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# A NOTE ON A CERTAIN CLASS OF FUNCTIONS RELATED TO HURWITZ ZETA FUNCTION AND LAMBERT TRANSFORM

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**Abstract**. In this paper we obtain multiple-series generating relations involving a class of function $\theta_{(p_n)}^{(\mu_n)}(s,a;x_1,\ldots,x_n)$ which are connected to the Hurwitz zeta function. Also, a new generalization of Lambert transform is introduced, and its relationship with the above class of functions further depicted.

## 1. Introduction and Preliminaries

The generalized (Hurwitz's) zeta function is defined by [3]

$$\zeta(s,a)=\sum_{n=0}^{\infty}(a+n)^{-s},\qquad (\mathrm{Re}(s)>1;\ a\neq 0,-1,\ldots) \tag{1.1}$$

and when $a=1$, we have

$$\zeta(s,1)=\sum_{n=1}^{\infty}n^{-s}=\zeta(s), \tag{1.2}$$

where $\zeta(s)$ is the Riemann zeta function.

The function $\phi(x,s,a)$ ([3, p.27]) extends (1.1) and is defined by

$$\phi(x,s,a)=\sum_{n=0}^{\infty}(a+n)^{-s}x^n.\qquad (\mathrm{Re}(a)>0;|x|<1) \tag{1.3}$$

The integral representation of $\phi(x,s,a)$ is of form

$$\phi(x,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-1}a^{-at}(1-xe^{-t})^{-1}dt, \tag{1.4}$$

provided that $R(a)>0$ (and either $|x|\leq 1$, $x\neq 1$, and $\mathrm{Re}(s)>0$, or $x=1$ and $\mathrm{Re}(s)>1$).

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We introduce a multivariable function $\theta^{(\mu_n)}_{(p_n)}(s,a;x_1,\ldots,x_n)$ which is defined by

$$\begin{aligned}\theta^{(\mu_n)}_{(p_n)}(s,a;x_1,\ldots,x_n) &= \theta^{(\mu_1,\ldots,\mu_n)}_{(p_1,\ldots,p_n)}(s,a;x_1,\ldots,x_n)\\ &= \sum_{m_1,\ldots,m_n=0}^{\infty}(a+\Omega)^{-s}\prod_{i=1}^{n}\left\{\frac{(\mu_i)_{m_i}}{m_i!}x_i^{m_i}\right\},\end{aligned} \tag{1.5}$$

where $\Omega=\sum_{i=1}^{n}p_i m_i$, $\mathrm{Re}(a)>0$, $\mu_i\geq 1$ (either $|x_i|<1$, $x_i\neq 1$; or $|x_i|=1$, $\mathrm{Re}(s)>n$, $\forall i=1,\ldots,n$).

Equivalently, the integral representation of $\theta^{(\mu_n)}_{(p_n)}(s,a;x_1,\ldots,x_n)$ is given by

$$\theta^{(\mu_n)}_{(p_n)}(s,a;x_1,\ldots,x_n)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\prod_{i=1}^{n}\left(1-x_ie^{-p_it}\right)^{-\mu_i}dt, \tag{1.6}$$

provided that $\mathrm{Re}(\mu_i)\geq 1$, $\mathrm{Re}(p_i)>0$; $(i=1,\ldots,n)$, $\mathrm{Re}(a)>0$ (and either $\max\{|x_i|\}\leq 1$, $x_i\neq 1$ $(i=1,\ldots,n)$, and $\mathrm{Re}(s)>0$; or $x_i=1 (i=1,\ldots,n)$ and $\mathrm{Re}(s)>\sum_{i=1}^{n}\mu_i$). As usual the symbol $(\lambda)_n$ stands for

$$(\lambda)_n=\begin{cases}1, & \text{if } n=0,\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n\in\mathbb{N}.\end{cases}\qquad (\lambda\neq 0,-1,-2,\ldots)$$

**Special cases of (1.5)**

(i) When $n=p=1$, we have

$$\phi_1^{\mu}(s,a;x)=\sum_{m=0}^{\infty}(a+m)^{-s}\frac{(\mu)_m x^m}{m!}=\phi^*_\mu(x,s,a). \tag{1.7}$$

The function $\phi^*_\mu(x,s,a)$ was studied recently by Goyal and Laddha [4].

(ii) For $n=p=\mu=1$, we have

$$\theta_1^1(s,a;1)=\sum_{m=0}^{\infty}(a+m)^{-s}x^m=\phi(x,s,a). \tag{1.8}$$

Evidently, the Hurwitz's zeta function (1.1) is given by the relation

$$\theta_1^1(s,a;1)=\zeta(s,a). \tag{1.9}$$

(iii) Corresponding to $\mu_i=x_i=1$ $(\forall i=1,\ldots,n)$, we have

$$\theta^{1,\ldots,1}_{p_1,\ldots,p_n}(s,a;1,\ldots,1)=\sum_{m_1,\ldots,m_n=0}^{\infty}(a+\Omega)^{-s}=\zeta_n(s,a;p_1,\ldots,p_n), \tag{1.10}$$

where $\Omega=\sum_{i=1}^{n}p_im_i$. The class of functions $\zeta_n(s,a;p_1,\ldots,p_n)$ is the $n$-tuple Hurwitz $\zeta$-function introduced by Barnes [1] (see also [9]).

(iv) For $\mu_i = p_i = x_i = 1$ ($\forall i = 1, \ldots, n$), we have

$$\theta^{1,\ldots,1}_{1,\ldots,1}(s, a; 1, \ldots, 1) = \sum_{m_1,\ldots,m_n=0}^{\infty} (a + \Omega^*)^{-s} = \zeta_n(s, a), \tag{1.11}$$

where $\Omega^* = \sum_{i=1}^{n} m_i$. The function $\zeta_n(s, a)$ is the multiple Hurwitz's zeta function (studied recently by Choi [2]).

In the present paper we first obtain certain multiple-series generating functions involving the multivariable function $\theta^{(\mu_n)}_{(p_n)}(s, a; x_1, \ldots, x_n)$ defined by (1.5) above. A new generalization of Lambert transform is introduced, and its inversion formula, and relationship with the function $\theta^{(\mu_n)}_{(p_n)}(s, a; x_1, \ldots, x_n)$ are also pointed out. The results presented provide extensions to some of the results in [4] and [6].

## 2. Generating Relations

Using (1.5) and the multinomial expansion ([7, p.329])

$$\sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_{\sum\limits_{i=1}^{r} k_i} \prod_{i=1}^{r} \left\{ \frac{x_i^{k_i}}{k_i!} \right\} = (1 - \sum_{i=1}^{r} x_i)^{-\lambda}, \tag{2.1}$$

provided that $|\sum_{i=1}^{r} x_i| < 1$, we easily obtain the generating function:

$$\sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_{\sum\limits_{i=1}^{r} k_i} \theta^{(\mu_n)}_{(p_n)}(\lambda + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{t_i^{k_i}}{k_i!} \right\} = \theta^{(\mu_n)}_{(p_n)}(\lambda, a - \sum_{i=1}^{r} t_i; x_1, \ldots, x_n), \tag{2.2}$$

provided that $|\sum_{i=1}^{r} t_i| < |a|$, $\lambda \neq 1$, and $|x_i| < 1$ ($i = 1, \ldots, n$).

The generating function (2.2) admits of a further extension. Indeed, in terms of the Lauricella's multiple hypergeometric series $F_D^r$ ([7, p.33]), which is defined by

$$F_D^{(r)}[a, b_1, \ldots, b_r; c; x_1, \ldots, x_r]$$
$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(a)_{m_1+\cdots+m_r}(b_1)_{m_1}\cdots(b_r)_{m_r}}{(c)_{m_1+\cdots+m_r}} \prod_{i=1}^{r} \left\{ \frac{x_i^{m_i}}{m_i!} \right\}, \quad (\max\{|x_1|, \ldots, |x_r|\} < 1) \tag{2.3}$$

it follows from (1.5) that

$$\sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\lambda)_{\sum\limits_{i=1}^{r} k_i}}{(\nu)_{\sum\limits_{i=1}^{r} k_i}} \prod_{i=1}^{r} \left\{ \frac{(\mu_i)_{k_i} t_i^{k_i}}{k_i!} \right\} \theta^{(\mu_n)}_{(p_n)}(\lambda + \sum_{i=1}^{r} (\mu_i + k_i) - \nu, a; x_1, \ldots, x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} (a + \Omega)^{-\lambda + \nu - \sum\limits_{i=1}^{r} \mu_i} \prod_{i=1}^{n} \left\{ \frac{(\mu_i)_{m_i} x_i^{m_i}}{m_i!} \right\} F_D^{(r)}\left[\lambda, \mu_1, \ldots, \mu_r; \nu; \frac{t_1}{a+\Omega}, \ldots, \frac{t_r}{a+\Omega}\right], \tag{2.4}$$

provided that $\max\{|\frac{t_i}{a}|\} < 1$, $\mu_i \geq 1$, $\forall i = 1, \ldots, r$; $\mathrm{Re}(\lambda + \sum_{i=1}^{r} \mu_i) > \mathrm{Re}(v) > 0$. More generally, (2.4) may further be extended in the form

$$\sum_{k_1,\ldots,k_r=0}^{\infty} \nabla(k_1, \ldots, k_r)\theta_{(p_n)}^{(\mu_n)}(\sigma + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{t_i^{k_i}}{k_1!} \right\}$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} (a + \Omega)^{-\sigma} \prod_{i=1}^{n} \left\{ \frac{(\mu_i)_{m_i} x_i^{m_i}}{m_i!} \right\} F_{l:s_1;\ldots;s_r}^{P:Q_1:\ldots;Q_r} \left[ \frac{t_1}{a+\Omega}, \ldots, \frac{t_r}{a+\Omega} \right], \quad (2.5)$$

provided that

(i) $1 + \sum_{i=1}^{r}(s_i - Q_i) + l - P \geq 0$ and either $P > l$ and $\sum_{i=1}^{r} |\frac{t_i}{a}|^{\frac{1}{P-l}}$ or $P \leq l$ and $\max\{|\frac{t_i}{a}|\} < 1$ $(i = 1, \ldots, r)$

(ii) $\mu_i \geq 1$, $\mathrm{Re}(a) > 0$

where

$$\nabla(k_1, \ldots, k_r) = \frac{\prod_{i=1}^{P} (a_i)_{\sum_{i=1}^{r} k_i} \prod_{i=1}^{Q_1} (b_i^1)_{k_1} \cdots \prod_{i=1}^{Q_r} (b_i^{(r)})_{k_r}}{\prod_{i=1}^{l} (\alpha_i)_{\sum_{i=1}^{r} k_i} \prod_{i=1}^{s_1} (\beta_i^1)_{k_1} \cdots \prod_{i=1}^{s_r} (\beta_i^{(r)})_{k_r}}. \quad (2.6)$$

The function $F_{l:s_1;\ldots;s_r}^{P:Q_1;\ldots;Q_r}[x_1, \ldots, x_r]$ occurring in (2.5) is the generalized Lauicella series in several variables defined as follows (see [7, p.38]):

$$F_{l:s_1;\ldots;s_r}^{P:Q_1;\ldots;Q_r}[x_1, \ldots, x_r] = F_{l:s_1;\ldots;s_r}^{P:Q_1;\ldots;Q_r} \left[ \begin{matrix} (a_P) : (b_{Q_1}^1); \ldots; (b_{Q_r}^{(r)}); \\ (\alpha_l) : (\beta_{s_1}^1); \ldots; (\beta_{s_r}^{(r)}); \end{matrix} x_1, \ldots, x_r \right]$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} \nabla(k_1, \ldots, k_r) \prod_{i=1}^{r} \left\{ \frac{x_i^{k_i}}{k_i!} \right\}, \quad (2.7)$$

where, $\nabla(k_1, \ldots, k_r)$ is defined above by (2.6).

Next, consider a set of polynomials $\{S_m^q(x)\}_{m=0}^{\infty}$ defined by [5, p.1, Eq. (1)]:

$$S_m^q(x) = \sum_{j=0}^{[m/q]} \frac{(-m)_{qj}}{j!} C(j) x^j, \qquad (q \in \mathbb{N}; m \in \mathbb{N}_0 = \mathbb{N}U\{0\}) \quad (2.8)$$

where $C(j)$ are arbitrary constants (real or complex).

Then, by simple series rearrangement method, and applying (2.2) in the process, we are lead to the following multiple generating relation:

$$\sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_{\sum_{i=1}^{r} k_i} \theta_{(p_n)}^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ S_{k_i}^{q_i}(y_i) \frac{t_i^{k_i}}{k_i!} \right\}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_{\sum_{i=1}^{r} k_i q_i} \theta_{(p_n)}^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i q_i, a - \sum_{i=1}^{r} t_i; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{C(k_i)\{y_i(-t_i)^{q_i}\}^{k_i}}{k_i!} \right\}. \quad (2.9)$$

By specializing the sequence $C(k_i)$ as follows:

$$C(k_i) = \frac{\prod_{j=1}^{P_i} \left(a_j^{(i)}\right)_{k_i}}{q_i^{k_i q_i} \prod_{j=1}^{Q_i} \left(b_j^{(i)}\right)_{k_i}}, \tag{2.10}$$

we then find from (2.9) that

$$\sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_{\sum_{i=1}^{r} k_i} \theta_{(p_n)}^{(\mu_n)}(\lambda+\sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ {}_{q_i+P_i}F_{Q_i} \left[ \begin{matrix} \Delta(q_i, -k_i), (a_{P_i}^{(i)}); \\ (b_{Q_i}^{i}) \quad ; \end{matrix} y_i \right] \frac{t_i^{k_i}}{k_i!} \right\}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_{\sum_{i=1}^{r} k_i q_i} \theta_{(p_n)}^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i q_i, a - \sum_{i=1}^{r} t_i; x_1, \ldots, x_n)$$

$$\cdot \prod_{i=1}^{r} \left\{ \frac{\prod_{j=1}^{P_i} (a_j^{(i)})_{k_i}}{k_i! \prod_{j=1}^{Q_i} (b_j^{(i)})_{k_i}} \left\{ y_i \left( \frac{-t_i}{q_i} \right)^{q_i} \right\}^{k_i} \right\}, \tag{2.11}$$

where $\Delta(m, \lambda)$ denotes the array of $m$-parameters

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \ldots, \frac{\lambda+m-1}{m} \quad (m \in \mathbb{N}).$$

**Example.** By involving the Laguerree polynomials (which occurs when $P_i = 0$, $Q_i = 1 = q_i$, $b_1^{(i)} = 1 + \alpha_i (i = 1, \ldots, r)$), (2.11) yields

$$\sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_{\sum_{i=1}^{r} k_i} \theta_{(p_n)}^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i, a; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{L_{k_i}^{\alpha_i}(y_i) t_i^{k_i}}{(1+\alpha_i)_{k_i}} \right\}$$

$$= \sum_{k_1,\ldots,k_r=0}^{\infty} (\lambda)_{\sum_{i=1}^{r} k_i} \theta_{(p_n)}^{(\mu_n)}(\lambda + \sum_{i=1}^{r} k_i, a - \sum_{i=1}^{r} t_i; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{(-y_i t_i)^{k_i}}{(1+\alpha_i)_{k_i}} \right\}. \tag{2.12}$$

Several other examples similar to (2.12) can be obtained from (2.11) by suitably specializing the sequence $C(k_i)$. We omit further details.

## 3. An Integral Transform

Let $f(t)(t \geq 0)$ be a continuous function, and

$$f(t) = O(e^{kt})(t \to \infty). \tag{3.1}$$

Then, the Lambert transform of $f(t)$ is defined by

$$F(s) = LM\{f(t)\} = \int_0^\infty \frac{st}{e^{st}-1} f(t)dt. \qquad (\mathrm{Re}(s) > 0) \tag{3.2}$$

We introduce a generalization of the Lambert transform (3.2) in the following form:

$$\begin{aligned} H^*\{f(t)\} &= H_{(p_n)}^{(\mu_n)}(x_1, \ldots, x_n; s) = H_{(p_1,\ldots,p_n)}^{(\mu_1,\ldots,\mu_n)}(x_1, \ldots, x_n; s) \\ &= \int_0^\infty \frac{st}{\prod_{i=1}^n (e^{p_i st} - x_i)^{\mu_i}} f(t)dt, \end{aligned} \tag{3.3}$$

provided that $\mathrm{Re}(s) > 0$, $p_i > 0$, $|\mu_i| \geq 1$, $\max|x_i| \leq 1$ $(\forall i = 1, \ldots, n)$, $f(t) \in A$ and $\mathrm{Re}(\gamma) > -2$, where $A$ denotes the class of functions $f(t)$ which are continuous for $t > 0$ and satisfy the order estimates:

$$f(t) = \begin{cases} O(t^\gamma) & (t \to 0_+), \\ O(t^\delta) & (t \to \infty). \end{cases} \tag{3.4}$$

The parameter $\delta$ is unrestricated, in general, since $\mathrm{Re}(s) > 0$, $p_i > 0$ $(i = 1, \ldots, n)$. We note that on putting $n = p = 1$, and setting $f = t^{k-1}g$ in (3.3), we have

$$H_1^\mu(x; s) = \int_0^\infty \frac{st^k}{(e^{st} - x)^\mu} g(t)dt, \tag{3.5}$$

which was recently studied by Goyal and Laddha [4]. Evidently,

$$H_1^1(x; s) = \int_0^\infty \frac{st}{(e^{st} - x)} f(t)dt, \tag{3.6}$$

the transform investigated by Raina and Srivastava [6]. It readily follows from (3.3) and (1.6) that

$$H^*\{t^{\alpha-1}e^{-\nu st}\} = \frac{\Gamma(\alpha+1)}{s^\alpha} \theta_{(p_n)}^{(\mu_n)}(\alpha + 1, \nu + \sum_{i=1}^n p_i\mu_i; x_1, \ldots, x_n), \tag{3.7}$$

provided that $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\nu) > 0$, $\mu_i \geq 1$, $p_i > 0$, and $|x_i| \leq 1$ $(\forall i = 1, \ldots, n)$.

Further, in view of (2.8) and (3.7) we obtain

$$\begin{aligned} &H^*\left\{t^{\alpha-1}e^{-\nu st} \prod_{i=1}^r \left(S_{m_i}^{q_i}(y_i t)\right)\right\} \\ &= \frac{1}{s^\alpha} \sum_{j_1=0}^{[m_1/q_1]} \cdots \sum_{j_r=0}^{[m_r/q_r]} \Gamma(1 + \alpha + \sum_{i=1}^r j_i) \prod_{i=1}^r \left\{\frac{(-m_i)_{q_i j_i}}{j_i!} C(j_i) \left(\frac{y_i}{s}\right)^{j_i}\right\} \\ &\cdot \theta_{(p_n)}^{(\mu_n)}(1 + \alpha + \sum_{i=1}^r j_i, \nu + \sum_{i=1}^n p_i\mu_i; x_1, \ldots, x_n) \end{aligned} \tag{3.8}$$

provided that $\mathrm{Re}(s) > 0$, $\mathrm{Re}(\alpha) > -1$, $\mathrm{Re}(\nu) > 0$ and $\mu_i \geq 1$, $p_i > 0$, $|x_i| \leq 1$ ($\forall i = 1, \ldots, n$) and $q_i \in \mathbb{N}$, $m_i \in \mathbb{N}_0$ ($\forall i = 1, \ldots, r$). In particular, when $C(j_i) = \frac{(1+\alpha_i)_{m_i}}{(1+\alpha_i)_{j_i}}$, $q_i = 1$ ($\forall i = 1, \ldots, r$), then (3.8) in terms of Laguerre polynomials gives

$$\begin{aligned} &H^* \left\{ t^{\alpha-1} e^{-\nu st} L_{m_1}^{\alpha_1}(x_1 t) \cdots L_{m_r}^{\alpha_r}(x_r t) \right\} \\ &= \prod_{i=1}^{r} \left\{ \frac{\Gamma(m_i + \alpha_i + 1)}{m_i!} \right\} s^{-\alpha} \sum_{j_1=0}^{m_1} \cdots \sum_{j_r=0}^{m_r} \Gamma(1 + \alpha + \sum_{i=1}^{r} j_i) \\ &\cdot \theta_{(p_n)}^{(\mu_n)}(1 + \alpha + \sum_{i=1}^{r} j_i, \nu + \sum_{i=1}^{n} p_i \mu_i; x_1, \ldots, x_n) \prod_{i=1}^{r} \left\{ \frac{(-m_i)_{j_i} \left(\frac{x_i}{s}\right)^{j_i}}{\Gamma(1 + \alpha_i + j_i) j_i!} \right\}. \end{aligned} \tag{3.9}$$

**Inversion formula for the transform (3.3)**

Applying the Mellin transform [8, p.46], (3.3) then gives

$$\begin{aligned} \Psi(k) &= \int_0^\infty s^{-k-1} H_{(p_n)}^{(\mu_n)}(x_1, \ldots, x_n; s) ds \\ &= \int_0^\infty s^{-k-1} \left\{ \int_0^\infty st \prod_{i=1}^{n} \left(e^{p_i st} - x_i\right)^{-\mu_i} f(t) dt \right\} ds \\ &= \int_0^\infty t f(t) \left\{ \int_0^\infty s^{-k} \prod_{i=1}^{n} \left(e^{p_i st} - x_i\right)^{-\mu_i} ds \right\} dt, && (3.10) \\ &= \Gamma(1-k) \theta_{(p_n)}^{(\mu_n)}(1 - k, \sum_{i=1}^{n} p_i \mu_i; x_1, \ldots, x_n) \int_0^\infty t^k f(t) dt, && (3.11) \end{aligned}$$

provided that, in addition to the existence and convergence conditions stated with (3.3), we also require that $\mathrm{Re}(k) < 1$, for the convergence of the inner $s$-integral in (3.10) above.

By the Mellin inversion theorem [8, p.46], we obtain the following inversion formula for the integral transform (3.3):

$$\begin{aligned} &\frac{1}{2}[f(t+0) + f(t-0)] \\ &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \left\{ \Gamma(1-k) \theta_{(p_n)}^{(\mu_n)}(1 - k, \sum_{i=1}^{n} p_i \mu_i; x_1, \ldots, x_n) \right\}^{-1} t^{-k-1} \Psi(k) dk, \end{aligned} \tag{3.12}$$

provided that $\sigma > 1/2$, $\mathrm{Re}(k) < 1$, $t^k f(t) \in L(0, \infty)$, $f(t)$ is of bounded variation in the neighbourhood of the point $t$, $\Psi(k)$ is given by (3.10), and $\mu_i \geq 1$, $p_i > 0$, $\max\{|x_i|\} \leq 1$ ($\forall i = 1, \ldots, n$).

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### References

[1] E. W. Barnes, *On the theory of the multiple gamma function*, Philos. Trans. Roy. Soc., **199**(1904), 374-439.

[2] J. Choi, *Multiple gamma function and their applications*, in Proc. Internat. Conf. on Analysis (editied by Y.C. Kim), Yeungnam Univeristy, Korea (1996), 73-84.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York, Toronto and Londom, 1953.

[4] S. P. Goyal and R. K. Laddha, *On the generalized Riemann zeta function and the generalized Lambert transform*, Ganita Sandesh, **11**(1997), 99-108.

[5] H. M. Srivastava, *A contour integral involving Fox's $H$-function*, Indian J. Math., **14**(1972), 1-6.

[6] R. K. Raina and H. M. Srivastava, *Certain results associated with the generalized Riemann zeta functions*, Rev. Tec. Ing. Univ. Zulia, **18**(1995), 301-304.

[7] H. M. Srivastava and P. W. Karlson, Multiple Gaussian Hypergeometric Series, John Wiley and Sons, New York, 1984.

[8] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Clarendon Press, Oxford, 1948.

[9] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (4th ed.), Cambridge University Press, 1963.

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