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On the sum of distance Laplacian eigenvalues of graphs

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Abstract. Let G be a connected graph with n vertices, m edges and having diameter d. The distance Laplacian matrix D^L is defined as $D^L = \text{Diag}(Tr) - D$, where $\text{Diag}(Tr)$ is the diagonal matrix of vertex transmissions and D is the distance matrix of G. The distance Laplacian eigenvalues of G are the eigenvalues of D^L and are denoted by $\delta_1,\ \delta_1, \dots, \delta_n$. In this paper, we obtain (a) the upper bounds for the sum of k largest and (b) the lower bounds for the sum of k smallest non-zero, distance Laplacian eigenvalues of G in terms of order n, diameter d and Wiener index W of G. We characterize the extremal cases of these bounds. Also, we obtain the bounds for the sum of the powers of the distance Laplacian eigenvalues of G. Finally, we obtain a sharp lower bound for the sum of the βth powers of the distance Laplacian eigenvalues, where $\beta \neq 0, 1$.

Keywords. Distance matrix, distance Laplacian matrix, distance Laplacian eigenvalues, diameter, Wiener index

1 Introduction

A graph $G = (V, E)$ consists of the vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(G)$. We assume all the graphs under consideration are simple and connected. Further, $|V(G)| = n$ is the *order* and $|E(G)| = m$ is the *size* of G. The *degree* of v, denoted by $d_G(v)$ (we simply write d_v) is the number of edges incident on the vertex v. As usual, K_n is a complete graph with n vertices and $K_{1,n-1}$ is a star graph with n vertices. Also, $K_{a,b}$ is a complete bipartite graph with two partite sets V_1 and V_2 of cardinalities a and b, respectively, such that each vertex of V_1 is adjacent to every vertex of V_2. For other standard definitions, we refer [12].

The adjacency matrix $A = (a_{ij})$ of G is an $n \times n$ matrix whose (i, j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. Let $Deg(G) = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_{v_i}$, $i = 1, 2, \dots, n$ of G. The positive semi-definite matrix $L(G) = \text{Deg}(G) - A(G)$ is the Laplacian matrix of G. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues of G. Let $S_k(G)$ be the sum of the k largest Laplacian eigenvalues of G. Several researchers have been investigating the parameter $S_k(G)$ because of its importance in dealing with many problems in the theory, for instance, Brouwer's conjecture, Laplacian energy. More recent work on $S_k(G)$ can be seen in [5, 6, 7, 8, 13].

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In G, the *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G is denoted by $D(G)$ and is defined as $D(G) = (d_{uv})_{u,v\in V(G)}$. The *vertex transmission* $Tr_G(v)$ of a vertex v is defined as the sum of the distances from v to all other vertices in G, that is, $Tr_G(v) = \sum\limits_{u\in V(G)} d_{uv}$. A graph G is said to be *k-transmission regular* if $Tr_G(v) = k$, for each $v \in V(G)$. The *Wiener index* (also called *transmission*) of a graph G, denoted by $W(G)$, is the sum of distances between all unordered pairs of vertices in G. Clearly, $W(G) = \frac{1}{2}\sum\limits_{v\in V(G)} Tr_G(v)$. For any vertex $v_i \in V(G)$, the vertex transmission $Tr_G(v_i)$ is also called the *transmission degree* of v_i.

Let $Tr(G) = diag(Tr_1, Tr_2, \ldots, Tr_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [1] introduced the Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = Tr(G) - D(G)$ (or simply D^L) is called the *distance Laplacian matrix* of G. The eigenvalues of D^L are called the distance Laplacian eigenvalues of G. Since $D^L(G)$ is a real symmetric positive semi-definite matrix, we denote its eigenvalues by δ_i's and order them as $0 = \delta_n \leq \delta_{n-1} \leq \cdots \leq \delta_1$. The largest distance Laplacian eigenvalue δ_1 is called the distance Laplacian spectral radius of G. More work on distance Laplacian eigenvalues can be found in [2, 9, 11].

Motivated by the parameter $S_k(G)$ of the Laplacian matrix, we introduce the following. For $1 \leq k \leq n-1$, let U_k denote the sum of the k largest distance Laplacian eigenvalues and L_k denote the sum of k smallest positive distance Laplacian eigenvalues of the graph G, that is,

$$U_k = \sum_{i=1}^{k} \delta_i \quad and \quad L_k = \sum_{i=1}^{k} \delta_{n-i}.$$

In Section 2, we obtain the upper bounds for U_k in terms of order n, diameter d and Wiener index W of G. Also, we find the lower bounds for L_k in terms of the same parameters. In particular, we obtain the bounds for U_k and L_k when G is a bipartite graph. We characterize the extremal cases of these bounds. We derive the bounds for the sum of the powers of the distance Laplacian eigenvalues of G. Also, we obtain a sharp lower bound for the sum of the βth powers of the distance Laplacian eigenvalues, where $\beta \neq 0, 1$.

2 On the sum of the distance Laplacian eigenvalues of graphs

We begin with the following observation due to Caen [3].

Lemma 2.1. [3] *Let $[n] = \{1, 2, \ldots, n\}$ be the canonical n-element set and let $[n]^{(2)}$ denote the set of 2-element subsets of [n], that is, the edge set of K_n. To each entry $\{i, j\} = ij$ in $[n]^{(2)}$, associate a real variable x_{ij}, then for $n \geq 2$, and for all $x'_{ij}s$, we have*

$$\Big(\sum_{ij} x_{ij}\Big)^2 + \binom{n-1}{2}\sum_{ij} x_{ij}^2 - \frac{n-1}{2}\sum_{i}\Big(\sum_{j\neq i} x_{ij}\Big)^2 \geq 0.$$

The following result gives the upper bound for the sum of the squares of the vertex transmissions in terms of the Wiener index W, the diameter d and the order n of the graph G.

Lemma 2.2. *Let G be a connected graph with n vertices having diameter d. Then*

$$\sum_i Tr^2(i) \leq \frac{2W^2}{n-1} + \frac{n(n-1)(n-2)d^2}{2}$$

with equality if and only if $G \cong K_n$.

Proof. Substituting d_{ij} for x_{ij} in Lemma 2.1 and noting that each $d_{ij} \leq d$, we have

$$\Big(\sum_{ij} d_{ij}\Big)^2 + \binom{n-1}{2} \sum_{ij} d_{ij}^2 - \frac{n-1}{2} \sum_i \Big(\sum_{j \neq i} d_{ij}\Big)^2 \geq 0$$

$$\text{or} \quad W^2 + \binom{n-1}{2} \sum_{ij} d_{ij}^2 - \frac{n-1}{2} \sum_i Tr^2(i) \geq 0.$$

$$\text{Then} \quad \sum_i Tr^2(i) \leq \frac{2W^2}{n-1} + \frac{2}{n-1} \frac{(n-1)(n-2)}{2} \Big(\frac{n(n-1)}{2} d^2\Big),$$

$$\text{or} \quad \sum_i Tr^2(i) \leq \frac{2W^2}{n-1} + \frac{n(n-1)(n-2)d^2}{2},$$

proving the inequality.

Assume that the equality holds in the above inequalities. Then clearly each $d_{ij} = d$. Since G is connected, there is at least one $d_{ij} = 1$ and thus $d = 1$ which clearly shows that $G \cong K_n$.

Conversely, if $G \cong K_n$, then it is easy to see that $d = 1$, $W = \frac{n(n-1)}{2}$ and $\sum_i Tr^2(i) = n(n-1)^2$. Substituting these values in the above inequalities, we observe that the equality holds. ☐

Now, we obtain an upper bound for U_k in terms of the Wiener index, the diameter and the order of the graph G. The proof follows by using similar techniques as in Zhou [14].

Theorem 2.1. *Let G be a connected graph with n vertices having diameter d. For $1 \leq k \leq n-2$, we have*

$$U_k \leq \frac{2Wk}{n-1} + \frac{\sqrt{k(n-k-1)(dn(n-1)-2W)(dn(n-1)+2W)}}{\sqrt{2}(n-1)}$$

with equality if and only if $G \cong K_n$. Equality always holds for $k = n-1$.

Proof. It is easy to see that

$$\sum_{i=1}^{n-1} \delta_i = \sum_{i \in V} Tr(i) = 2W \quad \text{and} \quad \sum_{i=1}^{n-1} \delta_i^2 = \sum_i Tr^2(i) + 2 \sum_{1 \leq i < j \leq n} d_{ij}^2.$$

For $1 \leq k \leq n-2$, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&(2W - U_k)^2 \\
&= (\delta_{k+1} + \cdots + \delta_{n-1})^2 \leq (n-k-1)(\delta_{k+1}^2 + \cdots + \delta_{n-1}^2) \\
&= (n-k-1)\Big(\sum_i Tr^2(i) + 2 \sum_{1 \leq i < j \leq n} d_{ij}^2 - (\delta_1^2 + \cdots + \delta_k^2)\Big) \\
&\leq (n-k-1)\Big(\sum_i Tr^2(i) + 2 \sum_{1 \leq i < j \leq n} d_{ij}^2 - \frac{U_k^2}{k}\Big),
\end{aligned}$$

which implies that

$$U_k^2 - \frac{4kWU_k}{n-1} + \frac{4kW^2}{n-1} - \frac{k(n-k-1)}{n-1}\Big(\sum_i Tr^2(i) + 2\sum_{1\le i<j\le n} d_{ij}^2\Big) \le 0.$$

Thus,

$$U_k \le \frac{2Wk + \sqrt{k(n-k-1)\Big[(n-1)\Big(\sum_i Tr^2(i) + 2\sum_{1\le i<j\le n} d_{ij}^2\Big) - 4W^2\Big]}}{n-1}.$$

Using Lemma 2.2, we get

$$U_k \le \frac{2Wk + \sqrt{k(n-k-1)\Big[(n-1)\Big(\frac{2W^2}{n-1} + \frac{n(n-1)(n-2)d^2}{2} + d^2n(n-1)\Big) - 4W^2\Big]}}{n-1}.$$

On further simplifications, we have

$$U_k \le \frac{2Wk + \sqrt{\frac{k(n-k-1)\Big(d^2n^2(n-1)^2 - 4W^2\Big)}{2}}}{n-1},$$

or,

$$U_k \le \frac{2Wk}{n-1} + \frac{\sqrt{k(n-k-1)(dn(n-1)-2W)(dn(n-1)+2W)}}{\sqrt{2}(n-1)},$$

proving the inequalities.

Assume that the equality hold in above inequalities. Then all the above inequalities have to be equalities and after using Cauchy-Schwarz theorem and Lemma 2.2, we observe that $G \cong K_n$.

Conversely, let $G \cong K_n$. Taking the characteristic polynomial of K_n into consideration, it is quite easy to check that $U_k = nk$, $W = \frac{n(n-1)}{2}$ and $d = 1$. Using these values in the main inequality, we observe that the equality holds.

Using the fact that trace of a matrix is equal to sum of its eigenvalues and noting that $2W = U_{n-1}$, we see that equality always holds in main inequality when $k = n-1$. □

From Theorem 2.1, we obtain the following upper bound for the spectral radius δ_1 of the distance Laplacian matrix of a graph.

Theorem 2.2. *Let G be a connected graph on n vertices having diameter d. Then*

$$\delta_1 \le \frac{2W}{n-1} + \frac{\sqrt{(n-2)(dn(n-1)-2W)(dn(n-1)+2W)}}{\sqrt{2}(n-1)}$$

with equality if and only if $G \cong K_n$.

Using arguments same as in Theorem 2.1, we have the following lower bound for L_k.

Theorem 2.3. *Let G be a connected graph with n vertices having diameter d. Then, for $1 \le k \le n-2$, we have*

$$L_k \ge \frac{2Wk}{n-1} - \frac{\sqrt{k(n-k-1)(dn(n-1)-2W)(dn(n-1)+2W)}}{\sqrt{2}(n-1)}$$

with equality if and only if $G \cong K_n$. Equality always holds for $k = n-1$.

As a consequence of Theorem 2.3, we get the following upper bound for the smallest non-zero distance Laplacian eigenvalue δ_{n-1} of G.

$$\delta_{n-1} \geq \frac{2W}{n-1} - \frac{\sqrt{(n-2)(dn(n-1)-2W)(dn(n-1)+2W)}}{\sqrt{2}(n-1)}$$

with equality if and only if $G \cong K_n$.

Now, we have the following observation about the sum of the squares of the distances in a graph.

Lemma 2.3. *Let G be a connected graph with $n \geq 2$ vertices and m edges having diameter d. Then*

$$\sum_{1\leq i<j\leq n} d_{ij}^2 \leq \frac{2m(1-d^2)+d^2n(n-1)}{2}$$

with equality if and only if $d \leq 2$.

Proof. Since G contains m edges, therefore there are exactly m distances equal to 1 and the remaining distances (if there are any) are greater or equal to 2. As each $d_{ij} \leq d$, we have

$$\begin{aligned}\sum_{1\leq i<j\leq n} d_{ij}^2 &\leq m + d^2\Big(\frac{n(n-1)}{2} - m\Big)\\ &= \frac{2m(1-d^2)+d^2n(n-1)}{2},\end{aligned}$$

proving the inequality.

Assume that the equality hold in above inequalities. If there are no non-adjacent pair of vertices, then clearly $d = 1$, so that G is a complete graph. If there are some non-adjacent pair of vertices, then from the above proof we observe that the distance between every non-adjacent pair of vertices is same and equals to d. Since G is connected, at least one such distance equals to 2, so that $d = 2$.

Conversely, if $d = 1$, then G is a complete graph and it is easy to check that the equality holds in this case. Similarly, we can easily see that the equality holds when $d = 2$. This proves the result. □

From Lemma 2.3, we obtain the following corollary for the class of connected bipartite graphs.

Corollary 2.4. *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and having diameter d. Then*

$$\sum_{1\leq i<j\leq n} d_{ij}^2 \leq \frac{2m(1-d^2)+d^2n(n-1)}{2}$$

with equality if and only if G is a complete bipartite graph.

We require the following lemma due to Zhou [14].

Lemma 2.4. [14] *Let G be a connected bipartite graph on n vertices and m edges and let d_v be the degree of any vertex v of G. Then*

$$\sum_{v\in V(G)} d_v^2 \leq mn$$

with equality if and only if G is a complete bipartite graph.

We have the following observation for the sum of the squares of the vertex transmissions of a bipartite graph.

Lemma 2.5. *Let G be a connected bipartite graph having $n \geq 3$ vertices, m edges and diameter d with bi-partition of vertex set as $\{V_1, V_2\}$ with $|V_1| = a$, $|V_2| = b$, $a \geq b \geq 1$ and $a + b = n$. Then*

$$\sum_i Tr^2(i) \leq mn(1-d)^2 + nd^2(n-1)^2 + 4md(1-d)(n-1)$$

with equality if and only if G is a complete bipartite graph.

Proof. Without loss of generality, let $i \in V_1$. Since G is bipartite we have the following inequality

$$Tr(i) \leq d_i + d(b - d_i) + d(a-1),$$

or,

$$Tr(i) \leq d_i(1-d) + d(n-1)$$

so that

$$Tr^2(i) \leq d_i^2(1-d)^2 + (d(n-1))^2 + 2dd_i(1-d)(n-1).$$

Taking the summation over all vertices in V, we have

$$\sum_i Tr^2(i) \leq \sum_i d_i^2(1-d)^2 + \sum_i (d(n-1))^2 + \sum_i 2dd_i(1-d)(n-1).$$

Using Lemma 2.4 and noting that $\sum\limits_i d_i = 2m$, from the above inequality, we have

$$\sum_i Tr^2(i) \leq mn(1-d)^2 + nd^2(n-1)^2 + 4md(1-d)(n-1).$$

Assume that the equality hold in above inequality. Then we observe that the distance between every non-adjacent pair of vertices is the same and equals to 2, which is possible only if G is complete bipartite.

Conversely, it is easy to check that the equality holds for complete bipartite graphs. □

We note the following observation.

Lemma 2.6. [1] *The distance Laplacian characteristic polynomial of a complete bipartite graph $K_{a,b}$ is $P_L^{K_{a,b}}(x) = x(x-n)(x-(2a+b))^{a-1}(x-(2b+a))^{b-1}$.*

Now, we obtain an upper bound for U_k when G is a connected bipartite graph.

Theorem 2.5. *Let G be a connected bipartite graph with $n \geq 3$ vertices, m edges and having diameter d. Then, for $1 \leq k \leq n-2$, we have*

$$U_k \leq \frac{2Wk + \sqrt{k(n-k-1)\Big[(n-1)\Big(m(1-d)(n+3dn-2d+2) + d^2n^2(n-1)\Big) - 4W^2\Big]}}{n-1}$$

with equality if and only if $k = n-2$ and $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$. Equality always holds for $k = n-1$.

Proof. Let $1 \le k \le n-2$. From the proof of Theorem 2.1 and using Corollary 2.4 and Lemma 2.5, we get the required inequality.

Let the equality hold in above inequality. Then we observe that

$$\delta_1 = \delta_2 = \cdots = \delta_k \quad and \quad \delta_{k+1} = \delta_{k+2} = \cdots = \delta_{n-1}$$

and G is a complete bipartite graph. Taking Lemma 2.6 into consideration, we have $k = n-2$ and $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$.

Conversely, if $k = n-2$ and $G \cong k_{1,n-1}$ or $G \cong k_{\frac{n}{2},\frac{n}{2}}$, then we can easily check that equality follows in both cases.

Using same argument as in Theorem 2.1, we see that equality holds when $k = n-1$. □

Some results for δ_1 in bipartite graphs can be seen in [10]. Using Theorem 2.5, we get the following strict upper bound for distance Laplacian spectral radius for the class of connected bipartite graphs with $n \ge 4$.

Theorem 2.6. *Let G be a connected bipartite graph with $n \ge 4$ vertices, m edges and having diameter d. Then*

$$\delta_1 < \frac{2W + \sqrt{(n-2)\Big[(n-1)\Big(m(1-d)(n+3dn-2d+2) + d^2n^2(n-1)\Big) - 4W^2\Big]}}{n-1}.$$

With the help of Theorem 2.5, we can easily obtain the following upper bound for L_k.

Theorem 2.7. *Let G be a connected bipartite graph with $n \ge 3$ vertices, m edges and having diameter d. Then, for $1 \le k \le n-2$, we have*

$$L_k \ge \frac{2Wk - \sqrt{k(n-k-1)\Big[(n-1)\Big(m(1-d)(n+3dn-2d+2) + d^2n^2(n-1)\Big) - 4W^2\Big]}}{n-1}$$

with equality if and only if $k = n-2$ and $G \cong K_{1,n-1}$ or $G \cong K_{\frac{n}{2},\frac{n}{2}}$. Equality always holds for $k = n-1$.

The following lemma will be used in the sequel.

Lemma 2.7. [1] *Let G be a connected graph on n vertices and $m \ge n$ edges. Let G^* be the connected graph obtained from G by the deletion of an edge. Let $\delta_1 \ge \delta_2 \ge \cdots \ge \delta_{n-1} \ge \delta_n = 0$ and $\delta_1^* \ge \delta_2^* \ge \cdots \ge \delta_{n-1}^* \ge \delta_n^* = 0$ denote the distance Laplacian spectra of G and G^*, respectively. Then, $\delta_i^* \ge \delta_i$, for all $i = 1, 2, \ldots, n$.*

Now, let β be a real number such that $\beta \ne 0, 1$ and let $S_\beta(G) = \sum_{i=1}^{n-1} \delta_i^\beta$, that is, the sum of the βth powers of distance Laplacian eigenvalues of G.

Lemma 2.8. *Let G be a connected graph with n vertices and $m \ge n$ edges. Let G^* be the connected graph obtained from G by the deletion of an edge. Let $\delta_1 \ge \delta_2 \ge \cdots \ge \delta_{n-1} \ge \delta_n = 0$ and $\delta_1^* \ge \delta_2^* \ge \cdots \ge \delta_{n-1}^* \ge \delta_n^* = 0$ be the distance Laplacian eigenvalues of G and G^*, respectively. Then*
(i) $S_\beta(G^*) > S_\beta(G)$ for $\beta > 0$ and $S_\beta(G^*) < S_\beta(G)$ for $\beta < 0$.
(ii) For any connected graph G with n vertices $S_\beta(G) \ge (n-1)n^\beta$ if $\beta > 0$, and $S_\beta(G) \le (n-1)n^\beta$ if $\beta < 0$, with either of the equality if and only if $G \cong K_n$.

Proof. Using Lemma 2.7, we have $\sum_{i=1}^{n-1} \delta_i^* - \sum_{i=1}^{n-1} \delta_i \geq 2$. Hence (i) follows directly from Lemma 2.7. We know that the eigenvalues of K_n are $n, n, \ldots, n, 0$, so that (ii) follows from (i). $\square$

The following lemma will be used in the proof of Theorem 2.8.

Lemma 2.9. [4] *Let G be a connected graph such that D^L has an eigenvalue with multiplicity $n-2$. Let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_{n-1} \geq \delta_n = 0$ be the eigenvalues of D^L. Then exactly one of the following condition holds.*
(i) $m(\delta_1) = n-2$ and $G \cong K_{1,n-1}$ or $G \cong K_{p,p}$.
(ii) $m(\delta_{n-1}) = n-2$ and $G \cong K_{n-2} \vee \overline{K_2}$.

Now, we have the following result.

Theorem 2.8. *Let β be a real number with $\beta \neq 0, 1$ and let G be a connected graph with $n \geq 3$ vertices. Let $R = \prod_{i=1}^{n-1} \delta_i$. Then*

$$S_\beta(G) \geq \delta_1^\beta + (n-2)\Big(\frac{R}{\delta_1}\Big)^{\frac{\beta}{n-2}}$$

with equality if and only if $G \cong K_n$ or $G \cong K_{n-2} \vee \overline{K_2}$.

Proof. By the arithmetic-geometric mean inequality, we have

$$\begin{aligned} S_\beta(G) &= \delta_1^\beta + \sum_{i=2}^{n-1} \delta_i^\beta \geq \delta_1^\beta + (n-2)\Big(\prod_{i=2}^{n-1} \delta_i^\beta\Big)^{\frac{1}{n-2}} \\ &= \delta_1^\beta + (n-2)\Big(\frac{\prod_{i=1}^{n-1} \delta_i}{\delta_1}\Big)^{\frac{\beta}{n-2}} = \delta_1^\beta + (n-2)\Big(\frac{R}{\delta_1}\Big)^{\frac{\beta}{n-2}} \end{aligned}$$

with equality if and only if $\delta_2 = \delta_3 = \cdots = \delta_{n-1}$.
Now, if $\delta_1 = \delta_2$, then $\delta_1 = \delta_2 = \delta_3 = \cdots = \delta_{n-1}$, so that $G \cong K_n$. If $\delta_1 \neq \delta_2$, by Lemma 2.9, we have $G \cong K_{n-2} \vee \overline{K_2}$. $\square$

Conclusions. The parameters U_k and L_k introduced in this paper will be of great importance in the investigation of the distribution of distance Laplacian eigenvalues of a graph. In particular, it will throw more light in the study of the distance Laplacian spectral radius and distance Laplacian energy of a graph. Although the bounds for U_k and L_k are obtained in terms of the order, Wiener index and diameter of the graph, there is enough scope to obtain more bounds for U_k and L_k in terms of several other parameters, which will lead to interesting discussion on the problem.

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