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$\lambda(P)$-NUCLEARITY OF LOCALLY CONVEX SPACES HAVING GENERALIZED BASES

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Abstract. It has been established that a DF-space having a fully-$\lambda(P)$-basis is $\lambda(P)$-nuclear wherein P is a stable nuclear power set of infinite type. It is shown that a barrelled G_1-space $\lambda(Q)$ is uniformly $\lambda(P)$-nuclear iff $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $\lambda(Q)$. Suppose λ is a μ-perfect sequence space for a perfect sequence space μ such that there exist $u \in \lambda^\mu$ and $v \in \mu^x$ with $u_i \geq \varepsilon > 0$ and $v_i \geq \iota > 0$ for some ε and ι and for all i. Then the following results are found to be true.

(i) A sequentially complete space having a fully-$(\lambda, \sigma\mu)$-basis is $\lambda(P)$-nuclear, provided μ is a DF-space in which $\{e_i, e_i\}$ is a semi-$\lambda(P)$-basis.

(ii) Suppose $\{e_i, e_i\}$ is a fully-$(\lambda, \sigma\mu)$-basis for a barrelled G_i-space $\lambda(Q)$. If μ is barrelled and $\{e_i, e_i\}$ is a semi-$\lambda(P)$-basis for μ then $\lambda(Q)$ is uniformly $\lambda(P)$-nuclear.

(iii) A DF-space with a fully-$(\lambda, \sigma\mu)$-basis is $\lambda(P)$-nuclear wherein $(\lambda, \sigma\mu)$ is barrelled in which $\{e_i, e_i\}$ is a semi-$\lambda(P)$-basis.

Notations and Preliminary Results

Through this Section not only it has been sought to familiarize the reader with the concepts used here but also we recall a few basic results from various investigations, which are to be used in the present discussions.

This article expects rudimentary familiarity with classical theory of locally convex spaces in general, (cf. [9], [13]) and nuclear spaces in particular (cf. [16], [24]). For various terms, definitions and notions concerning sequence space theory it is requested to have a glance at [10] and [21].

Towards the generalization of the normal topology (cf. [10], [13]) Ruckle [20] introduced the concept of *$\sigma\mu$-topology* associated with a sequence space μ on an arbitrary sequence space λ. Indeed, the *μ-dual* of λ is the subspace of ω, the vector space of all scalar valued sequences; defined by

$$\lambda^\mu = \{y \in \omega : xy \in \mu, \quad \forall x \in \lambda\}.$$

In a similar way we can define another subspace of ω, namely; the *μ-dual* $\lambda^{\mu\mu}$ of λ^μ, where

$$\lambda^{\mu\mu} = (\lambda^\mu)^\mu = \{z \in \omega : yz \in \mu, \forall y \in \lambda^\mu\}.$$

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λ is said to be μ-perfect if $\lambda = \lambda^{\mu\mu}$. In order to topologize the spaces λ and λ^μ let us assume that D_μ is the family of semi-norms, generating the topology on μ. For $y \in \lambda^\mu$ and $p \in D_\mu$, we define

$$p_y(x) = p(\{x_n y_n\}), \qquad x \in \lambda.$$

Then the topology generated by the family $\{p_y : p \in D_\mu, y \in \lambda^\mu\}$ of semi-norms on λ is called the *$\sigma\mu$-topology*. Similarly, the *$\sigma\mu$-topology* on λ^μ is generated by the collection $\{p_x : p \in D_\mu, x \in \lambda\}$ of semi-norms where

$$p_x(y) = p(\{x_n y_n\}), \qquad y \in \lambda^\mu.$$

Notice that this μ-dual λ^μ includes in particular, the well known duals namely; α-dual (or cross dual), β-dual and γ-dual (cf. [20], [21]) which are obtained from λ^μ by taking respectively $\mu = \iota^1$, $\mu = cs$ (convergent series) and $\mu = bs$ (bounded partial sum) (cf. [8]).

When the sequence space μ is equipped with the normal topology $\eta(\mu, \mu^x)$ (cf. [10], [13]), the $\sigma\mu$-topology on λ is given by the family $\{p_{y,z} : y \in \lambda^\mu, z \in \mu^x)$ of semi-norms, where

$$p_{y,z}(x) = \sum_{n\geq 1} |x_n y_n z_n| \qquad (x \in \lambda)$$

Similarly, the $\sigma\mu$-topology on λ^μ is defined by the family $\{p_{x,z} : x \in \lambda, z \in \mu^x\}$ of semi-norms where

$$p_{x,z}(y) = \sum_{n\geq 1} |x_n y_n z_n| \qquad (y \in \lambda^\mu)$$

Concerning the various aspects of μ-perfectness and the impact of the sequence space μ on λ and λ^μ one is requested to refer [1], [2] and [8].

Passing onto bases theory, we begin with the following definitions. Let E be an 1.c.TVS and λ be a locally convex sequence space. A Schauder base $\{x_i, f_i\}$ for E is said to be a *semi-λ-base*, if for each $p \in D_E$, the mapping $\psi : E \to \lambda$ is well defined where

$$\psi_p(x) = \{f_i(x)p(x_i)\} \qquad (x \in E)$$

(or equivalently, $\{f_i(x)p(x_i)\} \in \lambda$, $\forall p \in D_E$) and it is called a *Q-fully-λ-base* if there exists a permutation π such that for each $p \in D_E$ the map $\psi_p^\pi : E \to \lambda$ is continuous where

$$\psi_p^\pi(x) = \{f_{\pi(i)}(x)p(x_{\pi(i)})\} \qquad (x \in E)$$

When π is the identity permutation, one gets what is called a *fully-λ-base*. Thus, a fully-λ-base is a Q-fully-λ-base. However, the converse remains untrue (cf. (22], [12]). For the details of various types of bases and their applications related aspects we turn to [1], [2], [12] and [14].

The following result which is to be found in [1], identifies topologically a sequentially complete space having a fully-λ-basis (λ being equipped with $\sigma\mu$-topology), with a Köthe space.

Proposition 0.1. *Suppose E is a sequentially complete space having a fully-λ-base $\{x_i, f_i\}$. Let $y \in \lambda^\mu$ and $z \in \mu^x$ be such that $y_i \geq \varepsilon > 0$ and $z_i \geq \iota > 0$, $\forall i$, for some ε and ι. Then E can be topologically identified with a Köthe space $\lambda(P)$ where*

$$P = \{p(x_i)a_i b_i : p \in D_E,\ a \in \lambda^\mu_+, b \in \mu^x_+\}$$

Also, contained in [1] is the following wherein the μ-dual λ^μ, takes the place of λ.

Proposition 0.2. *Let $y \in \lambda$ and $z \in \mu^x$ be such that $y_i \geq \varepsilon > 0$ and $z_i \geq \iota > 0$ for all i, for some ε and l. If a sequentially complete space E possesses a fully-λ^μ-base $\{x_i, f_i\}$ then it can be identified topologically with a Köthe space $\lambda(P_0)$ where*

$$P_0 = \{p(x_i)a_i b_i : p \in D_E, a \in \lambda_+, b \in \mu^x_+\}.$$

For more details one can go through [1] and [2] in order to appreciate the subject matter of this article. Investigations regarding the structure of nuclear Frechet spaces (cf. [5]) has given us the generalized nuclearity.

Let $\lambda(P)$ be a fixed nuclear G_∞-space. A linear mapping T of a normed space E into another normed space F is called $\lambda(P)$-*nuclear* (cf. [12], [17], [24]) if it has a representation in the form

$$Tx = \sum_{i=0}^{\infty} \alpha_i f_i(x) y_i$$

where $\{\alpha_i\} \in \lambda(P)$ and $\{f_i\}$, $\{y_i\}$ are bounded sequences in E^* and F respectively.

A locally convex space E is called $\lambda(P)$-*nuclear* (cf. [12], [23], [24]) if for every absolutely convex and closed neighbourhood u there is another such neighbourhood v contained in u such that the canonical mapping of the associated Banach space E^Λ_v into the associated Banach space E^Λ_u is $\lambda(P)$-nuclear.

Suppose now $P = \{(a^k_i) : k \geq 1\}$ is a stable nuclear power set of infinite type (cf. [5], [7]). Then for $k \geq 1$ we have the associated sequence space

$$\lambda(P; k) = \{x \in \omega : \sum_{i \geq 1} |x_i| a^k_i < \infty\}.$$

Following [22] (cf. [5] also) we say that an 1.c.TVS E is $\lambda(P; \mathbb{N})$-nuclear (or $\Lambda_\mathbb{N}(P)$-nuclear) if it is a $\lambda(P; k)$-nuclear for each $k \geq 1$. Equivalently, E is $\lambda(P; \mathbb{N})$-nuclear iff for each $k \geq 1$, $u \in \bigcup_E$, there exists $v \in \bigcup_E$, $v < u$ with $\{\delta_i(v, u)a^k_i\} \in \iota^\infty$ (cf. [5]). Well known example of a $\lambda(P; \mathbb{N})$-nuclear space is provided by $\lambda(P)$ itself, while $\lambda(P)$ is never $\lambda(P)$-nuclear space is provided by $\lambda(P)$ itself, while $\lambda(P)$ is never $\lambda(P)$-nuclear (cf. [5], [7], [12]). This establishes that, in general $\lambda(P; \mathbb{N})$-nuclearity is a weaker property than $\lambda(P)$-nuclearity. The facts and results with respect to $\lambda(P; \mathbb{N})$-nuclearity are to be found in [5], [6] and [22] while for the stronger notion $\lambda(P)$-nuclearity we turn to [6], [15] and [23].

Taking $\lambda(P)$ to be a stable nuclear power series space of infinite type $\Lambda(\alpha)$ (cf. [16], [17]) we have $\Lambda(\alpha)$-nuclearity and $\Lambda_{\mathbb{N}}(\alpha)$-nuclearity which have been discussed prominently in [6], [17] and [18].

Then there is this $\Lambda_1(\alpha)$-nuclearity (cf. [19]) which is a study in contrast vis-a-vis $\Lambda(\alpha)$-nuclearity.

Pertaining to generalized bases theory in which the associated sequence space λ carries the usual normal topology, the reader is requested to refer [11], [12] and [14]. The deep rooted relation between λ-base and λ-nuclearity presents a pleasant scenario which can be viewed through [11], [12] and [14].

At this stage it will be befitting to recall the following important result from [2] wherein the impact of the associated sequence space μ on a space having a fully-$(\lambda, \sigma\mu)$-basis is displayed.

Proposition 0.3. *Let E be sequentially complete space with a fully-λ-base $\{x_i, f_i\}$. Suppose that there exist $a \in \lambda^\mu$ and $b \in \mu^x$ such that $a_i \geq \varepsilon > 0$, $b_i \geq \iota > 0$ for all i for some ε and ι. Then E is $\lambda(P)$-nuclear provided $(\mu, \eta(\mu, \mu^x))$ is $\lambda(P)$-nuclear.*

Lastly, we come down to $\lambda(P;\phi)$-nuclearity (cf. [3], [4] and [12]) and from [3] recall the famous Grothendieck-Pietsch criterion for $\hat{\lambda}(P;\phi)$-nuclearity of a sequence space equipped with $\sigma\mu$-topology.

Theorem 0.4. *Let μ be a perfect sequence space such that λ is μ-perfect. Then $(\lambda, \sigma\mu)$ is $\hat{\lambda}(P, \phi)$-nuclear iff to each $a \in \lambda^\mu$, $y \in \mu^x$; there correspond $b \in \lambda^\mu$ and $z \in \mu^x$ such that the sequence $\{a_n y_n / b_n z_n\}$ can be rearranged into a sequence of $\lambda(P;\phi)$.*

A similar procedure adopted in the proof of the above result in[3] clearly says that the following is also true;

Theorem 0.5. *Let μ be a perfect sequence space. Then the μ-dual λ^μ is $\hat{\lambda}(P;\phi)$-nuclear iff for each $a \in \lambda$, $y \in \mu^x$ there exist $b \in \lambda$, $z \in \mu^x$ such that the sequence $\{a_n y_n / b_n z_n\}$ can be rearranged into a sequence of $\lambda(P;\phi)$.*

Remarks 0.6. (i) The above two results yield the Grothendieck-Pietsch criterion for $\hat{\lambda}(P, \phi)$-nuclearly of a Köthe space λ and its cross dual λ^x.

(ii) λ and λ^μ are always $\hat{\lambda}(P;\phi)$-nuclear for a $\hat{\lambda}(P, \phi)$-nuclear space μ, no matter what sequence space is choosen for λ.

Theorem 0.4 and Theorem 0.5 yield in particular the Grothendieck-Pietsch criterion for $\lambda(P)$-nuclearity.

Corollary 0.7. *Let λ be a μ-perfect space for a perfect sequence space μ. Then $\lambda[resp.\lambda^\mu]$ is $\lambda(P)$-nuclear iff to each $a \in \lambda^\mu$ (resp. $a \in \lambda$), $y \in \mu^x$ there correspond a $b \in \lambda^\mu$ (resp. $b \in \lambda$), $z \in \mu^x$ such that the sequence $\{a_n y_n / b_n z_n\}$ can be rearranged into a sequence of $\lambda(P)$.*

Throughout $P = \{(a_i^k) k \geq 1\}$ will be taken as a stable nuclear power set of infinite type (cf. [7], [22]).

1. $\lambda(P)$-nuclearity of Locally Convex Spaces having a Fully-λ-basis; λ being Equipped with the Normal Topology

This Section confirms that the ramifications of presence of a fully-$\lambda(P)$-basis in DF-spaces is relatively wider as compared to Frechet spaces. It also sends a loud and clear message that the presence of a fully-$\lambda(P)$-basis in G_1-spaces is rather too strong a condition vis-a-vis G_∞-spaces.

The investigations carried out in [6] and [23] informs us about the rich and powerful structures available in $\lambda(P)$ as well as its strong dual $(\lambda(P))^*_\beta$ (cf. [6], [23]) which is the foundation for the discussion to be held in this Section. For instance, consider the

Example 1.1. In [12] it has been affirmed that $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $\lambda(P)$ as well as for $(\lambda(P))^*_\beta$ while $\lambda(P)$ appears to be far away from being $\lambda(P)$-nuclear, in contrast $(\lambda(P))^*_\beta$ is always $\lambda(P)$-nuclear as elucidated in [14] and [23]. The last part can also be derived by resorting to Proposition 5.1 [14].

Since $(\lambda(P))^*_\beta$ is a DF-space, there lurks the suspicion that whether presence of a fully-$\lambda(P)$-basis in a DF-space invariably adds up to the $\lambda(P)$-nuclearity. That this is indeed true, as borne out by

Proposition 1.2. *Let E be a DF-space with a fully-$\lambda(P)$-basis $\{x_i, f_i\}$. Then E is $\lambda(P)$-nuclear.*

Proof. Since bounded sets are simple in $\lambda(P)$, and E^*_β is a Freachet space, using Corollary 4.2 [14] we find that E is semi-reflexive. But semi-reflexive DF-spaces are complete. So in view of Proposition 2.5 [22] E can be topologically identified with the Köthe space $\lambda(Q)$, where

$$Q = \{p(x_i)a_i^k : p \in \mathbb{D}_E, \qquad k \geq 1\}$$

Since $\lambda(P)$ is $\lambda(P;\mathbb{N})$-nuclear (cf. [22]) it follows that $\lambda(Q)$ is $\lambda(P,\mathbb{N})$-nuclear by Corollary 2.7 [22]. However, Nelimarkka [15] informs that $\lambda(P;\mathbb{N})$-nuclear DF-spaces are $\lambda(P)$-nuclear.

Note: (1) One can directly apply Theorem 2.6 [22] to get $\lambda(P;\mathbb{N})$-nuclearity.

(2) Completeness of E can also be obtained by using the fact that an 1.c.TVS with a fully-$\lambda(P)$-basis is always nuclear (cf. [12]). But DF-nuclear spaces are complete Montel spaces (cf. [16]).

Remark 1.3. (i) In the light of Example 1.1, it stands to reason that DF-character is essential for the validity of above result.

(ii) Taking $P = \{(i^k) : k \geq 1\}$ in the aforementioned result what we come across is that a DF-space with a fully-$\lambda(P)$-basis is strongly nuclear.

(iii) Fully-$\lambda(P)$-bases stay out of infinite dimensional normed spaces. This is averred by

Corollary 1.4. *Suppose E is a normed space and $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis for E. Then E is finite dimensional.*

Proof. [12] informs that an 1.c.TVS E with a fully-$\lambda(P)$-basis is nuclear; while normed nuclear spaces are finite dimensional (cf. [24]).

Note: Indeed, $(\lambda(P))^*_\beta$ is a uniformly $\lambda(P)$-nuclear G_1-space.
Restrictions on P yields the following in view of proposition 1.2.

Corollary 1.5. *Suppose E is a DF-space and $\{x_i, f_i\}$ is a fully-$\Lambda(\alpha)$-basis for E. Then E is $\Lambda(\alpha)$-nuclear; $\Lambda(\alpha)$ being a stable nuclear power series space of infinite type.*

Note: (i) $(\Lambda(\alpha))^*_\beta$ is a uniformly $\Lambda(\alpha)$-nuclear G_1-space (cf. [17], [18]).
(ii) if a DF-space E contains a fully-$\lambda(P)$-basis $\{x_i, f_i\}$ and $\{e_i, e_i\}$ is a fully-$\Lambda(\alpha)$-basis for $\lambda(P)$, then E is $\Lambda(\alpha)$-nuclear because $\{x_i, f_i\}$ becomes a fully-$\Lambda(\alpha)$-basis.
By Proposition 2.12 [18] $\Lambda(\alpha)$ is $\Lambda_1(\alpha)$-nuclear which yields a variant of Corollary 1.5 contained in

Corollary 1.6. *Suppose E is a DF-space with a fully-$\Lambda(\alpha)$-basis $\{x_i, f_i\}$. Then E is $\Lambda_1(\alpha)$-nuclear.*

Remarks 1.7. (i) $(\Lambda(\alpha))^*_\beta$ is a uniformly $\Lambda_1(\alpha)$-nuclear G_1-space.
(ii) If $\Lambda_1(\alpha)$ is nuclear, then an 1.c.TVS with a fully-$\Lambda(\alpha)$-basis is $\Lambda(\xi)$-nuclear where $\xi = (\xi_i)$, $\xi_i = (\alpha_i \log i)^{1/2}$. Its proof follows the standard analysis laid down in [12] for $\Lambda(\alpha)$ is $\Lambda(\xi)$-nuclear in view of Proposition 2.12 [18] as $\{\xi_i/\alpha_\iota\} \in c_0$.

Since $\lambda(P)^*_\beta$ is a uniformly $\lambda(P)$-nuclear Montel G_1-space, it transmits sufficient signals to mull over whether fully-$\lambda(P)$-basis character of $\{e_i, e_i\}$ in a Montel G_1-space measures upto the uniform $\lambda(P)$-nuclearity. Not only this is true in a barrelled G_1-space but also the reverse implication holds. Thus, for a barrelled G_1-space fully-$\lambda(P)$-basis character of $\{e_i, e_i\}$ and $\lambda(P)$-nuclearity are identical. This is manifest in

Proposition 1.8. *Suppose $\lambda(Q)$ is a barrelled G_1-space. Then $\lambda(Q)$ is uniformly $\lambda(P)$-nuclear iff $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $\lambda(Q)$.*

Proof. Suppose $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $\lambda(Q)$. Then invoking Proposition 2.1 [22] $\lambda(Q)$ can be identified topologically with the Köthe space $\lambda(M)$;

$$M = \{b_i a_i^k : b \in Q,\ k \geq 1\}$$

But by a result of [22] $\lambda(P)$ is $\lambda(P, \mathbb{N})$-nuclear which in turn yields the $\lambda(P)$-nuclearity of $\lambda(Q)$ as $\lambda(P, \mathbb{N})$-nuclear G_1-spaces are uniformly $\lambda(P)$-nuclear (cf. [22]).
Conversely, if $\lambda(Q)$ is uniformly $\lambda(P)$-nuclear, then by using the criterion Theorem 3.2 [23] we find that $Q \subset \lambda(P)$. Now take any $x \in \lambda(Q)$, $b \in Q$ and $k \geq 1$ arbitrarily. That $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $\lambda(Q)$ is a consequence of the following inequality;

$$\Sigma | < x, e_i > | p_b(e_i) a_i^k = \Sigma |x_i| b_i a_i^k$$

$$\leq \Sigma |x_i| c_i \cdot \Sigma c_i a_i^k$$
$$= K p_c(x)$$

where $c \in Q$ is such that $b_i \leq c_i^2$ due to the G_1-character and $K \equiv \Sigma c_i a_i^k < \infty$ as $Q \subset \lambda(P)$.

Note: This underscores the prodigious impact of a fully-$\lambda(P)$-basis in a barrelled G_1-space.

Opting for a power series space of infinite type $\Lambda(\alpha)$, in view of above result it is evident that the following stands confirmed

Corollary 1.9. *Suppose $\lambda(Q)$ is a barrelled G_1-space. Then $\lambda(Q)$ is uniformly $\tilde{\Lambda}_j(\alpha)$-nuclear for some $j > 1$ iff $\{e_i, e_i\}$ is a fully-$\Lambda(\alpha)$-basis for $\lambda(Q)$.*

Proof. This follows from the fact that, a nuclear G_1-space $\lambda(Q)$ is $\tilde{\Lambda}_j(\alpha)$-nuclear for some $j > 1$ iff $\lambda(Q)$ is uniformly $\Lambda(\alpha)$-nuclear which is a consequence of Proposition 2.11 [18]. The remainder of the proof is just the application of the above result.

Imposition of suitable restrictions on $\lambda(Q)$ in the above result presents a very interesting situation namely;

Corollary 1.10. *A nuclear power series space of finite type $\Lambda_1(\beta)$ is uniformly $\tilde{A}_j(\alpha)$-nuclear for some $j > 1$ iff $\{e_i, e_i\}$ is a fully-$\Lambda(\alpha)$-basis for $\Lambda_1(\beta)$.*

At this stage one may be inclined to know whether there exists a non-DF-, non-G_1-space in which a fully-$\lambda(P)$-basis guarantees the $\lambda(P)$-nuclearity. Yes, there are such spaces; for instance consider

Example 1.11. Let P be the set of all increasing sequence of real numbers. Then $\lambda(P) = \phi$ with its usual direct sum topology. It is easy to visualize that $\{e_i, e_i\}$ is a fully-ϕ-basis for ω. In addition, ω is ϕ-nuclear. However, ω is neither a G_1-space (otherwise $Q \subset \phi$ if $\omega = \lambda(Q)$) nor a DF-space. Incidentally, ω is not a nuclear G_∞-space (otherwise $\omega \subset l^1$) (cf. [23]).

This Section concludes with

Proposition 1.12. *A DF-space E with a fully-$\lambda(P_0)$-basis $\{x_i, f_i\}$ is $\Lambda(\alpha)$-nuclear provided $\lambda(P_0)$ is uniformly $\tilde{\Lambda}_j(\alpha)$-nuclear for some $j > 1$ with $\lambda(P_0) \subseteq l^1$.*

Proof. Since $\lambda(P_0)$ is in particular nuclear, by making use of Corollary 4.2 [14] we find that E is semi-reflexive which in turn yields the completeness of E. Now by invoking Proposition 2.1 [22] we can identify E topologically with the Köthe space $\lambda(M)$; $M = \{p(x_i)b_i : p \in \mathbb{D}_E,\ b \in P_0\}$. Since $\lambda(P_0)$ is uniformly $\tilde{\Lambda}_j(\alpha)$-nuclear it is $\Lambda_{\mathbb{N}}(\alpha)$-nuclear by Proposition 2.11 [18]. Thus, $\Lambda_{\mathbb{N}}(\alpha)$-nuclearity of $\lambda(M)$ follows by appealing to Proposition 2.1 [18]. However, $\Lambda_{\mathbb{N}}(\alpha)$-nuclear DF-spaces are always $\Lambda(\alpha)$-nuclear by Proposition 2.5 [18].

Remarks 1.13. A cursory glance at the above proof reveals that a barrelled G_1-space $\lambda(Q)$ in which $\{e_i, e_i\}$ is a semi-$\lambda(P_0)$-basis is uniformly $\Lambda(\alpha)$-nuclear for a uniformly $\tilde{\Lambda}_j(\alpha)$-nuclear Köthe space $\lambda(P_0)$ with $\lambda(P_0) \subseteq l^1$.

2. $\lambda(P)$-nuclearity of Spaces having a Fully-$(\lambda, \sigma\mu)$-basis

As suggested vividly by the caption this Section makes the arrangements for the study of $\lambda(P)$-nuclearity of spaces admitting fully-$(\lambda, \sigma\mu)$-bases.

Throughout this Section λ will be a μ-perfect sequence space for a perfect sequence space μ such that there exist $u \in \lambda^\mu$ and $\nu \in \mu^x$ with $u_i \geq \varepsilon > 0$ and $\nu_i \geq 1 > 0$ for some ε and 1 for all i.

To begin with we have the

Proposition 2.1. *Let E be a sequentially complete space with a fully-$\lambda(, \sigma\mu)$-basis $\{x_i, f_i\}$. Suppose $(\lambda, \sigma\mu)$ is $\lambda(P)$-nuclear. Then E is $\lambda(P)$-nuclear.*

Proof. Appealing to Proposition 0.1 one can topologically identify E with the Köthe space $\lambda(M)$;

$$M = \{p(x_i)y_i z_i : p \in \mathbb{D}_E, y \in \lambda^\mu_+,\ z \in \mu^x_+\}$$

Since $(\lambda, \sigma\mu)$ is $\lambda(P)$-nuclear using the Grothendieck-Pietsch criterion Corollary 0.7 we find that $\lambda(M)$ is $\lambda(P)$-nuclear in view of Grothendick-Pietsch criterion for $\lambda(P)$-nuclearity of a Köthe space; Remarks 0.6 (ii) (cf. [23], [24]). Thus, E becomes $\lambda(P)$-nuclear.

Note: In the light of Remarks 0.6, the above result yields at once Proposition 0.3.

If we impose further restrictions on the space $\lambda(P)$ then we obtain

Corollary 2.2. *Suppose E is a sequentially complete space with a fully-$(\lambda, \sigma\mu)$-basis $\{x_i, f_i\}$. If $(\lambda, \sigma\mu)$ is $\Lambda(\alpha)$-nuclear then E is $\Lambda(\alpha)$-nuclear.*

Remarks 0.6 informs that for a $\lambda(P)$-nuclear space μ; $(\lambda, \sigma\mu)$ is always $\lambda(P)$-nuclear thereby leading the way to

Corollary 2.3. *Suppose E is a sequentially complete space with a fully-$(\lambda, \sigma\mu)$-basis $\{x_i, f_i\}$. If μ is $\lambda(P)$-nuclear [resp. $\Lambda(\alpha)$-nuclear] then E is $\Lambda(P)$-nuclear [resp. $\Lambda(\alpha)$-nuclear].*

Note: This includes in particular proposition 2.5 [2].

Also, Remarks 0.6 intimates that $\lambda(P)$-nuclearity of $(\mu, \eta(\mu, \mu^x))$ brings forth the $\lambda(P)$-nuclearity of the μ-dual λ^μ. This sets the stage for,

Proposition 2.4. *Suppose $\{x_i, f_i\}$ is a fully-λ^μ-basis for a sequentially complete space E where μ is $\lambda(P)$-nuclear and for some $\xi \in \lambda$, $\xi_i \geq \varepsilon > 0$, for all i, for some $\varepsilon > 0$. Then E is $\lambda(P)$-nuclear.*

Proof. By making use of Proposition 0.2 E can be identified topologically with the Köthe space $\lambda(M)$;

$$M = \{p(x_i)y_i z_i : p \in \mathbb{D}_E, y \in \lambda_+, z \in \mu_+^x\}$$

But μ is $\lambda(P)$-nuclear and hence by Remarks 0.6 λ^μ is $\lambda(P)$-nuclear. Then the proof follows mutatis mutandis on lines similar to that of Proposition 2.1. Of course, one needs to use Corollary 0.7 which says that λ^μ is $\lambda(P)$-nuclear iff for each $y \in \lambda_+$ and $z \in \mu_+^x$ there correspond $a \in \lambda_+$, $b \in \mu_+^x$ and a permutation $\pi = \pi(y, z)$ such that

$$\left\{\frac{y_{\pi(i)} z_{\pi(i)}}{a_{\pi(i)} b_{\pi(i)}}\right\} \in \lambda(P).$$

In view of the study carried out in Section 1 we arrive at

Corollary 2.5. *Suppose E is a sequentially complete space with a fully-$(\lambda, \sigma\mu)$-basis and if μ is a DF-space in which $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis then E is $\lambda(P)$-nuclear.*

Proof. This follows from Proposition 1.2 and Corollary 2.3.

If the DF-character is withdrawn from the hypothesis of Corollary 2.5 then the following enables us to obtain only the $\lambda(P; \mathbb{N})$-nuclearity of E.

Corollary 2.6. *Suppose $\{x_i, f_i\}$ is a fully-$(\lambda, \sigma\mu)$-basis for a sequentially complete space E. Further, μ is barrelled and $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for μ; then E is $\lambda(P; \mathbb{N})$-nuclear.*

Proof. Since $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for μ, it turns out that for each $z \in \mu^x$ and $k \geq 1$ we get $t \in \mu^x$ with $|z_i| a_i^k \leq |t_i|$. Now take any $p \in \mathbb{D}_E$, $y \in \lambda^\mu$, $z \in \mu^x$ and $k \geq 1$. Then we arrive at the inequality

$$\Sigma |f_i(x)| p(x_i) |y_i z_i| a_i^k \leq \Sigma |f_i(x)| p(x_i) |y_i t_i| \leq q(x)$$

as $\{x_i, f_i\}$ is a fully-$(\lambda, \sigma\mu)$-basis for E. By making use of the existence of $u \in \lambda^\mu$ and $\nu \in \mu^x$ with $u_i \geq \varepsilon > 0$ and $\nu_i \geq 1 > 0$ in the above inequality we find that $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis for E. But a sequentially completes space with a fully-$\lambda(P)$-basis is always $\lambda(P; \mathbb{N})$-nuclear which has been established in Corollary 2.7 [22].

Turning to DF-spaces we have the

Proposition 2.7. *Suppose E is a DF-space with a fully-$(\lambda, \sigma\mu)$-basis $\{x_i, f_i\}$. If μ is barrelled and $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for μ then E is $\lambda(P)$-nuclear.*

Proof. Proceeding exactly as in Corollary 2.6 we obtain that $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis. Then just apply Proposition 1.2.

Analogously, for G_1-spaces we have

Proposition 2.8. *Suppose $\lambda(Q)$ is a barrelled G_1-space in which $\{e_i, e_i\}$ is a fully-$(\lambda, \sigma\mu)$-basis. If $\{e_i, e_i\}$ is a semi-$\lambda(P)$-basis for the barrelled space μ, then $\lambda(Q)$ is uniformly $\lambda(P)$-nuclear.*

Proof. Following the lines as in the proof of Corollary 2.6 we find that $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $\lambda(Q)$. The rest is the application of Proposition 1.8.

Remarks 2.9. For $\mu = \lambda(P)$ if we take λ to be $\lambda(P)$ itself in Proposition 2.7 then what we find is precisely Proposition 1.2; while for the above choice of λ and μ, Proposition 2.8 yields in particular, Proposition 1.8 because of the following;

Proposition 2.10. $(\lambda(P), \sigma(\lambda(P)) = \lambda(P)$

Proof. Since $\lambda(P) \cdot \lambda(P)^x = \lambda(P)$ and $\lambda(P)^x$ is a nuclear G_1-space it follows that $\lambda(P)$-dual of $\lambda(P)$ is $\lambda(P)^x$. The $\sigma\mu$-topology on $\lambda(P)$ is given by the colletion $\{p_{y,k} : y \in \lambda(P)^x, k \geq 1\}$ of semi-norms where

$$\begin{aligned} p_{y,k}(x) &= \Sigma |x_i y_i| a_i^k, \qquad x \in \lambda(P) \\ &\leq c\Sigma |x_i| a_i^1 a_i^k \\ &\leq c p_t(x) \end{aligned}$$

where $|y_i| \leq c a_i^1$ for some $l \geq 1$ as $y \in \lambda(P)^x$ and $a_i^1 a_i^k \leq a_i^t$ for some $t \geq 1$. Conversely, observe that $l^\infty \subset \lambda(P)^x$ as $\lambda(P)^x$ is a nuclear G_1-space. So $y = (1, 1, \dots) \in \lambda(P)^x$, thereby leading to the inequality;

$$\begin{aligned} p_t(x) &= \Sigma |x_i| a_i^t = \Sigma |x_i y_i| a_i^t, \qquad x \in \lambda(P) \\ &= p_{y,t}(x) \end{aligned}$$

for $t \geq 1$. This completes the proof.

Note: Because of the above result once again it is fairly visible that Proposition 2.7 yields Proposition 1.2. What one is required to do is just take $\lambda = \lambda(P) = \mu$.

The penultimate result of this article is the following invariant of Proposition 2.7, namely,

Proposition 2.11. *Let $\{x_i, f_i\}$ be a fully-$(\lambda, \sigma\mu)$-basis for a DF-space E. Suppose $(\lambda, \sigma\mu)$ is barrelled and $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $(\lambda, \sigma\mu)$. Then E is $\lambda(P)$-nuclear.*

Proof. Owing to Proposition 1.2 it will be sufficient to show that $\{x_i, f_i\}$ is a fully-$\lambda(P)$-basis for E. For $y \in \lambda^\mu$, $z \in \mu^x$ and $k \geq 1$, since $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for λ, we get $s \in \lambda^\mu$ and $t \in \mu^x$ with

$$|y_i z_i| a_i^k \leq |s_i t_i|, \qquad \forall\ i \geq 1.$$

Choosing $u \in \lambda^\mu$ and $\nu \in \mu^x$ with $u_i \geq \varepsilon > 0$ and $\nu_i \geq 1 > 0$ we obtain that for each $k \geq 1$, there exists $s \in \lambda^\mu$ and $t \in \mu^x$ with

$$a_i^k \leq c|x_i t_i|$$

for some constant $c > 0$. Thus, for any $p \in \mathbb{D}_E$ and $k \geq 1$ we get the inequality

$$\begin{aligned}\Sigma|f_i(x)|p(x_i)a_i^k &\leq c\Sigma|f_i(x)|p(x_i)|s_it_i| \\ &\leq cq(x)\end{aligned}$$

for some $q \in \mathbb{D}_E$ as $\{x_i, f_i\}$ is a fully-$\lambda, \sigma\mu)$-basis for E.

We come to close our discussions with the following which is a variant of Proposition 1.8.

Proposition 2.12. *Let $\{e_i, e_i\}$ be a fully-$(\lambda, \sigma\mu)$-basis for a barrelled G_1-space $\lambda(Q)$ where $(\lambda, \sigma\mu)$ is barrelled. Suppose $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $(\lambda, \sigma\mu)$. Then $\lambda(Q)$ is uniformly $\lambda(P)$-nuclear.*

Proof. In view of Proposition 1.8 it will be enough to show that $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $\lambda(Q)$ which we achieve by following the method adopted in the proof of the above result.

Remarks 2.13. From the discussions in [4] it can be safely concluded that for a $\lambda(P_0)$-nuclear space $\lambda(P)$, a locally convex space having a fully-$\lambda(P)$-basis is $\lambda(P_0)$-nuclear. But the structure of a Frechet nuclear G_∞-space $\lambda(P)$, indicates that although $\{e_i, e_i\}$ is a fully-$\lambda(P)$-basis for $\lambda(P)$; $\lambda(P)$ is never $\lambda(P)$-nuclear (cf. [6], [23]). The present article not only restores the $\lambda(P)$-nuclearity of DF-spaces [or G_1-spaces] from the presence of a fully-$\lambda(P)$-basis but also provides a simple alternative method (to the procedure adopted in [4]) to bring out the $\lambda(P)$-nuclearity ($\lambda(P;\mathbb{N})$-nuclearity) of a locally convex space possessing a fully-$(\lambda, \sigma\mu)$-basis. While the impact of $(\lambda, \sigma\mu)$ on a locally convex space having a fully-λ-basis has been analyzed in [4], in the present situation we focus our attention primarily to the influence of the associated sequence space μ (for larger choices of λ). Not only, some results (including the main result) in [4] is sharpened and extended in the present investigations but also these discussions assert that the role of the associated sequence space μ is equally significant for an arbitrary choice of λ.

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