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REFINEMENTS OF JENSEN'S INEQUALITY

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Abstract. Some new refinements are presented for Jensen's inequality. These strengthen several results obtained in the recent literature.

1. Introduction

A central tool in analysis is Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}$$

for convex functions. Recently some improvements for this have been found by Yang and Wang [3]. In particular they established the following.

Theorem A. *Let $f:[a,b]\to\mathbf{R}$ be a convex function and $\alpha_i\in(0,1)$ $(i=1,\dots,n)$ with $\Sigma_{i=1}^n a_i=1$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(\sum_{i=1}^n a_i x_i\right)dx_1\cdots dx_n \\ &\le \sum_{i=1}^n \frac{1-\alpha_i}{(n-1)(b-a)^{n-1}}\int_a^b\cdots\int_a^b \left(\frac{\sum\limits_{j=1,j\ne i}^n \alpha_j x_j}{1-\alpha_i}\right)\times dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_n \\ &\le \frac{1}{b-a}\int_a^b f(x)dx. \end{aligned} \tag{1}$$

They also constructed a convex, increasing function which lies between the two sides of the first inequality in (1).

Theorem B. *Let $f:[a,b]\to\mathbf{R}$ be a convex function and $\alpha_i\in(0,1)$ $(i=1,\dots,n)$ with $\sum_{i=1}^n\alpha_i=1$. If $K:[0,1]\to\mathbf{R}$ is a function defined by*

$$K(t)=\frac{1}{(b-a)^n}\int_a^b\cdots\int_a^b f\left(t\sum_{i=1}^n \alpha_i x_i+(1-t)\frac{a+b}{2}\right)dx_1\cdots dx_n,$$

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then

(i) K *is convex on* $[0,1]$,

(ii)

$$\min_{t\in[0,1]} K(t) = K(0) = f\left(\frac{a+b}{2}\right),$$

$$\max_{t\in[0,1]} K(t) = K(1) = \frac{1}{(b-a)^n}\int_a^b \cdots \int_a^b f\left(\sum_{i=1}^n \alpha_i x_i\right) dx_1 \cdots dx_n,$$

(iii) K *is increasing on* [0,1].

In this paper we make some analogous improvements to Jensen's inequality. In Section 2 we derive the first of these. An important special case extends Theorem A. A related result provides a further interpolation of our first result. In Section 3 we present a convex function construction analogous to that of Theorem B. Finally in Section 4 we give a second improvement of Jensen's inequality. This extends a result of Yang and Wu [4].

We suppose without further comment the existence of all the integrals in our discussion. We also suppose $n \geq 2$ throughout, as the statements for $n = 1$ are either trivial or void.

2. First Refinement

Theorem 1. *Suppose I is a real interval. Let $f : I \to \mathbf{R}$ be a convex function, $g : [a,b] \to I$ a real function and $w : [a,b] \to \mathbf{R}$ a positive function. Let $\alpha_i \in (0,1)$ $(i = 1,\ldots,n)$ with $\sum_{i=1}^n \alpha_i = 1$ and $\overline{w} = \int_a^b w(x)dx$. Further, let*

$$F_{k,n} = \frac{1}{\binom{n-1}{k-1}\overline{w}^k} \sum_{1\leq i_1<\cdots<i_k\leq n} \sum_{r=1}^k \alpha_{i_r} \int_a^b \cdots \int_a^b \left[\prod_{s=1}^k w(x_{i_s})\right] f\left(\frac{\sum\limits_{j=1}^k \alpha_{i_j} g(x_{i_j})}{\sum\limits_{j=1}^k \alpha_{i_j}}\right) d_{x_{i_1}} \cdots dx_{i_k}.$$

Then

$$f\left(\frac{1}{\overline{w}}\int_a^b w(x)g(x)dx\right) \leq F_{n,n} \leq \cdots \leq F_{k+1,n} \leq F_{k,n} \leq \cdots \leq F_{1,n} = \frac{1}{\overline{w}}\int_a^b w(x)f(g(x))dx. \tag{2}$$

Proof. Using Jensen's inequality, we obtain

$$\begin{aligned} f\left(\frac{1}{\overline{w}}\int_a^b w(x)g(x)dx\right) &= f\left(\frac{1}{\overline{w}^n}\int_a^b \cdots \int_a^b \left[\prod_{j=1}^n w(x_j)\right] \sum_{j=1}^n \alpha_i g(x_i) dx_1 \cdots dx_n\right) \\ &\leq \frac{1}{\overline{w}}\int_a^b \cdots \int_a^b \left[\prod_{j=1}^n w(x_j)\right] f\left(\sum_{i=1}^n \alpha_i g(x_i)\right) dx_1 \cdots dx_n, \end{aligned}$$

which establishes the first inequality in (2).

For all $k = 1, \ldots, n$, define

$$f_{k,n}(x_1, \ldots, x_n) = \frac{1}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{r=1}^{k} \alpha_{i_r} f \left(\frac{\sum\limits_{j=1}^{k} \alpha_{i_j} x_{i_j}}{\sum\limits_{j=1}^{k} \alpha_{i_j}} \right).$$

It has been shown in [2] that

$$f_{k+1,n}(x_1, \ldots, x_n) \le f_{k,n}(x_1, \ldots, x_n)$$

for all $k = 1, \ldots, n-1$ and all real n-tuples $(x_1, \ldots, x_n)$, so in particular we have

$$f_{k+1,n}(g(x_1), \ldots, g(x_n)) \le f_{k,n}(g(x_1), \ldots, g(x_n)).$$

On multiplying by $\prod_{\ell=1}^{n} w(x_\ell)$ and integrating with respect to $x_1, x_2, \ldots, x_n$, we obtain

$$\frac{\overline{w}^{n-k-1}}{\binom{n-1}{k}} \sum_{1 \le i_1 < \cdots < i_{k+1} \le n} \sum_{r=1}^{k+1} \alpha_{i_r} \int_a^b \cdots \int_a^b \left[\prod_{s=1}^{k+1} w(x_{i_s}) \right] f \left(\frac{\sum\limits_{j=1}^{k+1} \alpha_{i_j} g(x_{i_j})}{\sum\limits_{j=1}^{k+1} \alpha_{i_j}} \right) dx_{i_1} \cdots dx_{i_{k+1}}$$

$$\le \frac{\overline{w}^{n-k}}{\binom{n-1}{k-1}} \sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{r=1}^{k} \alpha_{i_r} \int_a^b \cdots \int_a^b \left[\prod_{s=1}^{k} w(x_{i_s}) \right] f \left(\frac{\sum\limits_{j=1}^{k} \alpha_{i_j} g(x_{i_j})}{\sum_{j=1}^{k} \alpha_{i_j}} \right) dx_{i_1} \cdots dx_{i_k},$$

that is,

$$F_{k+1,n} \le F_{k,n},$$

which provides the remaining inequalities. The final equality is trivial.

Denote by

$$\overline{F}_{k,n} = \frac{1}{\binom{n-1}{k-1}(b-a)^k} \sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{r=1}^{k} \alpha_{i_r} \int_a^b \cdots \int_a^b f \left(\frac{\sum\limits_{j=1}^{k} \alpha_{i_j} x_{i_j}}{\sum_{j=1}^{k} \alpha_{i_j}} \right) dx_{i_1} \cdots dx_{i_k}$$

the value of $F_{k,n}$ obtained in the special case $w(x) := 1$ and $g(x) := x$. In particular we have

$$\overline{F}_{n,n} = \frac{1}{(b-a)^n} \int_a^b \cdots \int_a^b f \left(\sum_{i=1}^{n} \alpha_i x_i \right) dx_1 \cdots dx_n.$$

Then we have the following.

Corollary 1. *Let* $f : [a,b] \to \mathbf{R}$ *be convex and* $\alpha_i \in (0,1)$ $(i = 1,\dots,n)$ *with* $\sum_{i=1}^n \alpha_i = 1$. *Then*

$$f\left(\frac{a+b}{2}\right) \le \overline{F}_{n,n} \le \cdots \le \overline{F}_{k+1,n} \le \overline{F}_{k,n} \le \cdots \le \overline{F}_{1,n} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Remark 1. Corollary 1 provides a refinement of Theorem A, which may be written

$$f\left(\frac{a+b}{2}\right) \le \overline{F}_{n,n} \le \overline{F}_{n-1,n} \le \overline{F}_{1,n}.$$

We now establish some associated results.

Theorem 2. *Let the assumptions of Theorem* 1 *be fulfilled. Then*

$$F_{k,n} \le G_{k,n} := \frac{(n-k)F_{1,n} + (k-1)F_{n,n}}{n-1}.$$

Proof. Let $f_{k,n}$ be defined as in the proof of Theorem 1. It is known (see [1, p.173]) that

$$f_{k,n}(x_1,\dots,x_n) \le \frac{(n-k)f_{1,n}(x_1,\dots,x_n) + (k-1)f_{n,n}(x_1,\dots,x_n)}{n-1}$$

holds for all $k = 1,\dots,n$ and all real n-tuples $(x_1,\dots,x_n)$. As in the previous theorem we may replace x_i by $g(x_i)$. Multiplication by $\prod_{\ell=1}^n w(x_\ell)$ and integrating with respect to $x_1, x_2, \dots, x_n$ yields the desired result.

Proposition 1. *Under the assumptions of Theorem* 1,

$$F_{n,n} = G_{n,n} \le F_{n-1,n} \le G_{n-1,n} \le \cdots \le G_{k+1,n} \le G_{k,n} \le \cdots \le G_{1,n} = F_{1,n}.$$

Proof. By definition $F_{n,n} = G_{n,n}$ and $F_{1,n} = G_{1,n}$ and since $F_{n,n} \le F_{1,n}$, we have $G_{k+1,n} \le G_{k,n}$ for $1 \le k < n$.

Corollary 2. *Suppose the assumptions of Corollary* 1 *hold and that* $\overline{G}_{k,n}$ *is the value of* $G_{k,n}$ *when* $w(x) := 1$ *and* $g(x) := x$. *Then*

$$\overline{F}_{n-1,n} \le \overline{G}_{n-1,n} \le \cdots \le \overline{G}_{k+1,n} \le \overline{G}_{k,n} \le \cdots \le \overline{G}_{1,n} = \overline{F}_{1,n}.$$

Remark 2. Corollary 2 supplies an interpolation of the last inequality in (1).

3. Convex Function Construction

In this section we proceed to the construction of a convex, increasing function between the two sides of the first inequality in (2).

Theorem 3. *Let the assumptions of Theorem* 1 *be fulfilled and suppose the function* $H_f : [0,1] \to \mathbf{R}$ *is defined by*

$$H_f(t) = \frac{1}{\overline{w}^n} \int_a^b \cdots \int_a^b \left[\prod_{j=1}^n w(x_j) \right] f\left(t \sum_{i=1}^n \alpha_i g(x_i) + \frac{1-t}{\overline{w}} \int_a^b w(x) g(x) dx \right) dx_1 \cdots dx_n.$$

Then

(i) H_f *is convex on* [0,1],

(ii)

$$\min_{t\in[0,1]} H_f(t) = H_f(0) = f\left(\frac{1}{\overline{w}} \int_a^b w(x)g(x)dx \right),$$

$$\max_{t\in[0,1]} H_f(t) = H_f(1) = \frac{1}{\overline{w}^n} \int_a^b \cdots \int_a^b f \left[\prod_{j=1}^n w(x_j) \right] f\left(\sum_{i=1}^n \alpha_i g(x_i) \right) dx_1 \cdots dx_n,$$

(iii) H_f *is increasing on* [0,1].

Proof. (i) The function

$$\phi(t) := t \sum_{i=1}^n \alpha_i g(x_i) + \frac{1-t}{\overline{w}} \int_a^b w(x)g(x)dx$$

is linear for $t \in [0,1]$, so that $f \circ \phi$ is convex on [0,1]. Hence H_f is convex on [0,1].
(ii) From the convexity of f,

$$H_f(t) \le tH_f(1) + (1-t)H_f(0).$$

Also, by the first inequality in (2),

$$H_f(0) = f\left(\frac{1}{\overline{w}} \int_a^b w(x)g(x)dx \right) \le H_f(1).$$

Hence $H_f(t) \le H_f(1)$.

Set $e(x) := x$. By the convexity of f,

$$\begin{aligned} H_f(t) \ge f(H_e(t)) &= f\left(\frac{t}{\overline{w}} \int_a^b w(x)g(x)dx + \frac{1-t}{\overline{w}} \int_a^b w(x)g(x)dx \right) \\ &= f\left(\frac{1}{\overline{w}} \int_a^b w(t)g(t)dt \right) = H_f(0). \end{aligned}$$

(iii) Using the convexity of H_f and $H_f(t) \ge H_f(0)$,

$$\frac{H_f(u) - H_f(t)}{u-t} \ge \frac{H_f(t) - H_f(0)}{t} \ge 0$$

for $0 \le t < u \le 1$. Hence $H_f(t) \le H_f(u)$ for $t < u$.

4. Second Refinement

Theorem 4. *Suppose I is a real interval. Let $f : I \to \mathbf{R}$ be a convex function and let $h : [0,1] \to \mathbf{R}$ be defined by*

$$h(t) = \frac{1}{n} \sum_{i=1}^{n} f((1-t)x_i + t x_{i+1}),$$

where $x_i \in I$ $(1 \le i \le n)$ and $x_{n+1} = x_1$. Then h is convex on [0,1],

$$\min_{t\in[0,1]} h(t) = h(0) = f\left(\frac{\sum\limits_{i=1}^{n} x_i}{n}\right), \tag{3}$$

$$\max_{t\in[0,1]} h(t) = h(1) = \frac{\sum\limits_{i=1}^{n} f(x_i)}{n}. \tag{4}$$

Proof. Let $t_1, t_2 \in [0,1]$ and $a_1, a_2 > 0$ with $a_1 + a_2 = 1$. Then

$$\begin{aligned} h(a_1t_1 + a_2t_2) &= \frac{1}{n}\sum_{i=1}^{n} f\left(\left(1 - \sum_{j=1}^{2} a_j t_j\right) x_i + \sum_{j=1}^{2} a_j t_j x_{i+1}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n} f\left(\sum_{j=1}^{2} a_j[(1-t_j)x_i + t_j x_{i+1}]\right) \\ &\le \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{2} a_j f((1-t_j)x_i + t_j x_{i+1}) \\ &= a_1 h(t_1) + a_2 h(t_2), \end{aligned}$$

so h is convex on [0,1].

Consequently

$$f\left(\frac{\sum\limits_{i=1}^{n} x_i}{n}\right) \le h(t) \le \frac{\sum\limits_{i=1}^{n} f(x_i)}{n},$$

for all $x_i \in I$ $(i = 1, \ldots, n)$ and $t \in [0,1]$, whence (3) and (4) follow.

The special case $n = 3$ was shown by Yang and Wu [4] and used to establish [4, Theorem 2.2]. The present result can be used to generalize, [4, Theorem 2.2] in a natural way.

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