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Fitted operator average finite difference method for singularly perturbed delay parabolic reaction diffusion problems with non-local boundary conditions

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Abstract. This paper deals with numerical solution of singularly perturbed delay parabolic reaction diffusion problem having large delay on the spatial variable with non-local boundary condition. The solution of the problem exhibits parabolic boundary layer on both sides of the spatial domain and interior layer is also created. Introducing a fitting parameter into asymptotic solution and applying average finite difference approximation, a fitted operator finite difference method is developed for solving the problem under consideration. To treat the non-local boundary condition, Simpson's rule is applied. The stability and $\varepsilon-$ uniform convergence analysis has been carried out. To validate the applicability of the scheme, numerical examples are presented and solved for different values of the perturbation parameter ε and mesh sizes. The numerical results are tabulated in terms of maximum absolute errors and rate of convergence and it is observed that the present method is more accurate and shown to be second order Uniformly convergent in both direction, and it also improves the results of the methods existing in the literature.

Keywords. Delay parabolic problem, exponentially fitted operator, singular perturbation, non-local boundary condition

1 Introduction

Singularly perturbed delay differential equations are being used to model many practical problems in different branches of science and engineering including simulation of oil extraction from underground reservoirs, chemical processes, fluid flows, heat and mass transfer process in composite materials with small heat conduction or diffusion, control theory, population dynamics, and neuronal variability. For example (see Driver[1], Bellen and Zennaro [2], Cannon [3], Ewing and Linβ [4], Formaggia *et.al.*[5] the references therein).

Singularly perturbed parabolic partial differential equations with out delay (see Bullo*et.al.* [6], Kumar and Rao [7], Clavero *et.al.* [8], Miller *et.al.*[10]) and with delay (see Ansari *et.al.*[9], Woldaregay *et.al.*[12], Kumar and Kadalbajoo [15], Singh *et.al.* [16], Kumar and Kumari[17], Bashier and Patidar [18], Patidar and Sharma [19], Bansal and Sharma[20, 21], S.Kumar and M.Kumar[22] and the references therein) have received considerable attention over the past few

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decades. The existence and uniqueness of the second order parabolic delay differential equations with integral boundary conditions and its applications is discussed in Bahuguna and Dabas [13]. But, in recent years, singularly perturbed delay ordinary differential equations with integral boundary conditions have been developed extensively in literature (see Amiraliyev *et.al.* [23], Debela and Duressa [24, 25, 27], Sekar and Tamilselvan [26], Amiraliyev and Ylmaz [28], Kudu and Amiraliyev [29] and the references therein). Due to the presence of the small perturbation parameter ε in its leading spatial derivative term, there exist parabolic boundary layers which are located in the neighborhood of boundary of the domain, and interior layer is formed in side domain, where the solution changed rapidly. Since the traditional numerical methods for solving such problem are sometimes unstable and fail to give accurate results and unexpected oscillations occur when the parameter $\varepsilon \to 0$ Farrell *et.al.*[30].

Therefore, it is important to develop suitable numerical method that gives good results for small values of the perturbation parameter where others fails to give good results and uniformly convergent (whose accuracy does not depend on the parameter ε). However, to the best of the researchers' knowledge, except the work in Elango *et.al.* [31], and Gobena and Duressa [32], not much work has been done to solve the problem under consideration. Hence, in this paper, fitted operator finite difference method (FOFDM) on uniform mesh is proposed to solve singular perturbed delay parabolic differential equations with non-local boundary condition.

The present paper is organized as follows: In Section 2, description of the problem and bounds on the solution and its derivatives are given. The discretization and techniques of exponentially fitted finite difference scheme is described in Section 3. Uniform convergence analysis of the discrete scheme is delt in Section 4. Numerical examples and Discussion are given in Section 5. Finally, conclusion is given in Section 6.

Notations: Throughout this paper N and M denote the number of mesh elements in space and time direction respectively. The notation for jump in any function at a point "p"with $[\chi](p) = \chi(p^+) - \chi(p^-)$. Further, C is a positive constant independent of the singular perturbation parameter ε and the mesh sizes. The sup norm for a given function χ defined on the domain Ω is calculated by: $\|\chi\|_\Omega = sup_{(x,t)\in\Omega}|\chi(x,t)|$.

2 Statement of the problem

Consider the following singularly perturbed delay parabolic differential equation with non-local boundary condition on $\Omega = D \times (0, T]$ in space-time plane, where $D = (0, 2), T$ is some fixed positive number and $\Gamma = \Gamma_l \cup \Gamma_b \cup \Gamma_r$. where $\Gamma_l = [-1, 0] \times [0, T]$ and $\Gamma_r = \{2\} \times [0, T]$ are the left and the right sides of the rectangular domain Ω corresponding to $x = 0$ and $x = 2$, respectively and $\Gamma_b = \overline{D} = [0, 2]$.

$$L_\varepsilon u(x,t) \equiv \left(\frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + a(x,t)\right)u(x,t) + b(x,t)u(x-1,t) = f(x,t), (x,t) \in \Omega \tag{2.1}$$

subject to initial and boundary conditions

$$\begin{cases} u(x,t) = \phi_l(x,t), (x,t) \in \Gamma_l, \quad u(x,t) = \phi_b(x,t), (x,t) \in \Gamma_b, \\ Ku(x,t) = u(2,t) - \varepsilon \int\limits_0^2 g(x)u(x,t)dx = \phi_r, (x,t) \in \Gamma_r, \end{cases} \tag{2.2}$$

where $(x,t)\in\Omega$, $\overline{\Omega}=\overline{D}\times[0,T]$ and $\varepsilon, 0<\varepsilon\ll 1$ is given constant, $a(x,t), b(x,t),\ f(x,t)$ on $\overline{\Omega}$ and $\phi_l(x,t),\phi_r(x,t),\phi_b(x,t)$ on Γ are sufficiently smooth, bounded functions that satisfy

$$a(x,t)\geq\alpha>0,\quad b(x,t)\leq\beta<0,\quad \alpha+\beta>0,\quad on\ \overline{\Omega}. \tag{2.3}$$

Further, $g(x)$ is non-negative function and monotonic with $1-\int\limits_0^2 g(x)dx>0$.

The problem (2.1)-(2.2) can be re-written as,

$$L_\varepsilon u(x,t)=F(x,t), \tag{2.4}$$

where

$$L_\varepsilon u(x,t)=\begin{cases} L_{1,\varepsilon}u(x,t)=\left(\frac{\partial}{\partial t}-\varepsilon\frac{\partial^2}{\partial x^2}+a(x,t)\right)u(x,t),\ (x,t)\in\Omega_1,\\ L_{2,\varepsilon}u(x,t)=\left(\frac{\partial}{\partial t}-\varepsilon\frac{\partial^2}{\partial x^2}+a(x,t)\right)u(x,t)+b(x,t)u(x-1,t),\ (x,t)\in\Omega_2,\end{cases} \tag{2.5}$$

$$F(x,t)=\begin{cases} f(x,t)-b(x,t)\phi_l(x-1,t),\ (x,t)\in(0,1)\times[0,T],\\ f(x,t),\quad (x,t)\in(1,2)\times[0,T],\end{cases} \tag{2.6}$$

with boundary conditions

$$\begin{cases} u(x,t)=\phi_l(x,t),(x,t)\in\Gamma_l,\quad u(x,t)=\phi_b(x,t),(x,t)\in\Gamma_b,\\ u(1^-,t)=u(1^+,t),\quad u_x(1^-,t)=u_x(1^+,t),\\ Ku(x,t)=u(2,t)-\varepsilon\int\limits_0^2 g(x)u(x,t)dx=\phi_r(x,t),(x,t)\in\Gamma_r,\end{cases} \tag{2.7}$$

where

$$\Omega_1=(0,1]\times[0,T],\ \Omega_2=(1,2)\times[0,T],\ \Omega^*=\Omega_1\cup\Omega_2.$$

In this paper, we analyze fitted finite-difference numerical method on uniform mesh for the numerical solution of the problem (2.1)-(2.3). Uniform convergence is proved in the discrete maximum norm. Finally, we formulate the algorithm for solving the discrete problem and give the illustrative numerical results.

2.1 Bounds on the Solution and Its Derivatives

The existence and uniqueness of a solution for problem (2.5)-(2.7) can be established by assuming that the data are *Hölder* continuous and imposing appropriate compatibility conditions at the corner points $(0,0),(2,0),(-1,0)$ and $(1,0)$ (see Ladyzhenskaya *et.al.*[11]), using the assumptions of sufficiently smoothness of $\phi_l(x,t),\phi_r(x,t)$ *and* $\phi_b(x,t)$.

The required compatibility conditions are

$$\phi_b(0,0)=\phi_l(0,0),\quad \phi_b(2,0)=\phi_r(2,0). \tag{2.8}$$

and

$$\begin{cases} \frac{\partial\phi_l}{\partial t}|_{(0,0)}-\varepsilon\frac{\partial^2\phi_b}{\partial x^2}|_{(0,0)}+a(0,0)\phi_b(0,0)+b(0,0)\phi_l(-1,0)=f(0,0),\\ \frac{\partial\phi_r}{\partial t}|_{(2,0)}-\varepsilon\frac{\partial^2\phi_b}{\partial x^2}|_{(2,0)}+a(2,0)\phi_b(2,0)+b(2,0)\phi_b(1,0)=f(2,0),\end{cases} \tag{2.9}$$

so that the data matches at the corner points. The following theorem gives sufficient conditions for the existence of a unique solution of the problem (2.5)-(2.7).

Theorem 2.1. *For*

$$a(x,t), b(x,t), f(x,t) \in C^{(\beta_1,\beta_1/2)}(\overline{\Omega}), \phi_l, \phi_r \in C^{(1,\beta_1/2)}([0,T]), \phi_b \in C^{(2+\beta_1,1+\beta_1/2)}(\Gamma_b), \beta_1 \in (0,1).$$

Then, the problem (2.5)-(2.7) has a unique solution $u(x,t) \in C^{(2+\beta_1,1+\beta_1/2)}(\overline{\Omega})$. In particular, when the compatibility conditions (2.8) and (2.9) are not satisfied, a unique standard solution still exists but is not differentiable on all of $\partial\Omega$.

Proof. One may refer Ladyzhenskaya *et.al.*[11]. □

The reduced problem corresponding to singularly perturbed delay parabolic PDE (2.5)-(2.7) is given as

$$\begin{cases} \frac{\partial u_0}{\partial t} + a(x,t)u_0 + b(x,t)\phi_l(x-1,t) = f(x,t), (x,t) \in \Omega_1, \\ u_0(x,t) = \phi_b(x,t), (x,t) \in \Gamma_b. \end{cases} \tag{2.10}$$

$$\begin{cases} \frac{\partial u_0}{\partial t} + a(x,t)u_0 + b(x,t)u_0(x-1,t) = f(x,t), (x,t) \in \Omega_2, \\ u_0(x,t) = \phi_b(x,t), (x,t) \in \Gamma_b. \end{cases} \tag{2.11}$$

As $u_0(x,t)$ need not satisfy $u_0(0,t) = u(0,t)$ and $u_0(2,t) = u(2,t)$, the solution $u(x,t)$ exhibits boundary layers at $x = 0$ and $x = 2$. Further, as $u_0(1^-,t)$ need not be equal to $u_0(1^+,t)$, the solution $u(x,t)$ exhibits interior layers at $x = 1$.

Lemma 2.1. *The solution $u(x,t)$ of (2.5)-(2.7) satisfies the estimate*

$$|u(x,t) - \phi_b(x,0)| \leq Ct, \quad (x,t) \in \overline{\Omega} \tag{2.12}$$

where C is a constant independent of ε.

Proof. The result follows from the compatibility condition. The detailed proof in Roos*et.al.* [33]. □

The differential operator for the problem under consideration is given by $L_\varepsilon \equiv \frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + a$.

Lemma 2.2. Continuous maximum principle. *Let $\psi(x,t) \in U^* = C^{(0,0)}(\overline{\Omega}) \cap C^{(1,0)}(\Omega) \cap C^{(2,1)}(\Omega_1 \cup \Omega_2)$ such that $\psi(0,t) \geq 0, \psi(x,0) \geq 0, K\psi(2,t) \geq 0, L_{1,\varepsilon}\psi(x,t) \geq 0, \forall(x,t) \in \Omega_1, L_{2,\varepsilon}\psi(x,t) \geq 0, \forall(x,t) \in \Omega_2$, and $[\psi_x](1,t) = \psi_x(1^+,t) - \psi_x(1^-,t) \leq 0$ then, $\psi(x,t) \geq 0, \forall(x,t) \in \overline{\Omega}$.*

Proof. Define a test function

$$s(x,t) = \begin{cases} \frac{1}{8} + \frac{x}{2}, (x,t) \in (0,1) \times (0,T], \\ \frac{3}{8} + \frac{x}{4}, (x,t) \in (1,2) \times (0,T]. \end{cases} \tag{2.13}$$

Note that $s(x,t) > 0, \forall(x,t) \in \overline{\Omega}, Ls(x,t) > 0, \forall(x,t) \in (\Omega_1 \cup \Omega_2), s(0,t) > 0, s(x,0) > 0, Ks(2,t) > 0$, and $[s_x](1,t) < 0$. Let

$$\delta_1 = max\{-\frac{\psi(x,t)}{s(x,t)} : (x,t) \in \overline{\Omega}\}.$$

Then, there exists $(x_0,t_0) \in \overline{\Omega}$ such that $\psi(x_0,t_0) + \delta_1 s(x_0,t_0) = 0$ and $\psi(x,t) + \delta_1 s(x,t) \geq 0, \forall(x,t) \in \overline{\Omega}$. Therefore, the function $(\psi + \delta_1 s)$ attains its minimum at $(x,t) = (x_0,t_0)$. suppose the theorem does not hold true, then $\delta_1 > 0$.

Case (i): $(x_0,t_0) = (0,t_0)$

$0 < (\psi + \delta_1 s)(0, t_0) = \psi(0, t_0) + \delta_1 s(0, t_0) = 0$. It is a contradiction.
Case (ii): $(x_0, t_0) \in \Omega_1$
$0 < L_{1,\varepsilon}(\psi+\delta_1 s)(x_0, t_0) = (\psi+\delta_1 s)_t(x_0, t_0) - \varepsilon(\psi+\delta_1 s)_{xx}(x_0, t_0) + a(x_0, t_0)(\psi+\delta_1 s)(x_0, t_0) \leq 0$.
It is a contradiction.
Case (iii): $(x_0, t_0) = (1, t_0)$
$0 \leq [(\psi + \delta_1 s)^{'}](1, t_0) = [\psi'](1, t_0) + \delta_1[s'](1, t_0) < 0$. It is a contradiction.
Case (iv): $(x_0, t_0) \in \Omega_2$
$0 < L_{2,\varepsilon}(\psi + \delta_1 s)(x_0, t_0) = (\psi + \delta_1 s)_t(x_0, t_0) - \varepsilon(\psi + \delta_1 s)_{xx}(x_0, t_0) + a(x_0, t_0)(\psi + \delta_1 s)(x_0, t_0) + b(x_0, t_0)(\psi + \delta_1 s)(x_0 - 1, t_0) \leq 0$. It is a contradiction.
Case (v): $(x_0, t_0) = (2, t_0)$
$0 \leq K(\psi + \delta_1 s)(2, t_0) = (\psi + \delta_1 s)(2, t_0) - \varepsilon \int_0^2 g(x)(\psi + \delta_1 s)(x, t)dx \leq 0$. It is a contradiction.
Hence, the proof of the theorem. □

An immediate consequence of the maximum principle is the following stability result.

Lemma 2.3. Stability Result. *Let $u(x, t)$ be a solution of the problem (2.5)-(2.7), then*

$$\|u\|_{\overline{\Omega}} \leq Cmax\{\|u\|_{\Gamma_l}, \|u\|_{\Gamma_b}, \|\kappa u\|_{\Gamma_r}, \|L_\varepsilon u\|_{\Omega^*}\}, (x, t) \in \overline{\Omega}.$$

Proof. It can be easily proved using the maximum principle Lemma (2.2) and the barrier functions

$$\Theta^{\pm}(x, t) = CMs(x, t) \pm u(x, t), (x, t) \in \overline{\Omega},$$

where $M = max\{\|u\|_{\Gamma_l}, \|u\|_{\Gamma_b}, \|\kappa u\|_{\Gamma_r}, \|L_\varepsilon u\|_{\Omega^*}\}$ and $s(x, t)$ is the test functions as in Lemma (2.2). □

Theorem 2.2. *Let $a(x, t), b(x, t), f(x, t) \in C^{(2+\beta_1, 1+\beta_1/2)}(\overline{\Omega}), \phi_l \in C^{(2,\beta_1/2)}([0, T])$, $\phi_r \in C^{(2,\beta_1/2)}([0, T]), \phi_b \in C^{(4+\beta_1,2+\beta_1/2)}(\Gamma_b), \beta_1 \in (0, 1)$. Assume that the compatibility conditions are fulfilled. Then, the problem (2.5)-(2.7) has a unique solution $u(x, t)$ and $u \in C^{(4+\beta_1,2+\beta_1/2)}(\overline{\Omega})$. Furthermore, the derivatives of the solution u satisfy:*

$$\left\|\frac{\partial^{(i+j)}u(x, t)}{\partial x^i \partial t^j}\right\| \leq C\varepsilon^{-i/2}, \quad \forall i, j \in Z \geq 0 \text{ such that } 0 \leq i + 2j \leq 4,$$

where the constant C is independent of ε.

Proof. One may refer Elango *et.al.*[31] for the details. □

Theorem 2.3. *Let the data $a(x, t), b(x, t), f(x, t) \in C^{(4+\beta_1,2+\beta_1/2)}(\overline{\Omega}), \phi_l, \phi_r \in C^{(3,\beta_1/2)}([0, T]), \phi_b \in C^{(6+\beta_1,3+\beta_1/2)}(\Gamma_b), \beta_1 \in (0, 1)$. Assume that the compatibility conditions are satisfied. Then, we have*

$$\left\|\frac{\partial^{(i+j)}v}{\partial x^i \partial t^j}\right\|_{\overline{\Omega}} \leq C(1 + \varepsilon^{1-i/2}), \tag{2.14}$$

$$\left\|\frac{\partial^{(i+j)}w_l(x, t)}{\partial x^i \partial t^j}\right\| \leq \begin{cases} C\varepsilon^{-i/2}e^{-x/\sqrt{\varepsilon}}, & (x, t) \in \Omega_1, \\ C\varepsilon^{-i/2}e^{-(x-1)/\sqrt{\varepsilon}}, & (x, t) \in \Omega_2, \end{cases} \tag{2.15}$$

$$\left\|\frac{\partial^{(i+j)}w_r(x, t)}{\partial x^i \partial t^j}\right\| \leq \begin{cases} C\varepsilon^{-i/2}e^{-(1-x)/\sqrt{\varepsilon}}, & (x, t) \in \Omega_1, \\ C\varepsilon^{-i/2}e^{-(2-x)/\sqrt{\varepsilon}}, & (x, t) \in \Omega_2, \end{cases} \tag{2.16}$$

where the constant C is independent of $\varepsilon, \forall i, j \in Z \geq 0, \quad 0 \leq i + 2j \leq 4$.

Proof. One may refer Elango *et.al.*[31] for the details. □

Theorem 2.4. *The partial derivative of $w(x,t)$ satisfy*

$$\left\|\frac{\partial^{(i+j)}w}{\partial x^i \partial t^j}\right\| \leq \begin{cases} C\varepsilon^{-i/2}(e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\sqrt{\varepsilon}}), & (x,t) \in \Omega_1, \\ C\varepsilon^{-i/2}(e^{-(x-1)/\sqrt{\varepsilon}} + +e^{-(2-x)/\sqrt{\varepsilon}}), & (x,t) \in \Omega_2, \end{cases} \tag{2.17}$$

$\forall i, j \in Z \geq 0$, *such that* $i + 2j \in [0, 4]$.

Proof. The proof of the theorem is completed by using the estimates of (2.15),(2.16) and the decomposition $w = w_l + w_r$. □

3 Formulation of the method

For small values of ε, the boundary value problem (2.1), (2.2) exhibits strong boundary layer at $x = 0, 2$ and interior layer at $x = 1$ and cannot, in general, be solved analytically because of the dependence of $a(x,t)$ and $b(x,t)$ on the space-time plane (x,t). Let N and M be positive integers different from one, and these integers may or may not be equal. Then, discretize the solution domain Ω with uniform step length h and Δt in space and time direction respectively. Hence, the interval $[0,2]$ is portioned into N, and M equal sub-intervals correspondingly. Also, each nodal points satisfies $0 = x_0, x_1, ..., x_{\frac{N}{2}} = 1, x_{\frac{N}{2}+1}, ..., x_N = 2$, and $0 = t_0, t_1, ..., t_M = 2$. Thus, the nodal points in the solution domain are points of the form (x_i, t_j) using the mesh generation

$$x_i = ih, \quad h = \frac{2}{N}, \quad i = 0, 1, ..., N, \quad t_j = j\Delta t, \quad \Delta t = \frac{2}{M}, \quad j = 0, 1, ..., M. \tag{3.1}$$

If we consider, the interval $\Omega_1 = (0,1) \times [0,T]$ then we will obtain the following equation

$$\begin{cases} \left(\frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + a(x,t)\right)u(x,t) = f(x,t) - b(x,t)\phi_l(x-1,t), \ (x,t) \in (0,1) \times [0,T], \\ u(x,t) = \phi_l(x,t), (x,t) \in \Gamma_l, \quad u(1^-,t) = u(1^+,t), \quad u_x(1^-,t) = u_x(1^+,t). \end{cases} \tag{3.2}$$

Now, the domain $[0,1]$ is discretized into $\frac{N}{2}$ equal number of subintervals, each of length h.

Let $0 = x_0, x_1, ..., x_{\frac{N}{2}} = 1$, be the points such that $x_i = ih$, $i = 0, 1, ..., \frac{N}{2}$ and $\forall t_j = j\Delta t$, $j = 0, 1, ..., M$. For the discretization, we apply a exponentially fitted operator finite difference method (FOFDM).

From (3.2) we have

$$\left(\frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + a(x,t)\right)u(x,t) = F(x,t), \quad (x,t) \in (0,1) \times [0,T], \tag{3.3}$$

where $F(x,t) = f(x,t) - b(x,t)\phi_l(x-1,t)$.

To formulate the method, let us consider singularly perturbed homogeneous differential equation of the form:

$$\begin{cases} -\varepsilon u''(x) + a(x)u(x) = 0, \quad x \in (0,1), \\ u(0) = \phi_l(0), \ u(1^-) = u(1^+), \quad u'(1^-) = u'(1^+). \end{cases} \tag{3.4}$$

whose analytical solution is

$$u(x) = C\exp(\pm\sqrt{\frac{a(x)}{\varepsilon}}x), \tag{3.5}$$

where C is an arbitrary constant to be determined from the boundary conditions. Considering h is reasonably small and evaluating the result in (3.5) at each nodal points $x_i \in [0,1]$ gives

$$u_i = u(ih) = C\exp(\sqrt{a(x_i)}i\rho). \tag{3.6}$$

similarly, we have

$$\begin{cases} u_{i+1} = C\exp(\sqrt{a(x_i)}i\rho)\exp(\sqrt{a(x_i)}\rho), \\ u_{i-1} = C\exp(\sqrt{a(x_i)}i\rho)\exp(-\sqrt{a(x_i)}\rho), \end{cases} \tag{3.7}$$

where $\rho = \frac{h}{\sqrt{\varepsilon}}$

Consider a uniform grid $\overline{\Omega}_1^N = \{x_i\}_{i=0}^N$ and denote $h = x_{i+1} - x_i$. For any mesh function v_i, define the following difference operators

$$D^+v_i = \frac{v_{i+1} - v_i}{h},\ D^-v_i = \frac{v_i - v_{i-1}}{h},\ D^+D^-v_i = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}. \tag{3.8}$$

To handle the effect of the perturbation parameter,we multiply artificial viscosity (exponentially fitting factor $\sigma(\rho)$)on the diffusive term of the problem. Introducing an exponentially fitting factor $\sigma(\rho)$ in (3.4), and applying the central finite difference scheme (3.8) gives:

$$-\varepsilon\sigma(\rho)D^+D^-u(x_i) + a(x_i)u(x_i) = 0. \tag{3.9}$$

Evaluating the limit of (3.6) and (3.7) at each nodal points $x_i \in [0,1]$, we obtain:

$$\begin{cases} \lim_{h\to 0} u_i = C\exp(\sqrt{a(x_i)}i\rho), \\ \lim_{h\to 0} u_{i+1} = C\exp(\sqrt{a(x_i)}i\rho)\exp(\sqrt{a(x_i)}\rho), \\ \lim_{h\to 0} u_{i-1} = C\exp(\sqrt{a(x_i)}i\rho)\exp(-\sqrt{a(x_i)}\rho). \end{cases} \tag{3.10}$$

Now, for small values of h from (3.9) and (3.10) we get:

$$\begin{aligned} \sigma(\rho) =& \rho^2 \frac{a(x_i)\lim_{h\to 0} u_i}{\lim_{h\to 0}(u_{i+1} - 2u_i + u_{i-1})} \\ =& \frac{\rho^2 a(x_i)}{\exp(\sqrt{a(x_i)}\rho) - 2 + \exp(-\sqrt{a(x_i)}\rho)} \\ =& \frac{\rho^2 a(x_i)}{4}\left(\csc h\left(\frac{\rho}{2}\sqrt{a(x_i)}\right)\right)^2. \end{aligned} \tag{3.11}$$

Assume that $\overline{\Omega}^N$ denote partition of $[0,2]$ into N subintervals such that $0 = x_0 < x_1 < ... < x_{\frac{N}{2}} = 1$ and $1 < x_{\frac{N}{2}+1} < x_{\frac{N}{2}+2} < ... < x_N = 2$, with $x_i = ih,\ i = 0(1)N$ and $\forall t_j = j\Delta t,\ j = 0(1)M$.

Case 1: Consider (2.4) on the domain $\Omega_1 = (0,1) \times [0,T]$ which is given by:

$$L_{1,\varepsilon}u(x,t) \equiv \left(\frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + a(x,t)\right)u(x,t) = f(x,t) - b(x,t)\phi_l(x-1,t). \tag{3.12}$$

Introducing fitting parameter $\sigma(\rho)$ into (3.12) and re-write it at the nodal point $(x_i, t_{j+\frac{1}{2}})$ as:

$$\begin{cases} L_{1,\varepsilon}u_i^{j+\frac{1}{2}} \equiv \frac{\partial u_i^{j+\frac{1}{2}}}{\partial t} - \varepsilon\sigma(\rho)\frac{\partial^2 u_i^{j+\frac{1}{2}}}{\partial x^2} + a_i^{j+\frac{1}{2}}u_i^{j+\frac{1}{2}} = f_i^{j+\frac{1}{2}} - b_i^{j+\frac{1}{2}}\phi_l(x_{i-\frac{N}{2}}, t_{j+\frac{1}{2}}), \\ \quad \forall(x_i, t_{j+\frac{1}{2}}) \in \Omega_1^{N,M}, \\ u(x_i,0) = \phi_b(x_i,0),\ x_i \in (0,1), \quad u(0,t_{j+\frac{1}{2}}) = \phi_l(0,t_{j+\frac{1}{2}}), \\ u(1^-,t_{j+\frac{1}{2}}) = u(1^+,t_{j+\frac{1}{2}}), \quad u_x(1^-,t_{j+\frac{1}{2}}) = u_x(1^+,t_{j+\frac{1}{2}}). \end{cases} \tag{3.13}$$

To get the finite difference approximation for $\frac{\partial u_i^{j+\frac{1}{2}}}{\partial t}$, we considered the Taylor's series expansion:

$$U_i^{j+1} = u_i^{j+\frac{1}{2}} + \frac{\Delta t}{2}\frac{\partial u_i^{j+\frac{1}{2}}}{\partial t} + \frac{(\Delta t)^2}{8}\frac{\partial^2 u_i^{j+\frac{1}{2}}}{\partial t^2} + \frac{(\Delta t)^3}{48}\frac{\partial^3 u_i^{j+\frac{1}{2}}}{\partial t^3} + \frac{(\Delta t)^4}{384}\frac{\partial^4 u_i^{j+\frac{1}{2}}}{\partial t^4} + ... \tag{3.14}$$

and

$$U_i^{j} = u_i^{j+\frac{1}{2}} - \frac{\Delta t}{2}\frac{\partial u_i^{j+\frac{1}{2}}}{\partial t} + \frac{(\Delta t)^2}{8}\frac{\partial^2 u_i^{j+\frac{1}{2}}}{\partial t^2} - \frac{(\Delta t)^3}{48}\frac{\partial^3 u_i^{j+\frac{1}{2}}}{\partial t^3} + \frac{(\Delta t)^4}{384}\frac{\partial^4 u_i^{j+\frac{1}{2}}}{\partial t^4} + ... \tag{3.15}$$

Subtracting (3.15) from (3.14), gives the second-order finite difference approximation:

$$\frac{\partial u_i^{j+\frac{1}{2}}}{\partial t} = \frac{U_i^{j+1} - Ui^j}{\Delta t} + \tau_1, \tag{3.16}$$

where the truncation term $\tau_1 = \frac{(\Delta t)^2}{24}\frac{\partial^3 u_i^{j+\frac{1}{2}}}{\partial t^3}$.

Considering all terms of (3.13) except the derivative with respect to time variable, at the average of j^{th} and $(j+1)^{th}$ time level, we obtain:

$$-\varepsilon\sigma(\rho)\frac{\partial^2 u_i^{j+\frac{1}{2}}}{\partial x^2} + a_i^{j+\frac{1}{2}}u_i^{j+\frac{1}{2}} - f_i^{j+\frac{1}{2}} + b_i^{j+\frac{1}{2}}\phi_l(x_{i-\frac{N}{2}}, t_{j+\frac{1}{2}}) = \frac{L_{\varepsilon,x}^M U_i^{j+1} + L_{\varepsilon,x}^M U_i^j}{2}, \tag{3.17}$$

where

$$L_{1,\varepsilon,x}^M U_i^j = -\varepsilon\sigma(\rho)\frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{h^2} + a_i^j U_i^j - f_i^j + b_i^j\phi_l(x_{i-\frac{N}{2}}, t_j) + \tau_2,$$
$$L_{1,\varepsilon,x}^M U_i^{j+1} = -\varepsilon\sigma(\rho)\frac{U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}}{h^2} + a_i^{j+1}U_i^{j+1} - f_i^{j+1} + b_i^{j+1}\phi_l(x_{i-\frac{N}{2}}, t_{j+1}) + \tau_2^*,$$

for the trunction terms $\tau_2 = -\frac{\varepsilon\sigma(\rho)h^2}{12}\frac{\partial^4 u_i^j}{\partial x^4}$, and $\tau_2^* = -\frac{\varepsilon\sigma(\rho)h^2}{12}\frac{\partial^4 u_i^{j+1}}{\partial x^4}$.

Substituting (3.16) and (3.17) into (3.13) gives, for $i = 1, 2, ..., \frac{N}{2}$ and $j = 0, 1, ..., M$:

$$\begin{aligned} &-\varepsilon\sigma(\rho)\frac{U_{i+1}^{j+1} - 2U_i^{j+1} + U_{i-1}^{j+1}}{h^2} + a_i^{j+1}U_i^{j+1} - \varepsilon\sigma(\rho)\frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{h^2} + a_i^j U_i^j \\ &+ \frac{2}{\Delta t}(U_i^{j+1} - U_i^j) = f_i^{j+1} + f_i^j - b_i^{j+1}\phi_l(x_{i-\frac{N}{2}}, t_{j+1}) - b_i^j\phi_l(x_{i-\frac{N}{2}}, t_j) + \tau_3, \end{aligned} \tag{3.18}$$

where $\tau_3 = -(\tau_2 + \tau_2^* + 2\tau_1)$.
This scheme can be re-written as three-term recurrence relation in terms of the spatial direction and two-term recurrence relation interms of the temporal direction as:

$$L_{1,\varepsilon}^{N,M} U_i^{j+1} \equiv r_{i-}^{(j+1)}U_{i-1}^{j+1} + r_{ic}^{(j+1)}U_i^{j+1} + r_{i+}^{(j+1)}U_{i+1}^{j+1} = R_i^{j+1}, \tag{3.19}$$

where,

$$r_{i-}^{(j+1)} = -\frac{\varepsilon\sigma(\rho)}{h^2} = r_{i+}^{(j+1)}, \quad r_{ic}^{(j+1)} = 2\frac{\varepsilon\sigma(\rho)}{h^2} + \frac{2}{\Delta t} + a_i^{j+1},$$
$$\begin{aligned} R_i^{j+1} = &f_i^{j+1} + f_i^j + \varepsilon\sigma(\rho)\frac{U_{i+1}^n - 2U_i^j + U_{i-1}^j}{h^2} - a_i^j U_i^j + \frac{2}{\Delta t}U_i^j - b_i^{j+1}\phi_l(x_{i-\frac{N}{2}}, t_{j+1}) \\ &- b_i^j\phi_l(x_{i-\frac{N}{2}}, t_j). \end{aligned}$$

Case 2: Consider (2.4) on the domain $\Omega_2 = (1,2) \times [0,T]$ which is given by:

$$L_{2,\varepsilon}u(x,t) \equiv \left(\frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + a(x,t)\right)u(x,t) + b(x,t)u(x-1,t) = f(x,t). \quad (3.20)$$

Using the same procedure for the spatial discretization (3.20) by applying the exponential fitting factor (3.11), for $i = \frac{N}{2}+1, \frac{N}{2}+2, ..., N-1$ and $j = 1(1)M$, the fully discrete scheme becomes

$$\begin{cases} L_{2,\varepsilon}u_i^{j+\frac{1}{2}} \equiv \frac{\partial u_i^{j+\frac{1}{2}}}{\partial t} - \varepsilon\sigma(\rho)\frac{\partial^2 u_i^{j+\frac{1}{2}}}{\partial x^2} + a_i^{j+\frac{1}{2}}u_i^{j+\frac{1}{2}} + b_i^{j+\frac{1}{2}}u_{i-\frac{N}{2}}^{j+\frac{1}{2}} = f_i^{j+\frac{1}{2}}, \forall (x_i, t_{j+\frac{1}{2}}) \in \Omega_2^{N,M}, \\ u(x_i, 0) = \phi_b(x_i, 0), \quad x_i \in (1,2), \\ u(1^-, t_{j+\frac{1}{2}}) = u(1^+, t_{j+\frac{1}{2}}), \quad u_x(1^-, t_{j+\frac{1}{2}}) = u_x(1^+, t_{j+\frac{1}{2}}). \end{cases} \quad (3.21)$$

In explicit form, the scheme is rewritten as

$$L_{2,\varepsilon}^{N,M}U_i^{j+1} \equiv b_i^{j+1}U_k^{j+1} + r_{i-}^{(j+1)}U_{i-1}^{j+1} + r_{ic}^{(j+1)}U_i^{j+1} + r_{i+}^{(j+1)}U_{i+1}^{j+1} = R_i^{j+1}, \quad (3.22)$$

where

$$U_k^{j+1} = U(x_{i-\frac{N}{2}}, t_{j+1}), \; k = 1(1)\tfrac{N}{2} - 1$$

$$r_{i-}^{(j+1)} = -\frac{\varepsilon\sigma(\rho)}{h^2} = r_{i+}^{(j+1)}, \quad r_{ic}^{(j+1)} = 2\frac{\varepsilon\sigma(\rho)}{h^2} + \frac{2}{\Delta t} + a_i^{j+1},$$

$$R_i^{j+1} = f_i^{j+1} + f_i^j + \varepsilon\sigma(\rho)\frac{U_{i+1}^n - 2U_i^j + U_{i-1}^j}{h^2} - a_i^jU_i^j + \frac{2}{\Delta t}U_i^j - b_i^jU_k^j.$$

Case 3: For $i = N$, (Simpson's rule) Suppose $g(x)u(x,t)$ is a function defined on the interval $[0,2]$ and let (x_i, t_j) be a uniform partition with step length h. The composite Simpson's rule approximates the integral of $g(x)u(x,t)$ by

$$\begin{aligned} \int_0^2 g(x)u(x,t)dx =& \frac{h}{3}[g(0)u(0,t_{j+\frac{1}{2}}) + g(2)u(2,t_{j+\frac{1}{2}}) + 2\sum_{i=1}^{N-1} g(x_{2i})u(x_{2i}, t_{j+\frac{1}{2}})] \\ &+ \frac{4h}{3}\sum_{i=1}^{N} g(x_{2i-1})u(x_{2i-1}, t_{j+\frac{1}{2}}). \end{aligned} \quad (3.23)$$

Substituting (3.23) in to (2.2) gives:

$$\begin{aligned} K^{N,M}u(x_i, t_{j+\frac{1}{2}}) = u(2, t_{j+\frac{1}{2}}) - \frac{\varepsilon h}{3}\Big[g(0)u(0,t_{j+\frac{1}{2}}) + g(2)u(2,t_{j+\frac{1}{2}})\Big] \\ - \frac{2\varepsilon h}{3}\sum_{i=1}^{N-1} g(x_{2i})u(x_{2i}, t_{j+\frac{1}{2}}) - \frac{4\varepsilon h}{3}\sum_{i=1}^{N} g(x_{2i-1})u(x_{2i-1}, t_{j+\frac{1}{2}}) = \phi_r. \end{aligned} \quad (3.24)$$

Since $u(0,t_j) = \phi_l(0,t_j)$ and $u(0,t_{j+1}) = \phi_l(0,t_{j+1})$, from (2.2), this equation can be re-written as follows:

$$\begin{aligned} &-\frac{4\varepsilon h}{3}\sum_{i=1}^{N} g(x_{2i-1})u(x_{2i-1}, t_{j+1}) - \frac{2\varepsilon h}{3}\sum_{i=1}^{N-1} g(x_{2i})u(x_{2i}, t_{j+1}) + \left(1 - \frac{\varepsilon h}{3}g(2)\right)u(2, t_{j+1}) \\ &-\frac{\varepsilon h}{3}g(0)u(0, t_{j+1}) = \frac{4\varepsilon h}{3}\sum_{i=1}^{N} g(x_{2i-1})u(x_{2i-1}, t_j) + \frac{2\varepsilon h}{3}\sum_{i=1}^{N-1} g(x_{2i})u(x_{2i}, t_j) \\ &-\left(1 - \frac{\varepsilon h}{3}g(2)\right)u(2, t_j) + \frac{\varepsilon h}{3}g(0)u(0, t_j) = \phi_r. \end{aligned} \quad (3.25)$$

Therefore, on the given domain $\overline{\Omega} = \overline{D} \times [0,T] = [0,2] \times [0,T]$, the basic schemes to solve (2.1)-(2.2) are the schemes given in (3.19),(3.22) and (3.25) which gives $N \times N$ system of algebraic equations.

4 Uniform Convergence Analysis

In this section, we need to show the discrete scheme in (3.19),(3.22) and (3.25) satisfy the discrete maximum principle, uniform stability estimates, and uniform convergence.

Lemma 4.1. Discrete maximum principle. *Assume that*

$$\sum_{i=1}^{N} \frac{g_{i-1} + 4g_i + g_{i+1}}{3} h_i = \rho < 1$$

and a mesh function Ψ *satisfies* $\Psi(x_0,t_j) \geq 0, \Psi(x_i,t_0) \geq 0,\ K^{N,M}\Psi(x_N,t_j) \geq 0$, $L_{1,\varepsilon}^{N,M}\Psi(x_i,t_j) \geq 0, \forall (x_i,t_j) \in \Omega_1^{N,M}, L_{2,\varepsilon}^{N,M}\Psi(x_i,t_j) \geq 0, \forall (x_i,t_j) \in \Omega_2^{N,M}$, *and* $[D_x]\Psi(x_{\frac{N}{2}},t_j) = D_x^{+}\Psi(x_{N/2},t_j) - D_x^{-}\Psi(x_{N/2},t_j) \leq 0$ *then, prove that* $\Psi(x_i,t_j) \geq 0$, $\forall (x_i,t_j) \in \overline{\Omega}^{N,M}$.

Proof. Define a test function $S(x_i,t_j)$as

$$S(x_i,t_j) = \begin{cases} \frac{1}{8} + \frac{x_i}{2}, & (x_i,t_j) \in \Omega_1^N, \\ \frac{3}{8} + \frac{x_i}{4}, & (x_i,t_j) \in \Omega_2^N. \end{cases} \tag{4.1}$$

Note that $S(x_i,t_j) > 0,\ \forall (x_i,t_j) \in \overline{\Omega}^{N,M},\ L_{\varepsilon}^{N,M}S(x_i,t_j) > 0,\ \forall (x_i,t_j) \in (\Omega_1^{N,M} \cup \Omega_2^{N,M}),\ \ S(x_0,t_j) > 0,\ \ S(x_i,t_0) > 0,\ \ K^{N,M}S(x_N,t_j) > 0$, and $[D_x]S(x_{\frac{N}{2}},t_j) < 0$.
Let

$$\zeta = max\left\{ -\frac{\Psi(x_i,t_j)}{S(x_i,t_j)} : (x_i,t_j) \in \overline{\Omega}^{N,M} \right\}.$$

Then, there exists $(x_*,t_*) \in \overline{\Omega}^{N,M}$ such that $\Psi(x_*,t_*) + \zeta S(x_*,t_*) = 0$ and $\Psi(x_i,t_j) + \zeta S(x_i,t_j) \geq 0, \forall (x_i,t_j) \in \overline{\Omega}^{N,M}$. Therefore, the function attains its minimum at $(x,t) = (x_*,t_*)$. suppose the theorem does not hold true, then $\zeta > 0$.
Case (i): $(x_*,t_*) = (x_0,t_*)$, $0 < (\Psi + \zeta S)(x_0,t_*) = 0$. It is a contradiction
Case (ii): $(x_*,t_*) \in \Omega_1^{N,M}$, $0 < L_{1,\varepsilon}^{N,M}(\Psi + \zeta S)(x_*,t_*) \leq 0$, It is a contradiction.
Case (iii): $(x_*,t_*) = (x_{\frac{N}{2}},t_*)$, $0 \leq [D_x(\Psi + \zeta S)]_{\frac{N}{2}}(t_*) < 0$. It is a contradiction.
Case (iv): $(x_*,t_*) \in \Omega_2^{N,M}$, $0 < L_{2,\varepsilon}^{N,M}(\Psi + \zeta S)(x_*,t_*) \leq 0$. It is a contradiction.
Case (v): $(x_*,t_*) = (x_N,t_*)$

$$\begin{aligned} 0 <& K^{N,M}(\Psi + \zeta S)(x_N,t_*) \\ =& (\Psi + \zeta S)(x_N,t_*) \\ &- \varepsilon \sum_{i=1}^{N} \frac{g_{i-1}(\Psi + \zeta S)(x_{i-1},t_j) + 4g_i(\Psi + \zeta S)(x_i,t_j) + g_{i+1}(\Psi + \zeta S)(x_{i+1},t_j)}{3} h_i \leq 0. \end{aligned}$$

It is a contradiction. Hence, the proof of the theorem. □

Now, we will prove the uniform stability analysis of the discrete problem.

Lemma 4.2. *Let Ψ be any mesh function then,*

$$\|\Psi\|_{\overline{\Omega}^{N,M}} \leq C \max\left\{ \|\Psi\|_{\Gamma_l^{N,M}}, \|\Psi\|_{\Gamma_b^{N,M}}, \left\|K^{N,M}\Psi\right\|_{\Gamma_r^{N,M}}, \max_{(x_i,t_j)\in(\Omega^*)^{N,M}} \left\|L_\varepsilon^{N,M}\Psi\right\| \right\}.$$

Proof. It can be easily proved using maximum principle Lemma (4.1) and the barrier functions

$$\Theta^{\pm}(x_i,t_j) = \Xi M S(x_i,t_j) \pm \Psi(x_i,t_j), \quad (x_i,t_j) \in \overline{\Omega}^{N,M}, \tag{4.2}$$

where

$$M = \max\left\{ \|\Psi\|_{\Gamma_l^{N,M}}, \|\Psi\|_{\Gamma_b^{N,M}}, \left\|K^{N,M}\Psi\right\|_{\Gamma_r^{N,M}}, \max_{(x_i,t_j)\in(\Omega^*)^{N,M}} \left\|L_\varepsilon^{N,M}\Psi\right\| \right\}, (x_i,t_j) \in \overline{\Omega}^{N,M},$$

and $S(x_i,t_j)$is the test function as in Lemma (4.1). □

Theorem 4.1. *Let u be the solution to problem in (2.5)-(2.7) and U be the solution to discrete problem in (3.19),(3.22) and (3.25). Then, the overall error bound satisfies the following*

$$\sup_{0<\varepsilon\leq 1} \max_{1<i<N;0<j<M} \|U(x_i,t_j) - u(x_i,t_j)\| \leq C(h^2 + (\Delta t)^2), \tag{4.3}$$

where C is a constant independent of ε, h and Δt.

Proof. The solution U_i^j of (3.19) for $i = 1,2,..., \frac{N}{2}$ and $j = 0,1,...,M$ is decomposed into smooth and singular components analogous to continuous problem (See Elango *et.al.* [31]). Thus,

$$U_i^j = V_i^j + W_i^j, \tag{4.4}$$

where a smooth component V_i^j is the solution of the following problem

$$\begin{cases} L_{1,\varepsilon}^{N,M} V_i^j = R_i^j, & (x_i,t_j) \in \Omega_1^{N,M}, \\ V_i^j = \phi_b(x_i,t_j), & (x_i,t_j) \in \Gamma_b^{N,M}, \\ V_i^j = \phi_0(x_i,t_j), & (x_i,t_j) \in \Gamma_l^{N,M}, \end{cases} \tag{4.5}$$

and the singular component W_i^j must satisfy

$$\begin{cases} L_{1,\varepsilon}^{N,M} W_i^j = 0, (x_i,t_j) \in \Omega_1^{N,M}, \\ W_i^j = 0, \quad (x_i,t_j) \in \Gamma_b^{N,M}, \\ W_i^j = U_i^j - V_i^j, \quad (x_i,t_j) \in \Gamma_l^{N,M}. \end{cases} \tag{4.6}$$

Hence, the error can be written in the form

$$U_i^j - u_i^j = (V_i^j - v_i^j) + (W_i^j - w_i^j).$$

To estimate the error for the regular component, using (3.13) and (3.18), we have:

$$\begin{aligned} L_{1,\varepsilon}^{N,M}(V_i^j - v_i^j) =& f_i^{j+\frac{1}{2}} - b_i^{j+\frac{1}{2}}\phi_l(x_{i-\frac{N}{2}}, t_{j+\frac{1}{2}}) - L_1^{N,M} v_i^j, \\ =& \left(\frac{\partial v_i^{j+\frac{1}{2}}}{\partial t} - \varepsilon\sigma(\rho)\frac{\partial^2 v_i^{j+\frac{1}{2}}}{\partial x^2} + a_i^{j+\frac{1}{2}} v_i^{j+\frac{1}{2}} \right) \\ & - \left[\frac{v_i^{j+1} - v_i^j}{\Delta t} - \varepsilon\sigma(\rho)\left(\frac{v_{i+1}^{j+\frac{1}{2}} - 2v_i^{j+\frac{1}{2}} + v_{i-1}^{j+\frac{1}{2}}}{h^2} \right) + a_i^{j+\frac{1}{2}} v_i^{j+\frac{1}{2}} \right], \\ =& \frac{\partial v_i^{j+\frac{1}{2}}}{\partial t} - \varepsilon\sigma(\rho)\frac{\partial^2 v_i^{j+\frac{1}{2}}}{\partial x^2} - \left[\frac{v_i^{j+1} - v_i^j}{\Delta t} \right] + \varepsilon\sigma\left(\frac{v_{i+1}^{j+\frac{1}{2}} - 2v_i^{j+\frac{1}{2}} + v_{i-1}^{j+\frac{1}{2}}}{h^2} \right). \end{aligned} \tag{4.7}$$

By Taylor's series expansion, we have

$$\begin{cases} \frac{v_{i+1}^{j+1}-2v_i^{j+1}+v_{i-1}^{j+1}}{h^2} = \frac{\partial^2 v_i^{j+1}}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 v_i^{j+1}}{\partial x^4} + \frac{h^4}{360}\frac{\partial^6 v_i^{j+1}}{\partial x^6} + ... \\ \frac{v_{i+1}^{j}-2v_i^{j}+v_{i-1}^{j}}{h^2} = \frac{\partial^2 v_i^{j}}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 v_i^{j}}{\partial x^4} + \frac{h^4}{360}\frac{\partial^6 v_i^{j}}{\partial x^6} + ... \end{cases} \tag{4.8}$$

$$\frac{v_i^{j+1}-v_i^j}{\Delta t} = \frac{\partial v_i^{j+\frac{1}{2}}}{\partial t} - \frac{(\Delta t)^2}{24}\frac{\partial^3 v_i^{j+\frac{1}{2}}}{\partial t^3} + ... \tag{4.9}$$

Using (4.8)-(4.9) into (4.7), we get

$$L_{1,\varepsilon}^{N,M}(V_i^j - v_i^j) = -\frac{\varepsilon\sigma(\rho)h^2}{24}\left(\frac{\partial^4 v_i^{j+1}}{\partial x^4} + \frac{\partial^4 v_i^j}{\partial x^4}\right) + \frac{(\Delta t)^2}{24}\frac{\partial^3 v_i^{j+\frac{1}{2}}}{\partial t^3} + ...$$

Since the value $\sigma(\rho) > 0$, from the above equation we have

$$\begin{aligned} \left|\left|L_{1,\varepsilon}^{N,M}(V_i^j - v_i^j)\right|\right|_\infty &\le \left|\left| -\frac{\varepsilon\sigma(\rho)h^2}{24}\left(\frac{\partial^4 v_i^{j+1}}{\partial x^4} + \frac{\partial^4 v_i^j}{\partial x^4}\right)\right|\right| + \left|\left| \frac{(\Delta t)^2}{24}\frac{\partial^3 v_i^{j+\frac{1}{2}}}{\partial t^3}\right|\right|, \\ &\le C(h^2 + (\Delta t)^2), \end{aligned} \tag{4.10}$$

where, C is independent of h and Δt.
Next, we prove singular component error estimate. To decompose W into W_l and W_r

$$\begin{cases} L_{1,\varepsilon}^{N,M} W_l(x_i,t_j) = 0, \quad (x_i,t_j) \in \Omega_1^{N,M}, \\ W_l(x_i,t_j) = \phi_l(x_i,t_j) - v_0(x_i,t_j), \quad (x_i,t_j) \in \Gamma_l^{N,M}, \\ W_l(x_i,t_j) = 0, \quad (x_i,t_j) \in \Gamma_b^{N,M} \cup \Gamma_r^{N,M}, \end{cases}$$

and

$$\begin{cases} L_{1,\varepsilon}^{N,M} W_r(x_i,t_j) = 0, \quad (x_i,t_j) \in \Omega_1^{N,M}, \\ W_r(x_i,t_j) = \omega, \quad (x_i,t_j) \in \Gamma_r^{N,M}, \\ W_r(x_i,t_j) = 0, \quad (x_i,t_j) \in \Gamma_l^{N,M} \cup \Gamma_b^{N,M}. \end{cases}$$

The error of the singular component is equivalent to

$$W - w = (W_l - w_l) + (W_r - w_r).$$

The errors $(W_l - w_l)$ and $(W_r - w_r)$, associated respectively with the boundary layers on Γ_l and Γ_r , and can be estimated separately using (3.13) and (3.18), we obtain:

$$\begin{aligned} \left|\left|L_{1,\varepsilon}^{N,M}(W_i^j - w_i^j)\right|\right|_\infty &\le \left|\left| -\frac{\varepsilon\sigma(\rho)h^2}{24}\left(\frac{\partial^4 w_i^{j+1}}{\partial x^4} + \frac{\partial^4 w_i^j}{\partial x^4}\right)\right|\right| + \left|\left| \frac{(\Delta t)^2}{24}\frac{\partial^3 w_i^{j+\frac{1}{2}}}{\partial t^3}\right|\right| \\ &\le C(h^2 + (\Delta t)^2). \end{aligned} \tag{4.11}$$

Therefore,from (4.10) and (4.11), we have

$$\left|\left|L_{1,\varepsilon}^{N,M}(U_i^j - u_i^j)\right|\right|_\infty = \|L_{1,\varepsilon}^{N,M}(V_i^j - v_i^j)\|_\infty + \left|\left|L_{1,\varepsilon}^{N,M}(W_i^j - w_i^j)\right|\right|_\infty \le C(h^2 + (\Delta t)^2),$$

and

$$\left|\left|L_{1,\varepsilon}^{N,M}(U_i^j - u_i^j)\right|\right|_\infty \le \max_{i,j} |L_{1,\varepsilon}^{N,M}(U_i^j - u_i^j)| \le C(h^2 + (\Delta t)^2).$$

Hence, the required estimate.

Remark 1. A similar analysis for convergence may be carried out for the finite difference scheme (3.21) for $i = \frac{N}{2}+1, \frac{N}{2}+2, ..., N-1$ and $j = 1(1)M$.

The error bound at the right boundary $i = N$ is estimated as follows.

$$\begin{aligned}
K^{N,M}(U-u)(x_i,t_j) =& K^{N,M}U(x_i,t_j) - K^{N,M}u(x_i,t_j)\\
=& \phi_r - K^{N,M}u(x_i,t_j)\\
=& Ku(x_i,t_j) - K^{N,M}u(x_i,t_j)\\
=& u(x_i,t_j) - \int_{x_0}^{x_N} g(x)u(x,t)dx - u(x_i,t_j)\\
&+ \sum_{i=1}^{N} \frac{g_{i-1}u(x_{i-1},t_j) + 4g_i u(x_i,t_j) + g_{i+1}u(x_{i+1},t_j)}{3} h_i\\
=& \frac{g_0 u(x_0,t_j) + 4g_1 u(x_1,t_j) + g_2 u(x_2,t_j)}{3} h_1 + ...\\
&+ \frac{g_{N-1}u(x_{N-1},t_j) + 4g_N u(x_N,t_j) + g_{N+1}u(x_{N+1},t_j)}{3} h_N\\
&- \int_{x_0}^{x_1} g(x)u(x,t)dx - ... - \int_{x_{N-1}}^{x_N} g(x)u(x,t)dx,\\
\left|K^{N,M}(U-u)(x_N,t_{j+1})\right| \le& C\varepsilon(h_1^3 u''(\gamma_1,t_{j+1})) + ... + (h_N^3 u''(\gamma_N,t_{j+1}))\\
\le& C\varepsilon(h_1^3 + ... + h_N^3) \le Ch^2
\end{aligned}$$

where $x_{i-1} \le \gamma_i \le x_i,\ for\ i = 1,2,...,N$.
The discrete problem satisfy the following bound

$$\left|K^{N,M}(U-u)(x_i,t_{j+1})\right| \le Ch^2.$$

Using Lemma 4.2, we get the result

$$|U(x_i,t_{j+1}) - u(x_i,t_{j+1})| \le Ch^2, \tag{4.12}$$

where C is a constant independent of ε, N and M. □

5 Numerical Examples and Discussion

In this section, two model examples are considered to illustrate the proposed scheme discussed above. The exact solutions of the considered examples are not known. Therefore, double mesh principle is used to estimate the errors and compute the numerical rate of convergence to the computed solution. The double mesh formula to determine maximum absolute error ($E_\varepsilon^{N,\Delta t}$) is defined as follows

$$E_\varepsilon^{N,\Delta t} = \max_{i,j} \left|U_{i,j}^{N,\Delta t} - U_{i,j}^{2N,\Delta t/2}\right|$$

where $U_{i,j}^{N,\Delta t}$ denotes the numerical solution obtained by using N and Δt mesh points and $U_{i,j}^{2N,\Delta t/2}$ denotes the numerical solution at $2N$ and $\frac{\Delta t}{2}$ mesh points.

For any value of mesh points N and Δt , the numerical ε-uniform (parameter uniform) error estimate by using

$$E^{N,\Delta t} = \max_{\varepsilon} \left| E_{\varepsilon}^{N,\Delta t} \right|.$$

The rate of convergence of the scheme is calculated by the formula

$$r_{\varepsilon}^{N,\Delta t} = \log_2(E_{\varepsilon}^{N,\Delta t}) - \log_2(E_{\varepsilon}^{2N,\Delta t/2})$$

and ε- uniform rate of convergence is calculated by:

$$r^{N,\Delta t} = \log_2(E^{N,\Delta t}) - \log_2(E^{2N,\Delta t/2}).$$

The numerical results are presented for the value of the perturbation parameter $\varepsilon \in \{10^{-10}, 10^{-9}, ..., 10^{-6}\}$.

Example 1. From [31] consider the problem

$$\left(\frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + 5\right)u(x,t) - u(x-1,t) = e^{-x}, (x,t) \in (0,2) \times (0,2],$$

subject to initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \forall (x,t) \in \Gamma_l, \quad u(x,t) = 0, \forall (x,t) \in \Gamma_b \\ Ku(2,t) = u(2,t) - \varepsilon \int_0^2 \frac{x}{3} u(x,t)dx = 0, \forall (x,t) \in \Gamma_r. \end{cases}$$

Example 2. From [31] consider the problem

$$\left(\frac{\partial}{\partial t} - \varepsilon\frac{\partial^2}{\partial x^2} + 5\right)u(x,t) - xu(x-1,t) = 1, (x,t) \in (0,2) \times (0,2],$$

subject to initial and boundary conditions

$$\begin{cases} u(x,t) = 0, \forall (x,t) \in \Gamma_l, \quad u(x,t) = sin(\pi x), \forall (x,t) \in \Gamma_b \\ Ku(2,t) = u(2,t) - \varepsilon \int_0^2 \frac{1}{6} u(x,t)dx = 0, \forall (x,t) \in \Gamma_r. \end{cases}$$

From Table (1) and (2), it can be observed that the computed maximum point wise errors $E^{N,\Delta t}$ after a certain value of $\varepsilon = 10^{-6}$ (for both problems) are stable, and uniformly convergent. To observe the changes in the boundary layer width with respect to ε , and to show the physical behavior of the solution, the surface plots of the numerical solution Figure (1) have been plotted. From the figures, for small ε close to zero twin boundary layers at $x = 0$ and $x = 2$ further an interior layer at $x = 1$ can be seen from the solution. The numerical solutions obtained by the present method have been log-log plotted for singular perturbation parameter ranging from 10^{-6} to 10^{-10} in Figure (2) to indicate the maximum absolute errors decrease as the number of the mesh points increases and maximum absolute errors increases as the perturbation parameters decreases. This is one of the main results to be shown in this paper.

6 Conclusion

This study introduces Exponential fitted finite difference method (EFFDM) for solving singularly perturbed delay parabolic differential equations with non-local boundary condition. The behavior

Table 1: Maximum absolute errors $E_{\varepsilon}^{N,\Delta t}$ obtained by the proposed scheme for Example (1), at different values of N and Δt.

ε $\downarrow$	N=16 $\Delta t = \frac{0.1}{2}$	N=32 $\Delta t = \frac{0.1}{2^2}$	N=64 $\Delta t = \frac{0.1}{2^3}$	N=128 $\Delta t = \frac{0.1}{2^4}$	N=256 $\Delta t = \frac{0.1}{2^5}$	N=512 $\Delta t = \frac{0.1}{2^6}$
10^{-6}	2.5580e-04	6.7641e-05	1.7419e-05	4.4217e-06	1.1140e-06	2.7958e-07
10^{-7}	2.5580e-04	6.7641e-05	1.7419e-05	4.4217e-06	1.1140e-06	2.7958e-07
10^{-8}	2.5580e-04	6.7641e-05	1.7419e-05	4.4217e-06	1.1140e-06	2.7958e-07
10^{-9}	2.5580e-04	6.7641e-05	1.7419e-05	4.4217e-06	1.1140e-06	2.7958e-07
10^{-10}	2.5580e-04	6.7641e-05	1.7419e-05	4.4217e-06	1.1140e-06	2.7958e-07
$E^{N,\Delta t}$	2.5580e-04	6.7641e-05	1.7419e-05	4.4217e-06	1.1140e-06	2.7958e-07
$r^{N,\Delta t}$	1.9190	1.9572	1.9780	1.9889	1.9944	-

Table 2: Comparison of ε- uniform error ($E^{N,\Delta t}$) and ε- uniform rate of convergence ($r^{N,\Delta t}$) of our method and result in [31, 32] for Example 1.

ε $\downarrow$	N=16 $\Delta t = \frac{0.1}{2}$	N=32 $\Delta t = \frac{0.1}{2^2}$	N=64 $\Delta t = \frac{0.1}{2^3}$	N=128 $\Delta t = \frac{0.1}{2^4}$	N=256 $\Delta t = \frac{0.1}{2^5}$	N=512 $\Delta t = \frac{0.1}{2^6}$
Proposed method						
$E^{N,\Delta t}$	2.5580e-04	6.7641e-05	1.7419e-05	4.4217e-06	1.1140e-06	2.7958e-07
$r^{N,\Delta t}$	1.9190	1.9572	1.9780	1.9889	1.9944	-
Result in [32]						
$E^{N,\Delta t}$	3.5045e-03	2.0026e-03	1.0723e-03	5.5505e-04	2.8241e-04	1.4245e-04
$r^{N,\Delta t}$	0.80733	0.90117	0.95002	0.97483	0.98734	-
Result in [31]						
$E^{N,\Delta t}$	2.0615e-02	1.2534e-02	6.9738e-03	3.6873e-03	1.8972e-03	9.6241e-04
$r^{N,\Delta t}$	0.71783	0.84584	0.91937	0.95873	0.97912	-

Table 3: Maximum absolute errors $E_\varepsilon^{N,\Delta t}$ obtained by the proposed scheme for Example (2), at different values of N and Δt.

ε ↓	N=16 $\Delta t = \frac{0.1}{2}$	N=32 $\Delta t = \frac{0.1}{2^2}$	N=64 $\Delta t = \frac{0.1}{2^3}$	N=128 $\Delta t = \frac{0.1}{2^4}$	N=256 $\Delta t = \frac{0.1}{2^5}$	N=512 $\Delta t = \frac{0.1}{2^6}$
10^{-6}	2.5266e-03	6.3220e-04	1.5808e-04	3.9500e-05	9.8764e-06	2.4690e-06
10^{-7}	2.5266e-03	6.3220e-04	1.5808e-04	3.9500e-05	9.8764e-06	2.4690e-06
10^{-8}	2.5266e-03	6.3220e-04	1.5808e-04	3.9500e-05	9.8764e-06	2.4690e-06
10^{-9}	2.5266e-03	6.3220e-04	1.5808e-04	3.9500e-05	9.8764e-06	2.4690e-06
10^{-10}	2.5266e-03	6.3220e-04	1.5808e-04	3.9500e-05	9.8764e-06	2.4690e-06
$E^{N,\Delta t}$	2.5266e-03	6.3220e-04	1.5808e-04	3.9500e-05	9.8764e-06	2.4690e-06
$r^{N,\Delta t}$	1.9987	1.9997	2.0007	1.9998	2.0001	-

Table 4: Comparison of ε- uniform error ($E^{N,\Delta t}$) and ε- uniform rate of convergence ($r^{N,\Delta t}$) of our method and result in [31, 32] for Example 2.

ε ↓	N=16 $\Delta t = \frac{0.1}{2}$	N=32 $\Delta t = \frac{0.1}{2^2}$	N=64 $\Delta t = \frac{0.1}{2^3}$	N=128 $\Delta t = \frac{0.1}{2^4}$	N=256 $\Delta t = \frac{0.1}{2^5}$	N=512 $\Delta t = \frac{0.1}{2^6}$
Proposed method						
$E^{N,\Delta t}$	2.5266e-03	6.3220e-04	1.5808e-04	3.9500e-05	9.8764e-06	2.4690e-06
$r^{N,\Delta t}$	1.9987	1.9997	2.0007	1.9998	2.0001	-
Result in [32]						
$E^{N,\Delta t}$	2.9520e-02	1.5980e-02	8.3421e-03	4.2681e-03	2.1581e-03	1.0853e-03
$r^{N,\Delta t}$	0.88543	0.93778	0.96682	0.98383	0.99180	-
Result in [31]						
$E^{N,\Delta t}$	1.8765e-01	1.4776e-01	9.7571e-02	5.7092e-02	3.1057e-02	1.6222e-02
$r^{N,\Delta t}$	0.34479	0.59873	0.77316	0.87837	0.93697	-

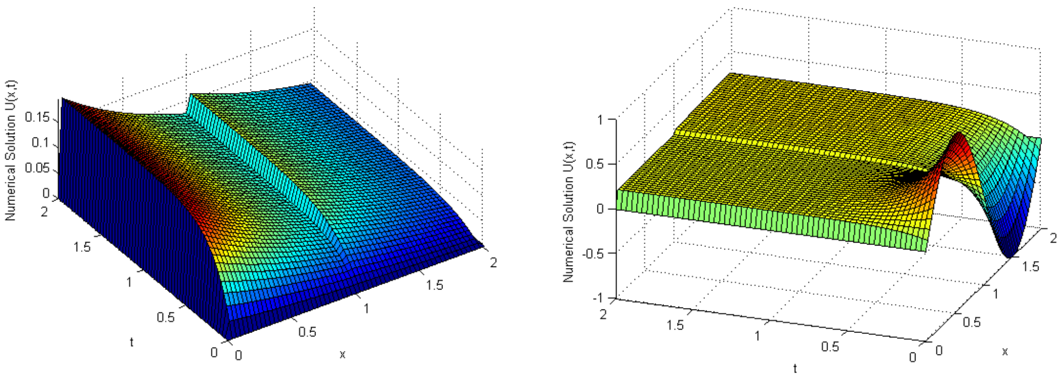

Figure 1: Surface plot of the Numerical Solution at $N, M = 64$ with boundary layer formation when $\varepsilon = 10^{-10}$ for Example (1) and (2) respectively

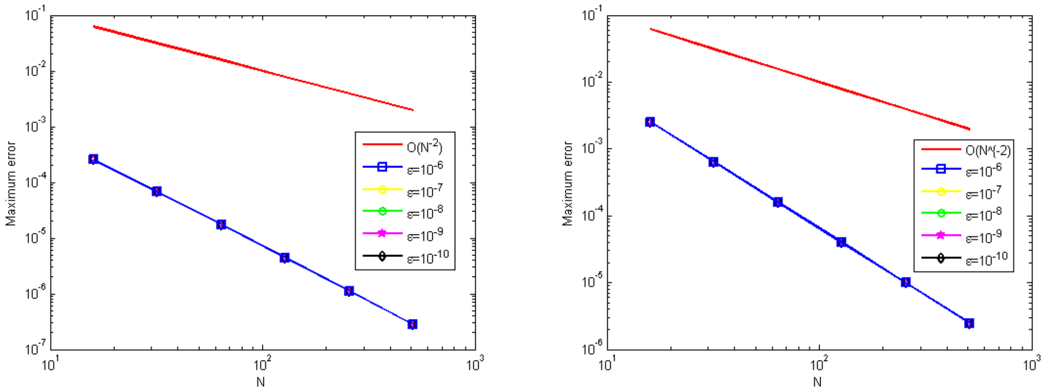

Figure 2: Log-Log scale plot of the maximum error for Example (1) and Example (2) for different values of ε respectively

of the analytic solution of the problem is studied and shown that it satisfies the continuous stability estimate and the derivatives of the solution are also bounded. Introducing fitted operator in the diffusive term, the numerical scheme is developed on uniform mesh for the problem under consideration. The stability and convergence of the proposed scheme are analyzed. Two model examples have been considered to validate the applicability of the scheme by taking different values for the perturbation parameter ε and mesh points. The computational results are presented in terms of tables (see Tables (1) and (3)) and figures (see Figure (1) and (2)) and compared with the results of the previously developed numerical methods existing in the literature Tables (2) and (4). Further, the uniformly convergence of the method is shown by the log-log plot of the ε-uniform error in Figure (2), it also improves the results of the methods existing in the literature.

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