TAMKANG JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 2000

# ON THE EXISTENCE OF PROJECTIVE AFFINE MOTION IN A $W$-RECURRENT FINSLER SPACE

A. KUMAR, H. S. SHULKA AND R. P. TRIPATHI

**Abstract**. The paper is devoted to study the properties of a $W$-$R$ $F_n$ space admitting an infinitesimal point transformation $\overline{x}^i = x^i + v^i(x)dt$ which satisfies the condition $L_v \lambda_s = 0$.

## 1. Introduction

Let us consider an $n$-dimensional affinely connected Finsler space $F_n$ [1][1] equipped with $2n$ line elements $(x^i, \dot{x}^i)$ and a fundamental metric function $F(x, \dot{x})$ positively homogeneous of degree one in its directional argument. The fundamental metric tensor $g_{ij}(x, \dot{x}) \overset{\text{def}}{=} \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, \dot{x})$[2] of the space is symmetric in its indices $i$ and $j$. Let $T^i_j(x, \dot{x})$ be any tensor field depending on both the positional and directional arguments. The covariant derivative of $T^i_j(x, \dot{x})$ with respect to $x^k$ in the sense of Berwald is given by

$$T^i_{j(k)} = \partial_k T^i_j - \dot{\partial}_h T^i_j G^h_k + T^h_j G^i_{hk} - T^i_h G^h_{jk} \tag{1.1}$$

where $G^i_{jk}(x, \dot{x})$ are Berwald's connection coefficients and satisfy the following relations:

$$\text{a) } \dot{\partial}_h G^i_{jk} = G^i_{hjk} \qquad \text{b) } G^i_{hjk} \dot{x}^h = 0 \quad \text{and} \tag{1.2}$$
$$\text{c) } G^i_{hk} = G^i_{kh}$$

The following commutation formulae involving the Berwald's covariant derivatives are given by

$$\dot{\partial}_h T^i_{j(k)} - (\dot{\partial}_h T^i_j)_{(k)} = T^s_j G^i_{shk} - T^i_s G^s_{jhk}, \tag{1.3}$$

$$[3]\; 2T^i_{j[(h)(k)]} = -\dot{\partial}_r T^i_j H^r_{hk} + T^s_j H^i_{shk} - T^i_s H^s_{jhk} \tag{1.4}$$

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Received February 12, 1998; revised October 30, 1999.
1991 *Mathematics Subject Classification.* 53C.
*Key words and phrases.* Finsler space, projective motion, recurrent, Lie-derivative, infinitesimal point transformation, connection coefficients.

[1] The numbers in square brackets refer to the references at the end of the paper.

[2] $\dot{\partial}_i \equiv \partial/\partial \dot{x}^i$ and $\partial_i \equiv \partial/\partial x_i$

[3] $2A_{[hk]} = A_{hk} - A_{kh}$

where

$$H^i_{hjk}(x,\dot{x}) \overset{def}{=} 2\left\{\partial_{[k}G^i_{j]h} + G^r_{h[j}G^i_{k]r} + G^i_{rh[k}G^r_{j]}\right\} \tag{1.5}$$

are Berwald's curvature tensor field and satisfies the following relations:

$$\begin{array}{lll} \text{a) } H^i_{hjk} = -H^i_{hkj}, & \text{b) } H^i_{hjk}\dot{x}^h = H^i_{jk}, & \text{c) } H^r_{rhj} = 2H_{[hj]} \\ \text{d) } H^r_{hjr} = H_{hj}, & \text{e) } H^i_i = (n-1)H \quad \text{and} & \text{f) } \dot{x}^j H^i_{jk} = H^i_k \end{array} \tag{1.6}$$

The projective deviation tensor field $W^i_{hjk}(x,\dot{x})$ of the space is given by

$$W^i_{hjk}(x,\dot{x}) = H^i_{hjk} + \frac{1}{(n+1)}\left\{\delta^i_h H^r_{rkj} + \dot{x}^i\dot{\partial}_h H^r_{rkj} + 2\delta^i_{[j}(H_{(h)k]} + \dot{\partial}_{k]}\dot{\partial}_h H)\right\} \tag{1.7}$$

which satisfies the following identity:

$$W^i_{hjk(\ell)} + W^i_{hk\ell(j)} + W^i_{h\ell j(k)} = 0 \tag{1.8}$$

If the projective deviation tensor field $W^i_{hjk}(x,\dot{x})$ satisfies the condition

$$W^i_{hjk(s)} = \lambda_s W^i_{hjk} \tag{1.9}$$

where $\lambda_s(x)$ means a non-zero covariant recurrence vector, the space is called a $W$-recurrent Finsler space or an $W$-$R$ $F_n$ space.

Let us consider an infinitesimal point transformation

$$\overline{x}^i = x^i + v^i(x)dt \tag{1.10}$$

where $v^i(x)$ is any vector field and $dt$ is an infinitesimal point constant. The above transformation which is considered at each point in the space is called a projective affine motion, when and only when

$$L_v G^i_{jk} = 0 \tag{1.11}$$

where $L_v$ denotes the well known Lie-derivative with respect to (1.10). The Lie-derivatives of the tensor field $T^i_j(x,\dot{x})$ and connection coefficient $G^i_{jk}(x,\dot{x})$ in view of (1.10) and the Berwald's covariant derivative are given by [2]

$$L_v T^i_j = T^i_{j(h)}v^h + T^i_h v^h_{(j)} - T^h_j v^i_{(h)} + \dot{\partial}_h T^i_j v^h_{(s)}\dot{x}^s \tag{1.12}$$

and

$$L_v G^i_{jk} = v^i_{(j)(k)} + H^i_{jkh}v^h + G^i_{sjk}v^s_{(r)}\dot{x}^r \tag{1.13}$$

respectively.

We have the following commutation formulae:

$$L_v(\dot{\partial}_\ell T^i_j) - \dot{\partial}_\ell L_v T^i_j = 0 \tag{1.14}$$

$$(L_v G^i_{jh})_{(k)} - (L_v G^i_{kh})_{(j)} = L_v H^i_{hjk} + 2\dot{x}^s G^i_{rh[j}L_v G^r_{k]s} \tag{1.15}$$

and

$$(L_vT^i_{jk(m)})-(L_vT^i_{jk})_{(m)}=T^s_{jk}L_vG^i_{sm}-T^i_{sk}L_vG^s_{jm}-T^i_{js}L_vG^s_{km}-\dot{\partial}_sT^i_{jk}L_vG^s_{rm}\dot{x}^r \quad (1.16)$$

Hence, for an infinitesimal projective affine motion the last relation shows that the two operators $L_v$ and $(k)$ are commutative with each other.

With the help of the equation (1.11) and (1.15), we get

$$L_vH^i_{hjk}=0 \quad (1.17)$$

In view of the equation (1.6) and the fact that the operations of contraction and Lie-derivation are commutative the above relation yields

$$\text{a) } L_vH^r_{rjk}=0, \qquad \text{b) } L_vH_{jk}=0 \quad \text{and} \quad \text{c) } L_vH=0 \quad (1.18)$$

Taking the Lie-derivative of the each side of (1.7) and using the equations (1.14), (1.17) and (1.18), we obtain

$$L_vW^i_{hjk}=0 \quad (1.19)$$

Applying $L_v$ to both sides of (1.9) and using the equations (1.11), (1.16) and (1.19), we have

$$(L_v\lambda_s)W^i_{hjk}=0 \quad (1.20)$$

Since the space is not an isotropic (i.e. $W^i_{hjk}\neq 0$), we have

$$L_v\lambda_s=0 \quad (1.21)$$

i.e. the recurrence vector $\lambda_s$ of the space must be Lie-invariant one.

In what follows, we shall study a *W-R Fn* space admitting an infinitesimal transformation $\overline{x}^i=x^i+v^i(x)dt$ which satisfies (1.21). We shall call such a restricted space, for brevity, as *S-WR Fn* space.

## 2. The Vanishing of $L_vW^i_{hjk}(x,\dot{x})$

First of all here we shall prove the following lemma:

**Lemma 2.1.** *In an S-WR Fn space if the recurrence vector $\lambda_s$ is a gradient one, we have $\lambda_sv^s=$ const.*

**Proof.** For brevity, let us put

$$\delta=\lambda_sv^s \quad (2.1)$$

Then, with the help of the equations (1.12) and (1.21), we have

$$L_v\lambda_s=\lambda_{s(m)}v^m+\lambda_mv^m_{(s)}=0 \quad (2.2)$$

By virtue of the assumption $\lambda_{s(m)} = \lambda_{m(s)}$ the above equation reduces to

$$\delta_{(m)} = 0 \tag{2.3}$$

which completes the proof.

In view of the basic condition (1.12), the Lie-derivative of $W^i_{hjk}(x, \dot{x})$ is given by

$$L_v W^i_{hjk} = W^i_{hjk(s)} v^s + W^i_{sjk} v^s_{(h)} + W^i_{hsk} v^s_{(j)} + W^i_{hjs} v^s_{(k)} - W^s_{hjk} v^i_{(s)} + \dot{\partial}_s W^i_{hjk} v^s_{(r)} \dot{x}^r \tag{2.4}$$

which by virtue of the equation (1.9) and (2.1) reduces to

$$L_v W^i_{h\dot{j}k} = \delta W^i_{h\dot{j}_k} + W^i_{sjk} v^s_{(h)} + W^i_{hsk} v^s_{(\dot{j})} + W^i_{h\dot{j}s} v^s_{(k)} - W^s_{h\dot{j}k} v^i_{(s)} + \dot{\partial}_s W^i_{h\dot{j}k} v^s_{(r)} \dot{x}^r. \tag{2.5}$$

Introducing the commutation formula (1.4) to the tensor field $W^i_{hjk}(x, \dot{x})$, we get

$$2W^i_{h\dot{j}k[(\ell)(m)]} = -\dot{\partial}_r W^i_{h\dot{j}k} H^r_{s\ell m} \dot{x}^s + W^s_{h\dot{j}k} H^i_{s\ell m} - W^i_{s\dot{j}k} H^s_{h\ell m} - W^i_{hsk} H^s_{j\ell m} - W^i_{h\dot{j}s} H^s_{k\ell m}. \tag{2.6}$$

In view of the definition (1.9), the above relation reduces to

$$(\delta_{\ell(m)} - \delta_{m(\ell)}) W^i_{h\dot{j}k} = -\dot{\partial}_r W^i_{h\dot{j}k} H^r_{s\ell m} \dot{x}^s + W^s_{h\dot{j}k} H^i_{s\ell m} - W^i_{s\dot{j}k} H^s_{h\ell m} - W^i_{hsk} H^s_{j\ell m} - W^i_{h\dot{j}s} H^s_{k\ell m}. \tag{2.7}$$

Next, let us assume that $\delta_m \neq$ const. Then, with the help of the Lemma (2.1), we get

$$N_{\ell m}(x) \overset{def}{=} (\delta_{\ell(m)} - \delta_{m(\ell)}) \neq 0 \tag{2.8}$$

Let us take

$$v^i_{(h)} = H^i_{h\dot{j}k} q^{jk}. \tag{2.9}$$

for a suitable non-symmetric tensor $q^{jk}$, then multiplying (2.7) by $q^{lm}$ and summing over 1 and $m$, we obtain

$$N_{\ell m} q^{\ell m} W^i_{hjk} = -\dot{\partial}_r W^i_{h\dot{j}k} v^r_{(s)} \dot{x}^s + W^s_{hjk} v^i_{(s)} - W^i_{s\dot{j}k} v^s_{(h)} - W^i_{hsk} v^s_{(\dot{j})} - W^i_{hjs} v^s_{(k)} \tag{2.10}$$

Comparing the last equation with (2.5), we get

$$L_v W^i_{h\dot{j}k} = (\delta - q^{\ell m} N_{\ell m}) W^i_{hjk} \tag{2.11}$$

The above equation shows that $L_v W^i_{hjk}$ vanishes when and only $\delta = q^{lm} N_{\ell m}$.

For $\delta \neq$ const. and $N_{\ell m} \neq 0$, from (2.5) and (2.7), we can construct the following identity

$$\begin{aligned} N_{\ell m} L_v W^i_{h\dot{j}k} = {} & W^s_{hjk}(\delta H^i_{s\ell m} - N_{\ell m} v^i_{(s)}) - W^i_{sjk}(\delta H^s_{h\ell m} - N_{\ell m} v^s_{(h)}) \\ & - W^i_{hsk}(\delta H^s_{j\ell m} - N_{\ell m} v^s_{(\dot{j})}) - W^i_{hjs}(\delta H^s_{k\ell m} - N_{\ell m} v^s_{(k)}) \end{aligned} \tag{2.12}$$

Thus, for $L_v W^i_{hjk} = 0$, the above equation yields [6]:

$$\delta H^i_{s\ell m} = N_{\ell m} v^i_{(s)} \tag{2.13}$$

where $v^i$ does not mean a parallel vector.

We define

**Definition 2.1.** A $S$-$WR\ Fn$ space satisfying $\lambda_m v^m \neq$ const. is called a special one of the first kind.

Next, let us go back to the case, $\lambda_m v^m =$ const. of the foregoing Lemma (2.1). Then, (2.7) is replaced by

$$-\dot{\partial}_r W^i_{h\dot{j}k} H^r_{s\ell m} \dot{x}^s + W^s_{h\dot{j}k} H^i_{s\ell m} - W^i_{s\dot{j}k} H^s_{h\ell m} - W^i_{hsk} H^s_{j\ell m} - W^i_{h\dot{j}s} H^s_{k\ell m} = 0 \tag{2.14}$$

Transvecting it by $q^{lm}$ and remembering the equation (2.9) we get

$$-\dot{\partial}_r W^i_{hjk} v^r_{(s)} \dot{x}^s + W^s_{h\dot{j}k} v^i_{(s)} - W^i_{sjk} v^s_{(h)} - W^i_{hsk} v^s_{(j)} - W^i_{h\dot{j}s} v^s_{(k)} = 0 \tag{2.15}$$

Substituting the above equation into the right hand side of (2.5), we obtain

$$L_v W^i_{h\dot{j}k} = \delta W^i_{h\dot{j}k} \tag{2.16}$$

Therefore, when the arbitrary constant $\delta$ vanishes, we have

$$L_v W^i_{h\dot{j}k} = 0 \tag{2.17}$$

We put the

**Definition.** An $S$-$WR\ Fn$ space is called a special one of the second kind when $\lambda_m v^m =$ const. holds good.

Then, summarizing the above results, we have the following theorems.

**Theorem 2.1.** *In a special $S$-$WR\ Fn$ space of the first kind, if the space has the resolved curvature $H^i_{hjk}$ of the form* (2.13)*, $L_v W^i_{hjk} = 0$ holds good.*

**Theorem 2.2.** *In a special $S$-$WR\ Fn$ space of the second kind, if the arbitrary constant $\lambda_m v^m$ vanishes, we have $L_v W^i_{hjk} = 0$. From the last theorem, if $\lambda_m = 0$, then with the help of the equation* (1.9)*, we have*

$$W^i_{hik(r)} = 0 \tag{2.18}$$

*Thus, we have*

**Corollary 2.1.** *In a symmetric Finsler space, $L_v W^i_{hjk} = 0$, is satisfied identically.*

## 3. Complete Condition

In this section we shall find the necessary and sufficient condition for (2.13). From the assumption (1.21), we have

$$L_v\lambda_m = \lambda_{m(s)}v^s + (\lambda_m v^s)_{(m)} - \lambda_{s(m)}v^s = 0 \tag{3.1}$$

By virtue of (2.1) and (2.8), the last equation reduces to

$$\delta_{(m)} + N_{ms}v^s = 0 \tag{3.2}$$

In view of the equation (1.12), the Lie-derivative of $N_{lm}(x)$ is given by

$$L_v N_{\ell m} = N_{\ell m(s)}v^s + N_{sm}v^s_{(\ell)} + N_{\ell s}v^s_{(m)} \tag{3.3}$$

Remembering the commutation formula (1.16), we have

$$L_v(\lambda_{m(s)} - (L_v\lambda_m)_{(s)} = -\lambda_r L_v G^r_{ms} \tag{3.4}$$

With the help of the equation (1.2c), (1.21) and (2.8), the above relation reduces to

$$L_v N_{sm} = 0. \tag{3.5}$$

Differentiating (2.7), convariantly with respect to $x^n$ and using the equations (1.3), (1.9), (2.7) and (2.8), we obtain

$$N_{\ell m(n)}W^i_{hjk} {=} \lambda_n W^i_{hjk}N_{\ell m} + H^r_{a\ell m}\dot{x}^a(W^s_{hjk}G^i_{srn} - W^i_{sjk}G^s_{hrn} - W^i_{hsk}G^s_{jrn} - W^i_{hjs}G^s_{krn}) \tag{3.6}$$

Transvecting it by $\dot{x}^n$ and noting the equations (1.2b), we get after a little simplification:

$$N_{\ell m(n)} = \lambda_n N_{\ell m} \tag{3.7}$$

Thus, by virtue of the equations (3.3), (3.5) and (3.7), we get

$$\delta N_{\ell m} + N_{sm}v^s_{(\ell)} + N_{\ell s}v^s_{(m)} = 0 \tag{3.8}$$

Next, from the equation (3.2), we have

$$\delta_{(m)(n)} - \delta_{(n)(m)} = -(N_{ms}v^s)_{(n)} + (N_{ns}v^s)_{(m)} \tag{3.9}$$

being $\delta$ a non-constant scalar function, the above equation reduces to

$$N_{ms}v^s_{(n)} - N_{sn}v^s_{(m)} = -\lambda_n N_{ms}v^s + \lambda_m N_{ns}v^s \tag{3.10}$$

where, we have used (3.7) and $N_{ms} = -N_{sm}$. Substituting the last equation into the left hand side of (3.8), we get

$$\delta N_{mn} = -\lambda_n\delta_{(m)} + \lambda_m\delta_{(n)} \tag{3.11}$$

In an affinely connected space the identify (1.8) reduces to

$$W^i_{h\dot{j}k(\ell)} + W^i_{hk\ell(\dot{j})} + W^i_{h\ell j(k)} = 0 \tag{3.12}$$

which in view of the definition (1.9) reduces to

$$\delta W^i_{hjk} = \lambda_k W^i_{h\dot{j}\ell} v^\ell - \lambda_{\dot{j}} W^i_{hks} v^s \tag{3.13}$$

Where, we have used (2.1) and $W^i_{hjk} = -W^i_{hkj}$. Hence, from (3.11) and (3.13), we can make the following identity:

$$\delta(\delta W^i_{h\dot{j}k} - N_{jk} v^s_{(h)}) = \lambda_k(\delta W^i_{h\dot{j}s} v^s + \delta_{(\dot{j})} v^i_{(h)}) - \lambda_j(\delta W^i_{hks} v^s + \delta_{(k)} v^i_{(h)}) \tag{3.14}$$

Consequently (2.13) follows when and only when, we have

$$\delta w^i_{h\dot{j}s} v^s + \delta_{(\dot{j})} v^i_{(h)} = \lambda_{\dot{j}} Q^i_h \tag{3.15}$$

where $Q^i_h$ means a suitable tensor. Transvecting the above equation by $v^j$ and summing over $j$ by virtue of $W^i_{hjk} v^j v^k = 0$ and $\delta_{(j)} v^j = 0$ derived from (3.2), we get

$$\delta Q^i_h = 0 \tag{3.16}$$

where we have used (2.1). Since $\delta \neq 0$, therefore, the last relation yields $Q^i_h = 0$. Thus, from (3.15), we have

$$W^i_{h\dot{j}s} v^s + \delta_j v^i_{(h)} = 0, \qquad (\delta_{\dot{j}} \equiv \delta_{(j)}/\delta) \tag{3.17}$$

In this way, we have the

**Theorem 3.1.** *In order that we have* (2.13), (3.17) *is necessary and sufficient.*

Now the equation (3.17) suggests the concrete form of the tensor $q^{lm}$ used in the first half of §2. In fact if $\delta_m \neq 0$ there exists a suitable vector $\rho^m$ such that

$$\delta_m \rho^m = 1 \tag{3.18}$$

Then transvecting (3.17) by $\rho^j$ and noting the above relation, we get

$$v^i_{(h)} = \delta W^i_{hs\dot{j}} v^s \rho^j \tag{3.19}$$

If, we introduce

$$q^{lm} = v^l \rho^m \tag{3.20}$$

then $N_{lm} q^{lm} = N_{lm} v^l \rho^m = \delta_{(m)} \rho^m = \delta \cdot \delta_m \rho^m = \delta$ That is from (3.17) and (2.13), we have

$$\delta = N_{lm} q^{lm} \tag{3.21}$$

straightway. Therefore, we can take (3.20) concretely. Hence in order to have the concrete form of $q^{lm}$, (3.17) should be taken as a basic condition in our theory. If this is done, we are able to have (2.13) always, so $L_v W^i_{hjk} = 0$ holds good.
Thus, we have

**Theorem 3.2.** *If we introduce* $v^i_{(h)}$ *by* (3.17), $L_v w^i_{hjk} = 0$ *is satisfied identically.*

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Department of Mathematics, DDU Gorakhpur University, Gorakhpur-273009, India.