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CERTAIN CLASSES OF MEROMORPHICALLY MULTIVALENT FUNCTIONS WITH FIXED ARGUMENT OF COEFFICIENTS

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Abstract. In this paper, we consider some classes of meromorphically multivalent functions with fixed argument of coefficients. In those classes, we determine coefficient estimates, distortion theorems and extreme points.

1. Introduction

Let Σ_p denote the class of functions f of the form

$$f(z)=\frac{1}{z^p}+\sum_{n=1}^{\infty}a_nz^n \qquad (p\in N=\{1,2,\ldots\}) \tag{1.1}$$

which are analytic in $U-\{0\}$, where $U=\{z:|z|<1\}$. For analytic functions f and g, we say that f is subordinate to g, written $f\prec g$, if there exists a Schwarz function w such that $f(z)=g(w(z))$ for $z\in U$.

Definition 1.1. Let $\Sigma_p^\theta(A,B)$ denote the class of functions f of the the form (1.1) such that

$$-z^{p+1}f'(z)\prec p\frac{1+Az}{1+Bz} \tag{1.2}$$

and $\arg a_n=\theta$ for $n\in\mathbb{N}$, where $0\le B\le 1$ and $-B\le A<B$ ($B\ne 1$ or $\cos\theta>0$).

We note that every function f belonging to the class $\Sigma_p^\theta(A,B)$ can be written as in the form

$$f(z)=\frac{1}{z^p}+e^{i\theta}\sum_{n=1}^{\infty}|a_n|z^n \quad (z\in U-\{0\}). \tag{1.3}$$

Definition 1.2. Let $\widetilde{\Sigma}_p^\theta(A,B)$ denote the class of functions f of the form (1.3) satisfying the following condition

$$\sum_{n=1}^{\infty}n|a_n|\le\delta(\theta,A,B), \tag{1.4}$$

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where

$$\delta(\theta, A, B) = \frac{p(B-A)}{\sqrt{1-B^2\sin^2\theta}+B\cos\theta}, \quad 0 \le B \le 1 \text{ and } -B \le A < B \ (B \ne 1 \text{ or } \cos\theta > 0). \tag{1.5}$$

Meromorphic univalent functions have been extensively studied by Clunie [1], Libera [2], Mogra, Reddy and Juneja [4], Pormmerenke [5] and others. In particular, the class $\Sigma_p^0(A, B)$ was studied by Mogra [3].

The object of the present paper is to obtain coefficient estimates, distortion theorems and extreme points for the classes of functions defined above.

2. Coefficient Estimates

Theorem 2.1. *If a function f of the form* (1.3) *belongs to the class $\Sigma_p^\theta(A, B)$, then it satisfies the condition* (1.4).

Proof. Let $f \in \Sigma_p^\theta(A, B)$. By Definition (1.1), we obtain

$$-z^{p+1}f'(z) = p\frac{1+Aw(z)}{1+Bw(z)},$$

where w is an analytic function in U such that $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$. Thus we have

$$\left|\frac{z^{p+1}f'(z)+p}{Ap+Bz^{p+1}f'(z)}\right| = |w(z)| < 1.$$

Then we have

$$\left|\sum_{n=1}^{\infty} n|a_n|z^{n+p}\right| < |Ap + Bz^{p+1}f'(z)|.$$

Putting $z = r(0 < r < 1)$, we obtain

$$|w| < |(B-A)p - Be^{i\theta}w|, \tag{2.1}$$

where

$$w = \sum_{n=1}^{\infty} n|a_n|r^{n+p}.$$

Since w is a real number, by (2.1) we have

$$(1-B^2)w^2 + 2pB(B-A)\cos\theta w - p^2(B-A)^2 < 0.$$

Solving this inequality with respect to w, we obtain

$$\sum_{n=1}^{\infty} n|a_n|r^{n+p} < \delta(\theta, A, B),$$

where $\delta(\theta, A, B)$ is defined by (1.5). Therefore, letting $r \to 1^-$, we have (1.4).

From Theorem 2.1, we have

Corollary 2.1. $\Sigma_p^\theta(A, B) \subset \widetilde{\Sigma}_p^\infty(A, B)$.

Corollary 2.2. *If a function f of the form* (1.3) *belongs to the class $\widetilde{\Sigma}_p^\theta(A, B)$, then*

$$|a_n| \le \frac{\delta(\theta, A, B)}{n} \quad (n \in N). \tag{2.2}$$

The result is sharp for the extremal functions f_n of the form

$$f_n(z) = \frac{1}{z^p} + e^{i\theta} \frac{\delta(\theta, A, B)}{n} z^n \quad (n \in N).$$

By Corollary 2.1 and Corollary 2.2, we obtain

Corollary 2.3. *If a function f of the form* (1.3) *belongs to the class $\Sigma_p^\theta(A, B)$, then*

$$|a_n| \le \frac{\delta(\theta, A, B)}{n} \quad (n \in N),$$

where $\delta(\theta, A, B)$ is defined by (1.5). *The result is sharp for $\theta = 0$. The extremal functions are functions f_n of the form*

$$f_n(z) = \frac{1}{z^p} + \frac{p(B - A)}{(1 + B)n} z^n \qquad (n \in N). \tag{2.3}$$

3. Distortion Theorems and Extreme Points

Theorem 3.1. *If $f \in \Sigma_p^\theta(A, B)$, then*

$$\frac{1}{|z|^p} - \delta(\theta, A, B)|z| \le |f(z)| \le \frac{1}{|z|^p} + \delta(\theta, A, B)|z| \tag{3.1}$$

and

$$\frac{p}{|z|^{p+1}} - \delta(\theta, A, B) \le |f'(z)| \le \frac{p}{|z|^{p+1}} + \delta(\theta, A, B), \tag{3.2}$$

where $\delta(\theta, A, B)$ is defined by (1.5). *The result is sharp for $\theta = 0$. The extremal function is function f_1 of the form* (2.3).

Proof. Let a function f of the form (1.3) belong to the class $\Sigma_p^\theta(A, B)$. By Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty} |a_n| \le \sum_{n=1}^{\infty} n|a_n| \le \delta(\theta, A, B). \tag{3.3}$$

Since

$$|f(z)| = \left| \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} |a_n| z^n \right| \leq \frac{1}{|z|^p} + \sum_{n=1}^{\infty} |a_n||z|^n \leq \frac{1}{|z|^p} + |z| \sum_{n=1}^{\infty} |a_n| \leq \frac{1}{|z|^p} + |z|\delta(\theta, A, B),$$

and

$$|f(z)| = \left| \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} |a_n| z^n \right| \geq \frac{1}{|z|^p} - \sum_{n=1}^{\infty} |a_n||z|^n \geq \frac{1}{|z|^p} - |z| \sum_{n=1}^{\infty} |a_n| \geq \frac{1}{|z|^p} - |z|\delta(\theta, A, B)$$

by (3.3), we obtain (3.1). Using (3.3), we prove the estimation (3.2) analogously.

Theorem 3.2. *Let $\delta(\theta, A, B)$ be defined by* (1.5) *and let*

$$f_0(z) = \frac{1}{z^p} \tag{3.4}$$

and

$$f_n(z) = \frac{1}{z^p} + e^{i\theta} \frac{\delta(\theta, A, B)}{n} z^n \qquad (n \in N). \tag{3.5}$$

Then a function f belongs to the class $\widetilde{\Sigma}_p^{\theta}(A, B)$ if and only if it is of the form

$$f(z) = \sum_{n=0}^{\infty} \gamma_n f_n(z) \quad (z \in U - \{0\}), \tag{3.6}$$

where $\sum_{n=0}^{\infty} \gamma_n = 1$ and $\gamma_n \geq 0$ $(n \in N \cup \{0\})$.

Proof. Let a function f of the form (1.3) belong to the class $\widetilde{\Sigma}_p^{\theta}(A, B)$. Put

$$\gamma_n = \frac{n}{\delta(\theta, A, B)} |a_n| \qquad (n \in N)$$

and

$$\gamma_0 = 1 - \sum_{n=1}^{\infty} \gamma_n.$$

By the assumption and Definition 1.2, we have $\gamma_n \geq 0$ $(n \in N)$ and $\gamma_0 \geq 0$. Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} \gamma_n f_n(z) &= \gamma_0 f_0(z) + \sum_{n=1}^{\infty} \gamma_n f_n(z) \\
&= \left(1 - \sum_{n=1}^{\infty} \gamma_n\right) \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{n}{\delta(\theta, A, B)} |a_n| \left(\frac{1}{z^p} + e^{i\theta} \frac{\delta(\theta, A, B)}{n} z^n \right) \\
&= \frac{1}{z^p} - \sum_{n=1}^{\infty} \frac{n}{\delta(\theta, A, B)} |a_n| \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{n}{\delta(\theta, A, B)} |a_n| \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} |a_n| z^n \\
&= f(z)
\end{aligned}$$

and the condition (3.6) follows. Conversely, let the function f satisfy (3.6). Since

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \gamma_n f_n(z) \\ &= \gamma_0 f_0(z) + \sum_{n=1}^{\infty} \gamma_n f_n(z) \\ &= \left(1 - \sum_{n=1}^{\infty} \gamma_n\right) \frac{1}{z^p} + \sum_{n=1}^{\infty} \left(\frac{1}{z^p} + e^{i\theta} \frac{\delta(\theta, A, B)}{n} z^n\right) \gamma_n \\ &= \frac{1}{z^p} + e^{i\theta} \sum_{n=1}^{\infty} \frac{\delta(\theta, A, B)}{n} \gamma_n z^n, \end{aligned}$$

we can write the function f in the form (1.3), where

$$|a_n| = \frac{\delta(\theta, A, B)}{n} \gamma_n.$$

Moreover,

$$\sum_{n=1}^{\infty} n|a_n| = \sum_{n=1}^{\infty} \gamma_n \delta(\theta, A, B) = \delta(\theta, A, B)(1 - \gamma_0) \leq \delta(\theta, A, B).$$

Thus we have $f \in \widetilde{\Sigma}_p^{\theta}(A, B)$, which completes the proof of our result.

By using the same method as in the proof of Theorem 3.2, we can prove the following.

Theorem 3.3. *Let* $f_0(z) = \frac{1}{z^p}$ *and let* $f_n (n \in N)$ *be defined by* (2.3). *Then a function* f *belongs to the class* Σ_p^0 *if and only if it is of the form* (3.6).

From Theorem 3.2 and Theorem 3.3, we obtain the following two corollaries.

Corollary 3.1. $\Sigma_p^0(A, B) = \widetilde{\Sigma}_p^0(A, B)$.

Corollary 3.2. *The class* $\Sigma_p^0(A, B)$ *is convex.*

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