. *Let (M^4, g) be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij} dx^i dx^j = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1 \sin x^2)^2 (dx^3)^2 + (dx^4)^2,$$
$$(i, j = 1, 2, \dots, 4),$$

where $x^1 \neq 0$ and $0 < x^2 < \frac{\pi}{2}$. Then (M^4, g) is a pseudo generalized quasi-Einstein manifold of non-vanishing scalar curvature which is neither quasi-Einstein nor generalized quasi-Einstein.

Example 2. We define a Riemannian metric on the 4-dimensional real number space R^4 by the formula

$$ds^2 = g_{ij} dx^i dx^j = e^{x^1} (dx^1)^2 + e^{x^2} (dx^2)^2 + e^{x^3} (dx^3)^2 + (\sin x^3)^2 (dx^4)^2,$$
$$(i, j = 1, 2, \dots, 4), \tag{7.6}$$

where $\frac{\pi}{4} < x^3 < \frac{\pi}{2}$. Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are

$$\Gamma^1_{11} = -\frac{1}{2} = \Gamma^2_{22} = \Gamma^3_{33}, \quad \Gamma^4_{34} = \cot x^3, \quad \Gamma^3_{44} = -e^{-x^3} \sin x^3 \cos x^3,$$
$$R_{3443} = 1 + \frac{3}{2} \sin x^3 \cos x^3 - 3(\cos x^3)^2$$

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

$$S_{33} = 1 + \frac{3}{2} \cot x^3 - 2(\cot x^3)^2,$$
$$S_{44} = e^{-x^3}[1 + \frac{3}{2} \sin x^3 \cos x^3 - 3(\cos x^3)^2].$$

Also it can be easily shown that the scalar curvature of the manifold is non-vanishing. Therefore R^4 with the considered metric is a Riemannian manifold (M^4, g) of non-vanishing scalar curvature. We shall now show that this M^4 is a $P(GQE)_4$ i.e., it satisfies the defining condition (1.3). Let us now consider the associated scalars, associated 1-forms and structure tensor as follows:

$$\alpha = -e^{-x^3}(\cot x^3)^2, \ \beta = 3e^{-x^3} \cot x^3, \ \gamma = e^{-x^3}, \ \delta = e^{-x^3}(\cot x^3)^2, \tag{7.7}$$

$$\begin{aligned} A_i(x) &= \sqrt{\frac{3e^{x^3} + 2e^{x^1} \cot x^3}{6}} \quad \text{for } i = 3, \\ &= \sqrt{\frac{3}{2}(\cot x^3)^2 + 1} \quad \text{for } i = 4, \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{7.8}$$

$$\begin{aligned} B_i(x) &= \sqrt{e^{x^3} - (\cot x^3)^2} \quad \text{for } i = 3, \\ &= 0 \quad \text{otherwise,} \end{aligned} \tag{7.9}$$

$$\begin{aligned} D_{ij}(x) &= e^{x^1} \quad \text{for } i = j = 1, \\ &= e^{x^2} \quad \text{for } i = j = 2, \\ &= -e^{x^1} \quad \text{for } i = j = 3, \\ &= \sqrt{\frac{[e^{x^3} - (\cot x^3)^2][3e^{x^3} + 2e^{x^1} \cot x^3]}{6}} \text{ for } i = 3, j = 4, \\ &= e^{-x^2} \quad \text{for } i = j = 4, \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{7.10}$$

at any point $x \in M$. In our M^4, (1.3) reduces with these associated scalars, 1-forms and structure tensor to the following equations

(i) $S_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta D_{11}$,
(ii) $S_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta D_{22}$,
(iii) $S_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta D_{33}$,
(iv) $S_{34} = \alpha g_{34} + \beta A_3 A_4 + \gamma B_3 B_4 + \delta D_{34}$,
(v) $S_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta D_{44}$,

since for the cases other than (i)−(v) the components of each term of (1.3) vanishes identically and the relation (1.3) holds trivially. By virtue of (7.7)−(7.10) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (iii):

$$\begin{aligned}\text{R.H.S. of (iii)} &= \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta D_{33} = 1 + \frac{3}{2}\cot x^3 - 2(\cot x^3)^2 \\ &= S_{33} = \text{ L.H.S. of (iii).}\end{aligned}$$

By a similar argument as in (iii) it can be shown that the relations (i), (ii), (iv) and (v) are true. Therefore, (M^4, g) is a $P(GQE)_4$ which is neither $(QE)_4$ nor $G(QE)_4$. Thus we can state the following:

Theorem 7.2. *Let (M^4, g) be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}(dx^1)^2 + e^{x^2}(dx^2)^2 + e^{x^3}(dx^3)^2 + (\sin x^3)^2(dx^4)^2,\ (i, j = 1, 2, \dots, 4)$$

where $\frac{\pi}{4} < x^3 < \frac{\pi}{2}$. Then (M^4, g) is a pseudo generalized quasi-Einstein manifold of non-vanishing scalar curvature which is neither quasi-Einstein nor generalized quasi-Einstein.

Example 3. We define a Riemannian metric on the 4-dimensional real number space R^4 by the formula

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],\ (i, j = 1, 2, \dots, 4) \tag{7.11}$$

where $p = \frac{e^{x^1}}{\rho^2}$ and ρ is a non-zero constant. Then the only non-vanishing components of the Christoffel symbols and the curvature tensor are

$$\begin{aligned}&\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{p}{1+2p},\ \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{p}{1+2p},\\ &R_{1221} = R_{1331} = R_{1441} = \frac{p}{1+2p}\end{aligned}$$

and the components which can be obtained from these by the symmetry properties. Using the above relations, we can find the non-vanishing components of Ricci tensor as follows:

$$\begin{aligned}S_{11} &= \frac{3p}{(1+2p)^2},\\ S_{22} &= \frac{p}{(1+2p)^2},\\ S_{33} &= \frac{p}{(1+2p)^2},\\ S_{44} &= \frac{p}{(1+2p)^2}.\end{aligned}$$

Also it can be easily shown that the scalar curvature of the manifold is non-vanishing. Therefore R^4 with the considered metric is a Riemannian manifold (M^4, g) of non-vanishing scalar curvature. We shall now show that this M^4 is a $P(GQE)_4$ i.e., it satisfies the defining condition (1.3). Let us now consider the associated scalars, associated 1-forms and structure tensor as follows:

$$\alpha = \frac{p}{(1+2p)^3},\ \beta = \frac{4p}{(1+2p)^3},\ \gamma = -\frac{2p}{(1+2p)^3},\ \delta = \frac{p}{(1+2p)^3}, \tag{7.12}$$

$$\begin{aligned} A_i(x) &= \sqrt{p} \quad \text{for } i=1,\\ &= \frac{1}{2} \quad \text{for } i=3,\\ &= \frac{p}{2} \quad \text{for } i=4,\\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{7.13}$$

$$\begin{aligned} B_i(x) &= \frac{p}{\sqrt{2}} \quad \text{for } i=2,\\ &= 0 \quad \text{otherwise}, \end{aligned} \tag{7.14}$$

$$\begin{aligned} D_{ij}(x) &= 1 \quad \text{for } i=j=1,\\ &= -2\sqrt{p} \quad \text{for } i=1,\ j=3,\\ &= -2p^{\frac{3}{2}} \quad \text{for } i=1,\ j=4,\\ &= p^2 \quad \text{for } i=j=2,\\ &= -1 \quad \text{for } i=j=3,\\ &= -p \quad \text{for } i=3,\ j=4,\\ &= -p^2 \quad \text{for } i=j=4,\\ &= 0 \quad \text{otherwise} \end{aligned} \tag{7.15}$$

at any point $x \in M$. In our M^4, scalars, 1-forms and structure tensor to the following equations

(i) $S_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta D_{11}$,
(ii) $S_{13} = \alpha g_{13} + \beta A_1 A_3 + \gamma B_1 B_3 + \delta D_{13}$,
(iii) $S_{14} = \alpha g_{14} + \beta A_1 A_4 + \gamma B_1 B_4 + \delta D_{14}$,
(iv) $S_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta D_{22}$,
(v) $S_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta D_{33}$,
(vi) $S_{34} = \alpha g_{34} + \beta A_3 A_4 + \gamma B_3 B_4 + \delta D_{34}$,
(vii) $S_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta D_{44}$,

since for the cases other than (i)−(vii) the components of each term of (1.3) vanishes identically and the relation (1.3) holds trivially. By virtue of (7.12)−(7.15) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (i):

$$\text{R.H.S. of (i)} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta D_{11} = \frac{3p}{(1+2p)^2} = S_{11} = \text{L.H.S. of (i)}.$$

By a similar argument as in (i) it can be shown that the relations (ii)−(vii) are true. Therefore, (M^4, g) is a $P(GQE)_4$ which is neither $(QE)_4$ nor $G(QE)_4$. Thus we can state the following:

Theorem 7.3. *Let (M^4, g) be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \; (i, j = 1, 2, \ldots, 4)$$

where $p = \frac{e^{x^1}}{\rho^2}$ and ρ is a constant. Then (M^4, g) is a pseudo generalized quasi-Einstein manifold of non-vanishing scalar curvature which is neither quasi-Einstein nor generalized quasi-Einstein.

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