TAMKANG JOURNAL OF MATHEMATICS
Volume 54, Number 1, 69-82, 2023
DOI:10.5556/j.tkjm.54.2023.4136

Relative essential ideals in N-groups

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Abstract. Let G be an N-group where N is a (right) nearring. We introduce the concept of relative essential ideal (or N-subgroup) as a generalization of the concept of an essential submodule of a module over a ring or a nearring. We provide suitable examples to distinguish between the notions relative essential and essential ideals. We prove the important properties and obtain equivalent conditions for the relative essential ideals (or N-subgroups) involving the quotient. Further, we derive results on direct sums, complement ideals of N-groups, and obtain their properties under homomorphism.

Keywords. Nearring, N-group, essential ideal, complement, direct summand.

1 Introduction

The notion 'essential submodule' of a module over a ring is analogue to the concept 'dense subspace' in a topological space [2]. As a topological space, the set of rational numbers is dense in the set of real numbers whereas the set of integers is not dense in the set of rational numbers. Unlike in topological spaces, in the case of algebraic systems such as modules over rings, there can be a situation that if a submodule is not essential in a given module, then it is possible to retain its essentiality with respect to (or relative to) a suitable proper submodule. Herein, we introduce and explore the properties of such essential ideals of N-group (also known as a module over a nearring) with respect to its arbitrary substructure. The role of an essential ideal is predominant to study the aspects of Goldie dimension in modules over rings, and over nearrings (generalized rings). The authors [8], [13], [15], [10] have studied uniform ideals, complement ideals, and corresponding Goldie dimension theorems in N-groups. Further, in [11], [17], linearly independent elements and u-linearly independent elements were introduced and obtained conditions for an N-group to have finite Goldie dimension. In [14], the concepts essential ideals, uniform ideals in modules over a matrix nearring were introduced and proved a characterization theorem for a module over a matrix nearring to have finite Goldie dimension. One can refer to [5], [6], [10], [16] for the developments of essential ideals and dimension concepts in N-groups, and [9], [12], [2] for essential submodules and related results on modules over rings.

An additive group N (not necessarily abelian) is said to be a (right) nearring if (i) $(N, \cdot)$ is a semigroup; and (ii) $(a+b)\cdot c = a\cdot c + b\cdot c$ for all $a, b, c \in N$. Obviously, if $(N, +, \cdot)$ is a right

Received date: Mar 4, 2021; Published online: Nov 8, 2021.
2010 *Mathematics Subject Classification*. 16Y30.
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nearring, then $0 \cdot a = 0$ for all $a \in N$, but $a \cdot 0 \neq 0$ for some $a \in N$. If $a \cdot 0 = 0$, for all $a \in N$, then N is called zero-symmetric (denoted as $N = N_0$). We denote ab instead of $a \cdot b$. We refer to Pilz [7] for the definitions such as N-group and ideal of an N-group. If $N = N_0$, then the modular law ([7], pg. 48): for any ideals I, J and K of G with K contained in I, then $I \cap (J + K) = (I \cap J) + K$. Further, for any ideals I and J of G, $I + J$ is an ideal of G ([7], Cor. 2.3).

For any subsets I_1, I_2 of G, $(I_1 : I_2) = \{n \in N \mid nI_2 \subseteq I_1\}$. For each $g \in G$, Ng is an N-subgroup of G. If I is an N-subgroup of G, then for each $g \in G$, $(I : g) = \{n \in N \mid ng \in I\}$, is a left N-subgroup of N.

An ideal H is essential in G (see, [8]) if for any ideal K of G, with $H \cap K = (0)$, then $K = (0)$. Let K be an ideal of G. If an ideal K' is maximal with respect to $K \cap K' = (0)$, then we say that K' is a complement of K (or a complement in G). An N-subgroup H is essential (resp. strictly essential) (see, [6]) in G denoted by $H_1 \leq^e G$ (resp. $H_1 \leq^{se} G$) if K is an ideal (resp. N-subgroup) of G, $H \cap K = (0)$, implies that $K = (0)$.

In section 2, we introduce the concept of relative essential ideal (or N-subgroup) as a generalization of the concept of an essential submodule of a module over a ring or a nearring. We exhibit possible illustrations of these notions to distinguish between relative essential and essential ideals. We prove the important properties and obtain equivalent conditions for the relative essential ideals (or N-subgroups) involving the quotient. In section 3, we prove the properties of essentiality and derive results on direct sums under homomorphisms. In section 4, we introduce the relative complement ideal and strictly relative complement ideal of an N-group and obtain its properties.

2 Relative essential ideals in N-groups and examples

We introduce different essentiality with respect to an arbitrary ideal and exhibit possible illustrations of the essentiality in various N-groups. We prove the fundamental properties, quotient preserving essentiality of ideals and related results.

Definition 2.1. Let H_1, H_2 be two ideals (or N-subgroups) of G. Then

(i) H_1 is said to be relative G-essential in H_2, if there exists a proper ideal Δ of G such that
(a) $H_1 \subseteq H_2$,
(b) $H_1 \nsubseteq \Delta$,
(c) for any ideal K of G, $K \subseteq H_2$, $H_1 \cap K \subseteq \Delta$ implies
$K \subseteq \Delta$.
We denote it by $H_1 \leq^e_\Delta H_2$, and read as H_1 is Δ-essential in H_2.

(ii) If $H_2 = G$ in (i), then we say that H_1 is relative essential in G, denoted by $H_1 \leq^e_\Delta G$.

Remark 2.2. 1. If H_1 and H_2 are ideals in definition 2.1(i) with $\Delta = (0)$, then the notion 'relative G-essential' coincides with 'G-essential', defined by [13], and if $\Delta = (0)$ in definition 2.1(ii), then the notion 'relative essential' coincides with 'essential', defined in [8].

2. If H_1 is an N-subgroup in definition 2.1(ii) with $\Delta = (0)$, then the notion 'relative essential' coincides with 'essential' defined by [6].

Definition 2.3. Let H_1, H_2 be two N-subgroups of G. Then

(i) H_1 is said to be strictly relative G-essential in H_2, if there exists a proper N-subgroup Δ of G such that

(a) $H_1 \subseteq H_2$,
(b) $H_1 \nsubseteq \Delta$,
(c) for any N-subgroup K of G, $K \subseteq H_2$, $H_1 \cap K \subseteq \Delta$ implies $K \subseteq \Delta$.
We denote it by $H_1 \leq^{se}_{\Delta} H_2$, and read as H_1 is strictly Δ-essential in H_2.

(ii) If $H_2 = G$ in (i), then we say that H_1 is strictly relative essential in G, denoted by $H_1 \leq^{se}_{\Delta} G$.

Remark 2.4. If $\Delta = (0)$ in definition 2.3(ii), then the notion 'strictly relative essential' coincides with 'strictly essential' defined by [6].

We provide explicit illustrations of different types of essential ideals and N-subgroups for a given N-group to distinguish the notion 'relative essential' introduced, and the notion 'essential' already exists. However, in some examples, we confine the computations only to such ideals and N-subgroups wherein the comparison between the types of essentiality is conveyed in the specified N-group.

Example 2.5. Let $(\mathbb{Z}_{12}, +_{12}, \cdot_{12})$ and $G = N$. Then the ideals and N-subgroups of G are $H_1 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}, H_2 = \{\bar{0}, \bar{6}\}, H_3 = \{\bar{0}, \bar{3}, \bar{9}\}$ and $H_4 = \{\bar{0}, \bar{4}, \bar{8}\}$. Then, we have

1. $H_4 \leq^{e}_{H_3} H_1, H_4 \leq^{se}_{H_3} H_1, H_4 \leq^{se}_{H_3} G$, and $H_4 \leq^{e}_{H_3} G$.
2. $H_4 \nleq^{e} H_1$ since $H_4 \cap H_2 \subseteq \{0\}$ and $H_2 \neq \{0\}$.
3. $H_4 \leq^{se} G$, whereas $H_4 \nleq^{e} G$. Also, $H_2 \nleq^{e} G, H_3 \nleq^{e} G$.

Example 2.6. Consider the nearring $N = (A_4, +, \cdot)$ listed as O(2), in ([7], page 423), and let $G = N$. The N-subgroups are $H_1 = \{0\}, H_2 = \{0, 1\}, H_3 = \{0, 2\}, H_4 = \{0, 2, 3\}, H_5 = \{0, 4, 8\}, H_6 = \{0, 1, 2, 3\}$ and the only proper ideal is H_6. Then we have the following.

1. $H_2 \leq^{se}_{H_4} H_6$. However, $H_2 \nleq^{se} H_6$, since $H_2 \cap H_4 \subseteq \{0\}$ and $H_4 \neq \{0\}$.
2. $H_5 \leq^{se}_{H_6} G$. However, $H_5 \nleq^{se} G$, since $H_5 \cap H_4 \subseteq \{0\}$ and $H_4 \neq \{0\}$.

Example 2.7. Consider the nearring N listed as K(139), page 418 of [7], and in [3]. Let $N = D_8 = \langle \{r, s \mid r^4 = s^2 = e,\ rs = sr^{-1}\} \rangle = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, where r is the rotation in an anti-clockwise direction about the origin through $\pi/2$ radians and s is the reflection about the line of symmetry, and $G = N$ with the addition and external multiplication are defined as follows. Then G is an N-group where N is non-abelian.

$+$	e	r	r^2	r^3	s	sr^3	sr^2	sr
e	e	r	r^2	r^3	s	sr^3	sr^2	sr
r	r	r^2	r^3	e	sr^3	sr^2	sr	s
r^2	r^2	r^3	e	r	sr^2	sr	s	sr^3
r^3	r^3	e	r	r^2	sr	s	sr^3	sr^2
s	s	sr	sr^2	sr^3	e	r^3	r^2	r
sr^3	sr^3	s	sr	sr^2	r	e	r^3	r^2
sr^2	sr^2	sr^3	s	sr	r^2	r	e	r^3
sr	sr	sr^2	sr^3	s	r^3	r^2	r	e

Table 1

$*$	e	r	r^2	r^3	s	sr^3	sr^2	sr
e	e	e	e	e	e	e	e	e
r	e	r	r^2	r^3	s	sr^3	sr^2	sr
r^2	e	r^2	e	r^2	e	e	e	e
r^3	e	r^3	r^2	r	s	sr^3	sr^2	sr
s	e	s	r^2	sr^2	s	e	sr^2	r^2
sr^3	e	sr^3	e	sr^3	e	sr^3	e	sr^3
sr^2	e	sr^2	r^2	s	s	e	sr^2	r^2
sr	e	sr	e	sr	e	sr^3	e	sr^3

Table 2

Then, $H_1 = \{e, sr^3\}, H_2 = \{e, r^2\}, H_3 = \{e, s\}, H_4 = \{e, sr^2\}, H_5 = \{e, r^2, sr^3, sr\}$ and $H_6 = \{e, r^2, s, sr^2\}$ are the N-subgroups, whereas the ideals are only H_2, H_5 and H_6. We have the following.

1. $H_2 \leq^e H_6, H_2 \leq^{se} H_5, H_2 \leq^e G, H_5 \leq^e G, H_6 \leq^e G$, but $H_1 \nleq^{se} H_5$, since $H_1 \cap H_2 \subseteq \{0\}$, and $H_2 \neq \{0\}$. Further, $H_5 \nleq^{se} G$, as $H_5 \cap H_3 = \{0\}$ and $H_3 \neq \{0\}$.
2. $H_1 \leq^{se}_{H_6} H_5, H_5 \leq^{se}_{H_6} G$ and $H_5 \leq^{e}_{H_6} G$.

Example 2.8. Let $N = \left(\begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}, +, \cdot \right)$, where N non-commutative, the Osofsky's 32-elements matrix ring and $G = N$. G is considered as an N-group. Ideals as well as N-subgroups are: $J_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $J_2 = \begin{pmatrix} 2\mathbb{Z}_4 & 0 \\ 0 & 0 \end{pmatrix}$, $J_3 = \begin{pmatrix} 0 & 0 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$, $J_4 = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$, $J_5 = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$, $J_6 = \begin{pmatrix} 0 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$, $J_7 = \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$, $J_8 = \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$, $J_9 = \begin{pmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$, $J_{10} = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 0 \end{pmatrix}$, $J_{11} = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 2\mathbb{Z}_4 \end{pmatrix}$, $J_{12} = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$. Then we have the following

1. $J_7 \leq^{e}_{J_3} J_{11}, J_7 \leq^{se}_{J_3} J_{11}, J_7 \leq^{se}_{J_3} G$ and $J_7 \leq^{e}_{J_3} G$.
2. $J_7 \leq^e J_{11}, J_7 \leq^{se} J_{11}, J_9 \leq^{se} G, J_9 \leq^e G$.
3. $J_6 \nleq^{se} J_9$, since $J_6 \cap J_2 \subset \{0\}$ and $J_2 \neq \{0\}$.
4. $J_7 \nleq^{se} G$ and $J_7 \nleq^{e} G$, since $J_7 \cap J_3 \subseteq \{0\}$ and $J_3 \neq \{0\}$.

Example 2.9. Let $N = (\mathbb{Z}_{24}, +_{24}, \cdot_{24})$ and $G = N$. The ideals and N-subgroups are $H_1 = \langle 2 \rangle H_2 = \langle 3 \rangle, H_3 = \langle 4 \rangle, H_4 = \langle 6 \rangle, H_5 = \langle 8 \rangle, H_6 = \langle 12 \rangle$. Then,

1. $H_3 \leq^{e}_{H_2} H_1$ and $H_5 \leq^{Ge}_{H_4} H_3$.
2. $H_3 \leq^{e}_{H_2} G$, but $H_5 \nleq^{se}_{H_4} G$, since $H_5 \cap H_2 = \{0\}$ and $H_2 \neq \{0\}$.
3. $H_3 \leq^{se} G, H_3 \leq^{e}_{H_2} G$ and $H_3 \nleq^{se} G$, but $H_5 \nleq^{se} G$, since $H_5 \cap H_6 = \{0\}$ and $H_6 \neq \{0\}$.

In the following examples 2.10, 2.11, we consider module G over the ring of integers, and hence the ideals and N-subgroups are the same. We refer to them as submodules.

Example 2.10. Take $G = (\mathbb{Z}_4 \times \mathbb{Z}_2, +)$. Then the submodules are $H_1 = \langle (0,0) \rangle, H_2 = \langle (2,0) \rangle, H_3 = \langle (2,1) \rangle, H_4 = \langle (1,0) \rangle, H_5 = \langle (1,1) \rangle, H_6 = \langle (0,1) \rangle$. It can be observed that,

1. $H_2 \leq^e_{H_3} H_5, H_2 \leq^e H_5$.
2. $H_2 \nleq^e_{H_3} G$, as $H_2 \cap H_6 \subseteq H_3$ but $H_6 \nsubseteq H_3$.
3. $H_2 \nleq^e G$ as $H_2 \cap H_3 \subseteq \{0\}$ but $H_3 \neq \{0\}$.
4. $H_2 \nleq^e G$ since $H_2 \cap H_3 \subseteq \{0\}$ and $H_3 \neq \{0\}$.

Example 2.11. Let $G = (\mathbb{Z} \times \mathbb{Z}_6, +)$. The submodules are $H_1 = \langle(0,0)\rangle, H_2 = \langle(0,2)\rangle, H_3 = \langle(0,3)\rangle, H_4 = \langle(1,0)\rangle$. Then we have the following.

1. $H_2 \leq^e_{H_3} H_4, H_2 \leq^{se}_{H_3} H_4, H_2 \leq^e_{H_3} G$ and $H_2 \leq^{se}_{H_3} G$.
2. $H_2 \nleq^e H_4, H_2 \nleq^e G$, since $H_2 \cap H_3 = \{0\}$ and $H_3 \neq \{0\}$.

Example 2.12. Let $(N = \{0,1,2,3,4,5,6,7,8,9,10,11\}, +, \cdot)$ ([1], Table no 12/3(23)) be a nearring. The operation table is given below:

+	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	3	2	5	4	7	6	9	8	11	10
2	2	10	0	4	3	6	5	8	7	11	1	9
3	3	11	1	5	2	7	4	9	6	10	0	8
4	4	9	10	6	0	8	3	11	5	1	2	7
5	5	8	11	7	1	9	2	10	4	0	3	6
6	6	7	9	8	10	11	0	1	3	2	4	5
7	7	6	8	9	11	10	1	0	2	3	5	4
8	8	5	7	11	9	1	10	2	0	4	6	3
9	9	4	6	10	8	0	11	3	1	5	7	2
10	10	2	4	0	6	3	8	5	11	7	9	1
11	11	3	5	1	7	2	9	4	10	6	8	0

Table 3

and

$$a \cdot b = \begin{cases} 0 & \text{if } b \neq 10, \\ b & \text{if } b = 10, \text{ for all } a, b \in N. \end{cases}$$

Then, $H_1 = \{0,1\}$, $H_2 = \{0,4\}$, $H_3 = \{0,8\}$, $H_4 = \{0,7\}$, $H_5 = \{0,2\}$, $H_6 = \{0,6\}$, $H_7 = \{0,11\}$, $H_8 = \{0,5,9\}$, $H_9 = \{0,1,6,7\}$, $H_{10} = \{0,4,7,11\}$, $H_{11} = \{0,2,7,8\}$, $H_{12} = \{0,1,4,5,8,9\}$, $H_{13} = \{0,3,5,7,9,10\}$ and $H_{14} = \{0,2,5,6,9,11\}$ are the N-subgroups, whereas the ideals are H_4, H_8, H_{12}, H_{13} and H_{14}. We have the following.

1. $H_4 \leq^e_{H_8} H_{13}$, but $H_4 \nleq^e H_{13}$, as $H_4 \cap H_8 = \{0\}$, and $H_8 \neq \{0\}$.
2. $H_{14} \leq^e_{H_4} G$, but $H_{14} \nleq^e G$, as $H_{14} \cap H_4 = \{0\}$ and $H_4 \neq \{0\}$.
3. $H_1 \leq^{se}_{H_{11}} H_{10}$, but $H_1 \nleq^{se} H_{10}$, as $H_1 \cap H_4 = \{0\}$, $H_4 \subseteq H_{10}$, and $H_4 \neq \{0\}$.

Proposition 2.13. Let H_i, $1 \leq i \leq 3$, be ideals of G, and Δ a proper ideal of G. Then

1. $H_1 \leq^e_\Delta H_3$, $H_2 \leq^e_\Delta H_3$ and $H_1 \cap H_2 \nsubseteq \Delta$ implies that $H_1 \cap H_2 \leq^e_\Delta H_3$.

2. Let $H_1 \subseteq H_2 \subseteq H_3$. Then $H_1 \leq^e_\Delta H_3$ if and only if $H_1 \leq^e_\Delta H_2$ and $H_2 \leq^e_\Delta H_3$.

Proof. (1) Suppose $H_1 \leq^e_\Delta H_3$, $H_2 \leq^e_\Delta H_3$ and $(H_1 \cap H_2) \nsubseteq \Delta$. Let $(H_1 \cap H_2) \cap K \subseteq \Delta$, where K is an ideal of G, $K \subseteq H_3$. Then $H_1 \cap (H_2 \cap K) \subseteq \Delta$. Since $(H_2 \cap K)$ is an ideal of G, $K \subseteq H_3$ and $H_1 \leq^e_\Delta H_3$, we get $H_2 \cap K \subseteq \Delta$. Again, since $H_2 \leq^e_\Delta H_3$, we have $K \subseteq \Delta$. Therefore, $(H_1 \cap H_2) \leq^e_\Delta H_3$.

(2) Suppose $H_1 \leq^e_\Delta H_3$. Let K be an ideal of G, $K \subseteq H_2$ such that $H_1 \cap K \subseteq \Delta$. Since $K \subseteq H_2 \subseteq H_3$ and $H_1 \leq^e_\Delta H_3$, we have $K \subseteq \Delta$, shows that $H_1 \leq^e_\Delta H_2$. Next, let L be an ideal of G, $L \subseteq H_3$ such that $H_2 \cap L \subseteq \Delta$. Now $H_1 \cap L \subseteq H_2 \cap L \subseteq \Delta$ and since $H_1 \leq^e_\Delta H_3$, we have $L \subseteq \Delta$. Therefore, $H_2 \leq^e_\Delta H_3$.

Conversely, let K be an ideal of G, $K \subseteq H_3$ such that $H_1 \cap K \subseteq \Delta$. Now $H_1 \cap (H_2 \cap K) \subseteq H_1 \cap K \subseteq \Delta$. Since $H_2 \cap K$ is ideal of G, $H_2 \cap K \subseteq H_2$, and $H_1 \leq^e_\Delta H_2$, we have $H_2 \cap K \subseteq \Delta$. Again since $H_2 \leq^e_\Delta H_3$, we get $K \subseteq \Delta$, proves $H_1 \leq^e_\Delta H_3$. □

Remark 2.14. The other implication of the Proposition 2.13 (1), need not be true, in general. Consider the Example 2.9, where $H_1 = \langle 2 \rangle$, $H_2 = \langle 3 \rangle$, $H_3 = \langle 4 \rangle$, $\Delta = H_4 = \langle 6 \rangle$, $H_5 = \langle 8 \rangle$ and $H_6 = \langle 12 \rangle$. Since $H_5 \cap H_1 = \langle 8 \rangle$, we have

(i) $H_5 \cap H_1 \subseteq H_3$

(ii) $H_5 \cap H_1 \nsubseteq \Delta$

Then $K = H_6$ is the only ideal satisfying $K \subseteq H_3$, $(H_5 \cap H_1) \cap K \subseteq \Delta$ implies that $K \subseteq \Delta$. Therefore, $H_5 \cap H_1 \leq^e_\Delta H_3$. However, since $H_1 \nsubseteq H_3$, we conclude that $H_1 \nleq^e_\Delta H_3$.

Lemma 2.15. Let $N = N_0$ and $\Delta \subseteq H_1 \subseteq H_2$ be ideals of G. Then $H_1 \leq^e_\Delta H_2$ if and only if $H_1/\Delta \leq^e H_2/\Delta$.

Proof. Suppose $H_1 \leq^e_\Delta H_2$. Let K/Δ be an ideal of G/Δ contained in H_2/Δ such that $H_1/\Delta \cap K/\Delta = (0)/\Delta$ in G/Δ. Then $H_1 \cap K/\Delta = (0)/\Delta$, implies $H_1 \cap K \subseteq \Delta$. Since $H_1 \leq^e_\Delta H_2$ and $K \subseteq H_2$, we get $K \subseteq \Delta$, means $K/\Delta = (0)/\Delta$.

Conversely, let K be an ideal of G, $K \subseteq H_2$ such that $H_1 \cap K \subseteq \Delta$. Then, $K + \Delta$ is an ideal of G and consequently, $(K+\Delta)/\Delta$ is an ideal of G/Δ contained in H_2/Δ. Now we show that $H_1/\Delta \cap (K+\Delta)/\Delta = (0)/\Delta$. For this, let $x + \Delta \in H_1/\Delta \cap (K+\Delta)/\Delta$. Then $x \in H_1$ and $x \in K + \Delta$, implies $x \in H_1 \cap (K + \Delta)$. Since N is zero-symmetric and $\Delta \subseteq H_1$, by the modular law, $x \in \Delta + (H_1 \cap K)$. Since $H_1 \cap K \subseteq \Delta$, we have $x \in \Delta + \Delta \subseteq \Delta$. Hence, $H_1/\Delta \cap (K+\Delta)/\Delta = (0)/\Delta$, which gives $(K+\Delta)/\Delta = (0)/\Delta$, by the converse hypothesis. Therefore $K \subseteq \Delta$, shows that $H_1 \leq^e_\Delta H_2$. □

Proposition 2.16. Let $N = N_0$ and let H_1, H_2, Δ be (proper) ideals of G. If $H_1 \leq^e_\Delta H_2$, then $(H_1+\Delta)/\Delta \leq^e H_2/\Delta$.

Proof. Let A/Δ be an ideal of G/Δ contained in H_2/Δ such that $A/\Delta \cap (H_1+\Delta)/\Delta = (0)/\Delta$. Then $(A \cap (H_1 + \Delta))/\Delta = (0)/\Delta$. Since N is zero-symmetric and $\Delta \subseteq A$, by modular law $((A \cap H_1) + \Delta)/\Delta = (0)/\Delta$. It follows that $(A \cap H_1) + \Delta \subseteq \Delta$, and hence $(A \cap H_1) \subseteq \Delta$. Since $H_1 \leq^e_\Delta H_2$, we have $A \subseteq \Delta$. Therefore, $A/\Delta = (0)/\Delta$, and thus $(H_1+\Delta)/\Delta \leq^e H_2/\Delta$. □

Theorem 2.17. Let H, Δ be N-subgroups of G and $1 \in N$. Then the following are equivalent.

1. $H \leq^{se}_\Delta G$
2. For each $g \in G \setminus \Delta$, there exist $n \in N$ such that $ng \in H \setminus \Delta$.

3. $(H : g) \leq^{se}_{(\Delta:g)\ N} N$, for each $g \in G \setminus \Delta$.

Proof. (1) $\Rightarrow$ (2): Let $g \in G \setminus \Delta$. Now Ng is an N-subgroup of G, $g \notin \Delta$ and $1 \in N$, we get $Ng \nsubseteq \Delta$. Since $H \leq^{se}_{\Delta} G$, we get $H \cap Ng \nsubseteq \Delta$. Let $x \in H \cap Ng$ such that $x \notin \Delta$. Then $x \in H$ and $x = ng$, for some $n \in N$. Therefore, $x = ng \in H$ and $x \notin \Delta$.

(2) $\Rightarrow$ (1): Let $H \cap K \subseteq \Delta$, where K be an N-subgroup of G. If $K \nsubseteq \Delta$, then there exists $a \in K \setminus \Delta \subseteq G \setminus \Delta$. Now by (2), $na \in H \setminus \Delta$, for some $n \in N$, whereas $na \in H \cap K$, a contradiction. Hence, $H \leq^{se}_{\Delta} G$.

(1) $\Rightarrow$ (3): Let $g \in G \setminus \Delta$. By (2), $ng \in H \setminus \Delta$, for some $n \in N$, and hence it follows that $(H : g) \nsubseteq (\Delta : g)$. Now let I be an N-subgroup of N such that $(H : g) \cap I \subseteq (\Delta : g)$. Clearly, Ig is an N-subgroup of G. If $H \cap Ig \nsubseteq \Delta$, then there exists $x \in H \cap Ig$, but $x \notin \Delta$. Then $x \in H$ and $x = ig$, for some $i \in I$. Hence, $i \in (H : g)$ and $i \in I$, but $i \notin (\Delta : g)$, a contradiction to the assumption. Therefore, $H \cap Ig \subseteq \Delta$. Since $H \leq^{se}_{\Delta} G$, we have $Ig \subseteq \Delta$. Thus, $(H : g) \leq^{se}_{(\Delta:g)\ N} N$.

(3) $\Rightarrow$ (1): Suppose that $H \cap K \subseteq \Delta$, where K be an N-subgroup of G. If $K \nsubseteq \Delta$, then there exists $x \in K \setminus \Delta \subseteq G \setminus \Delta$. Now by (3), we get $(H : x) \leq^{se}_{(\Delta:x)\ N} N$. Since $(H : x) \nsubseteq (\Delta : x)$, there exists $a \in (H : x) \subseteq N$, but $a \notin (\Delta : x)$. That is, $ax \in H$, but $ax \notin \Delta$. Now since K be an N-subgroup of G, and $a \in N$, $x \in K$, we get $ax \in K$. Then $ax \in H \cap K$, but $ax \notin \Delta$, a contradiction. Hence, $H \leq^{se}_{\Delta} G$. □

Remark 2.18. Observe that Theorem 2.17 is proved for 'strictly essential'. These results may not satisfy for the notion 'essential', since for any $g \in G$, Ng is an N-subgroup of G but not an ideal of G, in general.

Consider the following example.

Example 2.19. Consider the nearring $N = (S_3, +, \cdot)$, given in H(37), p. 411 of [7], and let $G = N$. Let $c \in G$. Then $Ng = S_3 g = \{0, c\}$, which is an N-subgroup of S_3 but it is not even a normal subgroup of S_3, since $a + c - a = b \notin \{0, c\}$.

3 N-Homomorphisms of relative essential ideals

We prove homomorphism results of essentiality and the direct sums with respect to arbitrary ideal of an N-group.

Theorem 3.1. Let $f : G_1 \to G_2$ be an N-homomorphism, and let Δ be a proper ideal of G_2 such that $f(G_1) \nsubseteq \Delta$. Then $f(G_1) \leq^{e}_{\Delta} G_2$ if and only if for any homomorphism ϕ, whenever $\phi^{-1}(0) \cap f(G_1) \subseteq \Delta$, we have $\phi^{-1}(0) \subseteq \Delta$.

Proof. Suppose $f(G_1) \leq^{e}_{\Delta} G_2$. Let $\phi : G_1 \to G_2$ be an N-homomorphism such that $\phi^{-1}(0) \cap f(G_1) \subseteq \Delta$. Since $f(G_1) \leq^{e}_{\Delta} G_2$, we have $\phi^{-1}(0) \subseteq \Delta$.

Conversely, let K be an ideal of G_2 such that $f(G_1) \cap K \subseteq \Delta$. Since $f^{-1}(\Delta)$ is an ideal of G_1, ${G_1}/{f^{-1}(\Delta)}$ is a quotient N-group.
Define $\phi : (f(G_1) + K) \to {G_1}/{f^{-1}(\Delta)}$ with $\phi(f(g_1) + k) = g_1 + f^{-1}(\Delta)$, for each $g_1 \in G_1,\ k \in K$. To show ϕ is well-defined, suppose $f(g_1) + k_1 = f(g_2) + k_2$. This implies $f(g_1) - f(g_2) = k_2 - k_1$. Since f is homomorphism, we get $f(g_1 - g_2) = k_2 - k_1 \in K \cap f(G_1) \subseteq \Delta$. Therefore, $g_1 - g_2 \in f^{-1}(\Delta)$, hence $g_1 + f^{-1}(\Delta) = g_2 + f^{-1}(\Delta)$. Thus $\phi(f(g_1) + k_1) = \phi(f(g_2) + k_2)$.

To show ϕ is an N-homomorphism,

(i) $\phi((f(g_1)+k_1)+(f(g_2)+k_2))= \phi(f(g_1)+f(g_2)+k_3+k_2)$, for some $k_3 \in K$. Since f is homomorphism, we get,

$$\begin{aligned}\phi(f(g_1)+f(g_2)+k_3+k_2) &= \phi(f(g_1+g_2)+k_3+k_2)\\ &= (g_1+g_2)+f^{-1}(\Delta)\\ &= (g_1+f^{-1}(\Delta))+(g_2+f^{-1}(\Delta))\\ &= \phi(f(g_1)+k_1)+\phi(f(g_2)+k_2)\end{aligned}$$

(ii) Let $n \in N$. Since K is an ideal of G_2, we have $n(f(g_1)+k) - nf(g_1) = k_1$, for some $k_1 \in K$. Then $n(f(g_1)+k) = k_1 + nf(g) = nf(g_1)+k_2$, for some $k_2 \in K$.
Now

$$\begin{aligned}\phi(n(f(g_1)+k)) &= \phi(nf(g_1)+k_2)\\ &= \phi((f(ng_1)+k_2))\\ &= ng_1+f^{-1}(\Delta)\\ &= n(g_1+f^{-1}(\Delta))\\ &= n(\phi(f(g_1)+k))\end{aligned}$$

Therefore, ϕ is an N-homomorphism. Now to show $\phi^{-1}(0)\cap f(G_1) \subseteq \Delta$, let $x \in \phi^{-1}(0)\cap f(G_1)$. Then $\phi(x) = 0$ in ${}^{G_1}/_{f^{-1}(\Delta)}$, where $x = f(g)+k$, for some $g \in G,\ k \in K$, implies $g + f^{-1}(\Delta) = f^{-1}(\Delta)$. Therefore, $g \in f^{-1}(\Delta)$ and hence $f(g) \in \Delta$. Also $x \in f(G_1)$ implies that $x = f(g_1)$, for some $g_1 \in G_1$, and it follows that $f(g)+k = f(g_1)$. Now $k = f(g_1) - f(g)= f(g_1-g) \in f(G_1)$. So, $k \in f(G_1)\cap K \subseteq \Delta$. Therefore, $x = f(g)+k \in \Delta+\Delta \subseteq \Delta$. By hypothesis we conclude that $\phi^{-1}(0) \subseteq \Delta$. Now let $x \in K$. Then $x = 0+k = f(0)+k$, since f is a homomorphism. Now $\phi(x) = \phi(f(0)+k) = 0+f^{-1}(\Delta)$ in ${}^{G_1}/_{f^{-1}(\Delta)}$. Therefore, $x \in \phi^{-1}(0)$. Thus $K \subseteq \phi^{-1}(0) \subseteq \Delta$, proves $f(G_1) \leq^e_\Delta G_2$. □

Proposition 3.2. Let $f \in Hom_N(G_1, G_2)$ where $N = N_0$, Δ and H be ideals of G_2. If $H \leq^e_\Delta G_2$, then $f^{-1}(H) \leq^e_{f^{-1}(\Delta)} G_1$.

Proof. Since Δ, H are ideals of G_2, ([7], Prop. 2.17), $f^{-1}(H)$ and $f^{-1}(\Delta)$ are ideals of G_1. Let L be an ideal of G_1 such that $f^{-1}(H)\cap L \subseteq f^{-1}(\Delta)$. To show $H \cap f(L) \subseteq \Delta$, let $x \in H\cap f(L)$. Then $x \in H$ and $x = f(l)$ for some $l \in L$, implies $l = f^{-1}(x) \in f^{-1}(H)\cap L \subseteq f^{-1}(\Delta)$, and hence $x = f(l) \in \Delta$. Since $H \leq^e_\Delta G_2$, we have $f(L) \subseteq \Delta$, and so $L \subseteq f^{-1}(\Delta)$. Hence, $f^{-1}(H) \leq^e_{f^{-1}(\Delta)} G_1$. □

Theorem 3.3. Let K be an ideal of G and $\pi : G \to {}^G/_K$ be an N-epimorphism. If S is an ideal of G such that $\pi(S) \leq^e_{f(\Delta)} \pi(G)$ then $S+K \leq^e_\Delta G$, where Δ is a proper ideal of G containing K.

Proof. Define $\pi\colon G \to G/K$ by $\pi(g) = g+K$ and let S be an ideal of G such that $\pi(S) \leq^e_{f(\Delta)} \pi(G)$. To show $S+K \leq^e_\Delta G$, let I be an ideal of G such that $(S+K)\cap I \subseteq \Delta$. Let $a+K \in ({}^{(S+K)}/_K)\cap({}^{(I+K)}/_K)$. Now $a+K = s+K = y+K$ for some $s \in S,\ y \in I$. Then $s-y = x$ for some $x \in K$. So, $y = s-x \in S+K$, hence $y \in (S+K)\cap I \subseteq \Delta$. Now $a+K = y+K \subseteq \Delta+K \subseteq \Delta$. Therefore, $({}^{(S+K)}/_K)\cap({}^{(I+K)}/_K) \subseteq \Delta+K \subseteq \Delta$. Since ${}^{(S+K)}/_K = \pi(S) \leq^e_\Delta \pi(G)$, we have ${}^{(I+K)}/_K \subseteq \Delta$, implies that $I \subseteq \Delta+K \subseteq \Delta$. Therefore, $S+K \leq^e_\Delta G$. □

Remark 3.4. Converse of Theorem 3.3 need not be true, in general. Consider an N-group G, where $G = N = \mathbb{Z}$, the nearring of integers. Clearly, $S = 2\mathbb{Z}$ and $K = 6\mathbb{Z}$ are ideals of G. Consider the canonical map $\pi : G \to {}^{G}/_{K}$. Now it can be observed that $S + K = 2\mathbb{Z} \leq^{e}_{3\mathbb{Z}} \mathbb{Z}$. However, $\pi(S) = \pi(2\mathbb{Z}) = {}^{2\mathbb{Z}}/_{6\mathbb{Z}} = \{\bar{0}, \bar{2}, \bar{4}\}$ is not $\Delta = 3\mathbb{Z}$ essential in ${}^{G}/_{K} = {}^{\mathbb{Z}}/_{6\mathbb{Z}} \cong \mathbb{Z}_6 = \{\bar{0}, \bar{3}\} \oplus \{\bar{0}, \bar{2}, \bar{4}\}$. That is, $\{\bar{0}, \bar{2}, \bar{4}\} \nleq^{e}_{\Delta/K} \mathbb{Z}_6$.

Definition 3.5. An ideal I of G is said to be a relative direct summand if there is an ideal J, and a proper ideal Δ of G such that $I + J = G$ and $I \cap J \subseteq \Delta$. In this case, we say that $I + J$ is Δ-direct (or Δ-direct sum) in G.

If $\Delta = (0)$, then I is a direct summand of G defined in [7].

Example 3.6. (i) Consider the ideals $\Delta = H_1 = \{1, -1\}$, $H_2 = \{1, -1, i, -i\}$ and $H_3 = \{1, -1, j, -j\}$ in the N-group $\mathbb{Q}_8$ over itself given in L(1), pg. 418 of [7]. Here H_2 is Δ-direct summand of H_3, but H_2 is not a direct summand of H_3, since $H_2 \cap H_3 = \{1, -1\} \neq \{1\}$, identity in $\mathbb{Q}_8$.

(ii) Consider the ideals $\Delta = H_2 = \{\bar{0}, \bar{6}\}$, $H_3 = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ and $H_4 = \{\bar{0}, \bar{4}, \bar{8}\}$ in the N-group $\mathbb{Z}_{12}$ over itself given in the Example 2.5. Here H_3 is a Δ-direct summand of H_4, and also, H_3 is a direct summand of H_4.

Definition 3.7. A family $\{I_i\}_{i \in I}$ of ideals of G is said to be relative direct if there exists a proper ideal Δ of G such that $I_i \cap (\sum\limits_{j \neq i} I_j) \subseteq \Delta$ and $\sum\limits_{i \in I} I_i = G$. In this case, we call $\sum\limits_{i=1}^{n} I_i$ as Δ-direct sum.

Example 3.8. Consider the ideals of N-group D_8 given in Example 2.7. Take $\Delta = H_1$. Then $H_2 \cap (H_3 + H_4) \subseteq \Delta$, $H_3 \cap (H_2 + H_4) \subseteq \Delta$ and $H_4 \cap (H_2 + H_3) \subseteq \Delta$. Also $H_2 + H_3 + H_4 = G$. Therefore, $\{H_2, H_3, H_4\}$ is Δ-direct.

Theorem 3.9. Let $f \colon G \to G'$ be an N-isomorphism. Suppose that I_i, $1 \leq i \leq n$ are ideals of G, and Δ a proper ideal of G such that $I_i \nsubseteq \Delta$ for all i. Then

(i) $\sum\limits_{j=1}^{n} I_j$ is Δ-direct in G if and only if $\sum\limits_{j=1}^{n} f(I_j)$ is $f(\Delta)$-direct in G'; and

(ii) $K_1 \leq^{e}_{\Delta} K_2$ if and only if $f(K_1) \leq^{e}_{f(\Delta)} f(K_2)$.

Proof. (i) Suppose $\sum\limits_{j=1}^{n} I_j$ is Δ-direct.

We show that $f(I_i) \cap \left(\sum\limits_{j=1, j\neq i}^{n} f(I_j) \right) \subseteq f(\Delta)$, let $y \in f(I_i) \cap \left(\sum\limits_{j=1, j\neq i}^{n} f(I_j) \right)$.

Then, $y = f(x_i) = \sum\limits_{j \neq i} f(x_j)$, $x_j \in I_j$, $1 \leq j \leq n$. Since f is homomorphism, we get $f(x_i) = f\left(\sum\limits_{j \neq i} x_j \right)$, $x_j \in I_j$, $1 \leq j \leq n$. Since f is one-one, we have $x_i = \left(\sum\limits_{j \neq i} x_j \right)$, $x_j \in I_j$, $1 \leq j \leq n$. Now since $\sum\limits_{j=1}^{n} I_j$ is Δ-direct, $x_i \in I_i \cap \left(\sum\limits_{j=1, j \neq i}^{n} I_j \right) \subseteq \Delta$. Therefore, $y = f(x_i) \in f(\Delta)$, and $\sum\limits_{j=1}^{n} f(I_j)$ is $f(\Delta)$-direct. Next to show $\sum\limits_{i \in I} f(I_i) = f(G)$. For any $x \in \sum\limits_{i \in I} I_i$, $x = x_1 + \cdots + x_n$, where $x_i \in I_i$ for $1 \leq i \leq n$. Then $f(x) = f(x_1 + \cdots + x_n)$. Since f is homomorphism, we have $f(x) = f(x_1) + \cdots + f(x_n) \in \sum\limits_{i \in I} f(I_i) \subseteq f(G)$. Since x_i's are distinct and f is one-one, we have $f(x_i)$'s are distinct.

Conversely, suppose that $f(I_i)$, $1 \leq i \leq n$ is $f(\Delta)$-direct.

Let $x \in I_i \cap \left(\sum\limits_{j=1, j \neq i}^{n} I_j \right)$. Now $f(x) \in f(I_i) \cap \left(f\left(\sum\limits_{j=1, j \neq i}^{n} (I_j) \right) \right) = f(I_i) \cap \left(\sum\limits_{j=1, j \neq i}^{n} f(I_j) \right) \subseteq f(\Delta)$. Then $f(x) = f(\delta)$ for some $\delta \in \Delta$, and since f is one-one, we get $x = \delta$. Therefore, $I_i \cap \left(\sum\limits_{j=1, j \neq i}^{n} I_j \right) \subseteq \Delta$, for $1 \leq i \leq n$. Let $x \in G$. To show $x \in \sum\limits_{j=1}^{n} I_j$. Since $x \in G$, $f(x) \in f(G) = G' = \sum\limits_{j=1}^{n} f(I_j)$. Therefore $f(x) = f(x_1) + \cdots + f(x_n) = f(x_1 + \cdots, x_n)$. Since f is one-one, $x = x_1 + \cdots, x_n \in \sum\limits_{j=1}^{n} I_j$.

(ii) Suppose $K_1 \leq^e_\Delta K_2$. In a contrary way, suppose that $f(K_1) \not\leq^e_{f(\Delta)} f(K_2)$. Then there exists an ideal I of G' contained in $f(K_2)$ such that $f(K_1) \cap I \subseteq f(\Delta)$ and $I \not\subseteq f(\Delta)$. Now $I \not\subseteq f(\Delta)$ implies that $f^{-1}(I) \not\subseteq \Delta$. Write $K = f^{-1}(I)$. Since K is an ideal of G, $f(K)$ is an ideal of G' and $f(K) = I \subseteq f(K_2)$. Therefore, $f(K_1) \cap f(K) \subseteq f(\Delta)$ and $f(K) \not\subseteq f(\Delta)$. This shows that $f(K)$ and $f(K_1)$ are $f(\Delta)$- direct in G'. By (i), $K \cap K_1 \subseteq \Delta$ and since $K \not\subseteq \Delta$, we get K_1 is not Δ- essential in K_2, a contradiction. Therefore, $f(K_1) \leq^e_{f(\Delta)} f(K_2)$. To prove the converse, we assume the contrary $K_1 \not\leq^e_\Delta K_2$. Then there exists an ideal I of G contained in K_2 such that $K_1 \cap I \subseteq \Delta$ and $I \not\subseteq \Delta$. Since $I \not\subseteq \Delta$, we have $f(I) \not\subseteq f(\Delta)$, also since $I \subseteq K_2$, we have $f(I) \subseteq f(K_2)$. Now from $K_1 \cap I \subseteq \Delta$, we get $f(K_1) \cap f(I) \subseteq f(\Delta)$ but $f(I) \not\subseteq f(\Delta)$, a contradiction to the converse hypothesis. □

4 Relative complement ideal in N-groups

We define the complement ideal of an N-group with respect to an arbitrary ideal and obtained some important results.

Definition 4.1. Let H be an ideal (resp. N-subgroup) of G. An ideal (resp. N-subgroup) H' of G is called a relative complement (resp. strictly relative complement) of H if there exists a proper ideal (resp. N-subgroup) Δ of G such that H' is maximal with respect to $H \cap H' \subseteq \Delta$. In this case, we call H' as Δ-complement of H.

If $\Delta = (0)$, then the Δ-complement corresponds to just the complement (resp. strictly complement) defined in [8] (resp. [6]). We denote the complement (or strictly complement) of H by H^c.

Example 4.2. Consider the Example 2.7.

(i) Let $\Delta = H_2$. Then H_5 is Δ-complement of H_6, whereas, with respect to the N-subgroups $\Delta = H_5$, it can be seen that H_6 is maximal with respect to $H_2 \cap H_6 \subseteq H_5$. Hence, H_6 is strictly Δ-complement of H_2.

(ii) H_5 is a strictly complement as well as strictly H_1-complement of H_4.

Proposition 4.3. Let H_1, H_2 be ideals of G, Δ a proper ideal of G such that $\Delta = H_1 \cap H_2$. If H_2 is a Δ-complement of H_1, then $H_1 + H_2 \leq^e_\Delta G$.

Proof. Suppose $(H_1 + H_2) \cap K \subseteq \Delta$, where K is an ideal of G. To show that $K \subseteq \Delta$. First we show that $H_1 \cap (H_2 + K) \subseteq \Delta$. Let $x \in H_1 \cap (H_2 + K)$. Then $x = h_1$ and $x = h_2 + k$, for some $h_2 \in H_2$, $k \in K$. Now $h_1 = h_2 + k$, implies $k = h_1 - h_2 \in K \cap (H_1 + H_2) \subseteq \Delta$. Therefore, $k \in \Delta$, and so $h_1 = h_2 + k \in H_2 + \Delta = H_2$ (since $H_1 \cap H_2 = \Delta$, we get $\Delta \subseteq H_2$), implies $h_1 \in H_1 \cap H_2 = \Delta$, hence $x \in \Delta$. Therefore $H_1 \cap (H_2 + K) \subseteq \Delta$. Since H_2 is a Δ-complement of H_1, we have that $H_2 + K = H_2$. This means that $K \subsetneqq H_2$, and since $K \subseteq H_1 + H_2$, we have $K = (H_1 + H_2) \cap K \subseteq \Delta$. □

Proposition 4.4. Let H and Δ be (proper) N-subgroups of G. If $H \leq_\Delta^{se} G$, then $H^c \subseteq \Delta$. Further, if $H \cap \Delta = (0)$, then $H^c = \Delta$.

Proof. By definition, H^c is maximum with respect to $H \cap H^c = (0)$. Since H and H^c are N-subgroups of G with $H \cap H^c = (0) \subseteq \Delta$, and $H \leq_\Delta^{se} G$, we have $H^c \subseteq \Delta$. Now suppose that $H \cap \Delta = (0)$. Again by definition, since H^c is maximum with respect to $H \cap H^c = (0)$, we have $\Delta \subseteq H^c$. Therefore, $H^c = \Delta$. □

Proposition 4.5. The following are equivalent for an N-subgroup H of G.

1. $H \leq_{H^c}^{se} G$.
2. For each N-subgroup K of G, $H \cap K = (0)$ implies $K \subseteq H^c$.
3. For each $x \in G \setminus H^c$, there exists $n \in N$ such that $0 \neq nx \in H$.

Proof. (1) ⇒ (2): Let K be an N-subgroup of G such that $H \cap K = (0)$ and $H \nsubseteq H^c$.
Since H^c is complement of H, we have $K \subseteq H^c$.
(1) ⇒ (3): Let $x \in G \setminus H^c$. By Theorem 2.17, there exists $n \in N$ such that $nx \in H \setminus H^c$. If $nx = 0$, then since H^c is an N-subgroup of G, we get $nx \in H^c$, a contradiction. Therefore, $0 \neq nx \in H$.
(2) ⇒ (1): Let K be an N-subgroup of G such that $H \cap K \subseteq H^c$. Since $H \cap K \subseteq H \cap H^c = (0)$, by (2), $K \subseteq H^c$.
(3) ⇒ (1): Follows by Theorem 2.17. □

Proposition 4.6. Let H and Δ $(\neq G)$ be ideals of G such that $\Delta \subseteq H$. Then there exists an ideal H' of G such that $H + H' \leq_\Delta^e G$ and $(H + H')/\Delta = H/\Delta \oplus (H' + \Delta)/\Delta$, where '$\oplus$' denotes the direct sum.

Proof. Let $\mathcal{S}$=$\{K : K \cap H \subseteq \Delta, K$ is an ideal of $G\}$. Clearly, $(0) \in S$, hence $\mathcal{S} \neq \phi$. Let $\{L_i\}_{i \in I}$ be a non empty family of ideals of G in $\mathcal{S}$. Define $K_i \sim K_j \Leftrightarrow K_i \subseteq K_j$. Clearly, $\sim$ is a partial order on $\mathcal{S}$, in which every chain has an upper bound, say $\bigcup_{i \in I} L_i$. By Zorn's lemma, $\mathcal{S}$ has a maximal element, say H'. We prove $H + H' \leq_\Delta^e G$. Let K be an ideal of G such that $(H+H') \cap K \subseteq \Delta$. We must show that $K \subseteq \Delta$, for this, first we show that $H \cap (H' + K) \subseteq \Delta$. Let $x \in H$, $y \in H'$, $z \in K$ such that $x = y + z$. Then by supposition $x - y = z \in (H + H') \cap K \subseteq \Delta$. Since $\Delta \subseteq H$, we get $y = x - (x - y) \in H$. Since $x - z = y \in H'$ and $y \in H$, it follows that $y = x - z \in H \cap H' \subseteq \Delta$. Since $x - y \in \Delta$, $y \in \Delta$, we get $x \in \Delta$, shows that $H \cap (H' + K) \subseteq \Delta$. Since H' is maximal such that $H \cap H' \subseteq \Delta$, we have $H' + K = H'$. This shows that $K \subseteq H'$. Consequently, $K = K \cap (H + H') \subseteq \Delta$, and hence, $H + H' \leq_\Delta^e G$.

Now to show $H/\Delta \cap (H' + \Delta)/\Delta = (0)/\Delta$, for any $x + \Delta \in H/\Delta \cap (H' + \Delta)/\Delta$, $x \in H$ and $x \in H'$. Hence $x \in H \cap H' \subseteq \Delta$, implies that $x + \Delta = (0) + \Delta$. Therefore, $H/\Delta \cap (H' + \Delta)/\Delta = (0)/\Delta$. Since $H \subseteq H + H'$, we have $H/\Delta \subseteq (H + H')/\Delta$. Further, since $\Delta \subseteq H$, we have $H' + \Delta \subseteq H' + H$, which shows that $(H' + \Delta)/\Delta \subseteq (H' + H)/\Delta$. Let $x + \Delta \in (H + H')/\Delta$. Then $x + \Delta = z + \Delta$, where

$z \in H + H'$, implies that $x + \Delta = (h + h') + \Delta$, for some $h \in H$, $h' \in H'$. Therefore, $x + \Delta = (h + \Delta) + (h' + \Delta) \in {}^{H}/_{\Delta} + {}^{(H' + \Delta)}/_{\Delta}$. Hence ${}^{(H + H')}/_{\Delta} = {}^{H}/_{\Delta} \oplus {}^{(H' + \Delta)}/_{\Delta}$. □

Definition 4.7. ([4]) We say that N distributes over G if $d(g_1 + g_2) = dg_1 + dg_2$ for all $d \in N, g_1, g_2 \in G$.

Evidently, if N distributes over G then aG is an ideal of G for any $a \in N$. The authors provided the classes of nearrings wherein each N-subgroup is an ideal ([16], Remark 5.3.39).

Corollary 4.8. Let N distributes over G, let H, and let Δ $(\neq G)$ be N-subgroups of G, such that $\Delta \subseteq H$. Then there exists an N-subgroup H' of G such that $H + H' \leq^{e}_{\Delta} G$ and ${}^{(H + H')}/_{\Delta} = {}^{H}/_{\Delta} \oplus {}^{(H' + \Delta)}/_{\Delta}$.

Proof. Follows from Proposition 4.6 and definition 4.7. □

Proposition 4.9. Let π: $G \to {}^{G}/_{K}$ be the canonical N-epimorphism, where K is an ideal of G. Let Δ be a proper ideal of G contained in K. Consider the following statements.

1. K is Δ-complement.
2. For any ideal K' of G with $K \subseteq K'$, K' is a Δ-complement of G.
3. $\pi(K')$ is a $\pi(\Delta)$-complement in ${}^{G}/_{K}$.

Then the conditions (1) and (2) imply (3).

Proof. Suppose K is a Δ-complement of an ideal J of G, and K' is a Δ-complement ideal of G such that $K' \supseteq K$. Then there exists an ideal I of G such that K' is maximal with respect to $K' \cap I \subseteq \Delta$. To show $\pi(K')$ is a $\pi(\Delta)$-complement of $\pi(I)$. That is, to show $\pi(K')$ is maximal with respect to $\pi(K') \cap \pi(I) \subseteq \pi(\Delta)$. Let $x \in \pi(K') \cap \pi(I)$. Then, $x = k' + K$ and $x = i + K$, where $k' \in K$ and $i \in I$, implies that $i - k' \in K \subseteq K'$. Hence $i \in K' \cap I \subseteq \Delta$. Therefore, $x = i + K \in {}^{\Delta}/_{K} = \pi(\Delta)$. Next we show that $\pi(K')$ is maximum with respect to the above property. Let T be any ideal of ${}^{G}/_{K}$ such that $\pi(K') \subsetneqq T$. Then $T = \pi(K'')$ for some ideal K'' of G with $K' \subseteq K''$. If $K' = K''$ then $T = \pi(K'') = \pi(K') \subsetneqq T$, a contradiction. Therefore, $K'' \supsetneqq K'$. Since K' is a Δ-complement of I, we have $K'' \cap I \nsubseteq \Delta$. Let $y \in K'' \cap I$, $y \notin \Delta$. Then $y + K \in \pi(K'') \cap \pi(I)$. Now if $y + K \in {}^{\Delta}/_{K}$, then $y + K = \delta + K$ for some $\delta \in \Delta$, and so $y - \delta \in K$. This implies that $y = (y - \delta) + \delta \in K$, hence $y \in K \cap I \subseteq K'' \cap I \subseteq \Delta$, a contradiction. Therefore, $y + K \notin {}^{\Delta}/_{K}$. Hence $y + K \in \pi(K'') \cap \pi(I) \nsubseteq \pi(\Delta)$, thus $T \cap \pi(I) \nsubseteq \pi(\Delta)$, proves $\pi(K')$ is $\pi(\Delta)$-complement of $\pi(I)$. □

5 Conclusion

We have defined the concept of relative essential ideal (or N-subgroup) of an N-group, as a generalization of essential submodules of modules over rings or nearrings. We have obtained various properties and proved results on homomorphism, relative complements, and relative direct sums. This concept can be extended to study various generalizations of closed submodules, extending submodules, uniform submodules, and their links to finite Goldie dimensional aspects.

Acknowledgments

The corresponding author[5] thanks the Indian National Science Academy (INSA), Govt. of India and the Iran Academy of Science (IAS), Iran (ref. no. IA/Iran/2018/ Oct. 29, 2018) for selecting under Bilateral Exchange program to visit Yazd University, Iran. Author[4] acknowledges the Yazd university for the hospitality during the visit. Further, the authors[1,3,4,5] acknowledge Manipal Institute of Technology (MIT), Manipal Academy of Higher Education, Manipal, India for their kind encouragement. The authors thank the referees for their suggestions.

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