ON THE DIOPHANTINE EQUATION $px^2 + 3^n = y^p$

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Abstract. Let $p$ be a prime. In this paper we prove that: (1) the equation $px^2 + 3^m = y^p$, $p \not\equiv 7 \pmod{8}$ has no solutions in positive integers $(x, y, p)$ with $\gcd(3, y) = 1$. (2) if $3 | y$ then the equation has at most one solution when $p = 3$ and for $p > 3$ it may have a solution only when $p \equiv 2 \pmod{3}$ and $\gcd(p, m) = 1$. (3) the equation $px^2 + 3^{2m+1} = y^p$, $p \not\equiv 5 \pmod{8}$, has a solution only when $p = 3$, and this solution exists only when $m = 3 + 3M$, and is given by $x = 46.3^{3M+1}$, $y = 13.3^{2M+1}$.

1. Introduction

Let $Z$, $Q$, $N$ be the sets of integers, rational numbers and positive integers respective. Let $p$ be an odd prime, in [5] Maohua proved two theorems concerning the equation $px^2 + 2^n = y^p$. In this paper we consider the same kind of diophantine equation $px^2 + 3^n = y^p$. In the beginning we will consider $n$ even, say $n = 2m$, where $m \in N$ and we prove the following two results:

**Theorem 1.** The diophantine equation

$$px^2 + 3^m = y^p, \quad p \not\equiv 7 \pmod{8} \quad (1)$$

has no solution with $\gcd(3, y) = 1$.

**Theorem 2.** Equation (1) has at most one solution when $p = 3$ and this solution exists only when $2m = 8 + 6M$, and then $x = 10.3^{3M+1}$, $y = 7.3^{2M+1}$. If $p > 3$ then equation (1) may have a solution only when $3 | y$, $p \equiv 2 \pmod{3}$ and $\gcd(p, m) = 1$.

For $n$ odd we get the following result

**Theorem 3.** The diophantine equation

$$px^2 + 3^{2m+1} = y^p, \quad p \not\equiv 5 \pmod{8}, \quad m \in N \quad (2)$$

has a solution only when $p = 3$ and $m = 3 + 3M$, and this solution is given by $x = 46.3^{3M+1}$, $y = 13.3^{2M+1}$.

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2. Lemmas

**Lemma 1** (Nagell [6]). Let $n$ be an odd integer $\geq 3$, and let $A \in N$, be an odd square free. If the class number of the field $Q(\sqrt{-A})$ is not divisible by $n$, then the diophantine equation $Ax^2 + 1 = y^n$, has no solutions in positive integers $x$ and $y$ for $y$ odd.

**Lemma 2** (Arif and Abu Muriefah [1]). The diophantine equation $x^2 + 3^k = y^n$, where $k$ is odd and $n \geq 3$, has only one solution given by $k = 5 + 6K$, $x = 10.3^{3K}$, $y = 7.3^{2K}$ and $n = 3$.

**Lemma 3** (Abu Muriefah and Arif [2]). The diophantine equation $x^2 + 3^{2k} = y^n$, where $n \geq 3$, has only one solution given by $k = 2 + 3K$, $x = 46.3^{3K}$, $y = 13.3^{2K}$ and $n = 3$.

3. Proofs

**Proof of Theorem 1.** By Lemma 1, it is sufficient to prove the case $m \in N$. Let $(x, y, p)$ be a solution of (1) with $\gcd(y, 3) = 1$, then $p > 3$. If $y$ is even then from (1) we get $p \equiv 7 \pmod{8}$, so we conclude that $y$ is odd. Since the class number of the field $Q(\sqrt{-p})$ is less than $p$, we factorize equation (1) to obtain

$$\sqrt{-px} + 3^m = (a + b\sqrt{-p})^p,$$

where $a, b \in Z$ and $y = a^2 + pb^2$ is odd, so $a$ and $b$ have opposite parity. On equating the real parts we get

$$3^m = a \sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r} a^{p-2r-1} (-pb^2)^r$$

(3)

From (3) we deduce that $a | 3^m$.

If $a = \pm 1$, then $y = 1 + pb^2$, but $\gcd(3, y) = 1$, so $pb^2 \equiv 0, 1 \pmod{3}$. If $pb^2 \equiv 0 \pmod{3}$ then equation (3) is impossible modulo 3. So $pb^2 \equiv 1 \pmod{3}$ and since $a^2 \equiv 1 \pmod{3}$ therefore the right hand side in (3) becomes

$$\sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r+1} (-1)^r = \frac{(1+i)^p - (1-i)^p}{2i} = 2^{p/2} \sin \frac{p\pi}{4} \neq 0$$

where the left hand side is congruent to 0 modulo 3, which is a contradiction. So $a = \pm 3^j$, $0 < j \leq m$, and $\gcd(3, b) = 1$, where $b$ is even. Then considering (3) modulo 3 we get $j = m$, hence equation (3) becomes

$$\pm 1 = \sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r} 3^{m(p-2r-1)} (-pb^2)^r.$$
Then $1 \equiv 3^{m(p-1)}(\mod p)$ and so the lower sign is impossible. That is

$$1 = \sum_{r=0}^{\frac{p-1}{2}} \binom{p}{2r} 3^{m(p-2r-1)}(-pb^2)^r,$$

(4)

Now considering (4) modulo 3 we get $p \equiv 1(\mod 12)$, say $p = 1 + 3^k u$, where $\gcd(u, 3) = 1$, $4 | u$ and $k \geq 1$. Then from (4) we obtain

$$1 \equiv \left(\frac{p}{3}\right) 3^{2m(-pb^2)} + pb^{p-1}(-p)^{\frac{p-1}{2}}(\mod 3^{k+1})$$

$$\equiv \frac{p(p-2)}{2} \cdot 3^{k+2m-1} u(-pb^2) + pb^{p-1}(-p)^{\frac{p-1}{2}}(\mod 3^{k+1}),$$

since $m \geq 1$, therefore $k + 2m - 1 \geq k + 1$, so from the above we obtain

$$1 \equiv pb^{p-1}(-p)^{\frac{p-1}{2}}(\mod 3^{k+1}).$$

Hence $p \equiv 1(\mod 3^{k+1})$ which is a contradiction.

**Proof of Theorem 2.** Let $p = 3$, then of course $m > 1$, and 3 divides both $x$ and $y$, say $x = 3x'$ and $y = 3y'$ then dividing (1) by 27, we get

$$x'^2 + 3^{2m-3} = y'^3,$$

(5)

from Lemma 2, equation (5) has a unique solution given by $2m - 3 = 5 + 6M$, $x' = 10.3^M$ and $y' = 7.3^M$.

So when $p = 3$, equation (1) has a solution only if $2m = 8 + 6M$ and the unique solution is given by $x = 10.3^{3M+1}$, $y = 7.3^{2M+1}$.

Now let $p > 3$, then from Theorem 1, it is sufficient to suppose $3 | y$, so let $x = 3^u X$, $y = 3^\nu Y$ where $u > 0$, $\nu > 0$ and $\gcd(3, X) = \gcd(3, Y) = 1$. Then (1) becomes

$$p(3^u X)^2 + 3^{2m} = 3^{\nu p} Y^p$$

(6)

We have three cases:-

1- $p\nu > 2u = 2m$. Then cancelling $3^{2m}$ in (6) we get

$$pX^2 + 1 = 3^{\nu p - 2m} Y^p$$

(7)

equation (7) implies that $p \equiv 2(\mod 3)$. Let $p | m$, then (7) becomes

$$pX^2 + 1 = (3^{\nu - 2m} Y)^p$$

and this equation is known to have no solution (Lemma 1).

2- $2u > 2m = \nu p$. Then cancelling $3^{2m}$ in (6) we get $p(3^{u-m} X)^2 + 1 = Y^p$. This equation from lemma 1 has no solution.
3- \(3m > 2u = pv\). Then
\[ pX^2 = 3^{2(m-u)} = Yp, \]
and this equation has no solution from Theorem 1.

Summarizing the above two theorems we get

**Conclusion**

The diophantine equation \(px^2 + 3^{2m} = y^p\), \(p > 3\) may have a solution only when \(p \equiv 23(\text{mod } 24)\), \(\gcd(p, m) = 1\) and \(3|y\).

**Proof of Theorem 3.** Let \((x, y, p)\) be a solution of (2) with \(\gcd(y, 3) = 1\), then \(p > 3\). If \(y\) even then \(p \equiv 5(\text{mod } 8)\), so we conclude that \(y\) is odd. Since the class number of the field \(Q(\sqrt{-3p})\) is less than \(p\). Then, according to the analysis in [4], we have

\[
\sqrt{-px} + 3^m \sqrt{3} = (a\sqrt{3} + b\sqrt{-p})^p,
\]
where \(a, b \in \mathbb{Z}\) and \(y = 3a^2 + pb^2\) is odd, so \(a\) and \(b\) have opposite parity. On equating the coefficient of \(\sqrt{3}\) in both sides, we get

\[3^m = a \sum_{r=0}^{p-1} \left( \binom{p}{2r} (3a^2)^{p-2r-1} (-pb^2)^r \right)\] (8)

From (8) we deduce that \(a = \pm 3^m\), hence \(\gcd(3, b) = 1\), and \(b\) even. So dividing (8) by \(3^m\) we get

\[\pm 1 = \sum_{r=0}^{p-1} \left( \binom{p}{2r} (3^{2m+1})^{p-2r-1} (-pb^2)^r \right)\] (9)

If the negative sign holds in (9), then

\[-1 = \sum_{r=0}^{p-1} \left( \binom{p}{2r} (3^{2m+1})^{p-2r-1} (-pb^2)^r \right)\] (10)

Now considering (10) modulo 4, we get \(-1 \equiv (3^{2m+1})^{p-1} \equiv (\text{mod } 4)\), so \(p \equiv 3(\text{mod } 4)\). Also modulo \(p\) we get \(-1 \equiv (3^{2m+1})^{p-1} \equiv (\text{mod } p)\), that is \(-1 \equiv (3)^{\frac{p-1}{2}}(\text{mod } p)\). But Legendre symbol \((3/p) = -1\) if and only if \(p \equiv 5, 7(\text{mod } 12)\), and since \(p \equiv 3(\text{mod } 4)\), we get \(p \equiv 7(\text{mod } 12)\). Now let \(p = 1 + 2.3^k\), where \(\gcd(u, 6) = 1\) and \(k \geq 1\), then from (10) we get

\[-1 \equiv pb^{p-1}(-p)^{\frac{k-1}{2}} \equiv (\text{mod } 3^{k+1}).\]

But \(\gcd(3, b) = 1\), so \(-1 \equiv p(-p)^{\frac{k-1}{2}} \equiv (\text{mod } 3^{k+1})\) hence \(1 \equiv (p)^{\frac{k-1}{2}} \equiv (\text{mod } 3^{k+1})\), squaring this equation we get \(1 \equiv (p)^{2+2.3^k} \equiv (\text{mod } 3^{k+1})\) and this implies that \(1 \equiv p^2(\text{mod } 3^{k+1})\), which is a contradiction.
ON THE DIOPHANTINE EQUATION \( px^2 + 3^m = y^p \)

So in eq. (9) only the positive sign holds, that is

\[
1 = \sum_{r=0}^{p-1} \left( \frac{p}{2r} \right) (3^{2m+1})^{\frac{p-2r-1}{2}} (-pb^{2r})^r
\]  

(11)

considering eq. (11) modulo 4, we obtain \( p \equiv 1 \pmod{4} \). Also modulo \( p \), eq (11) implies 
\( 1 \equiv (3)^{\frac{p-1}{2}} \pmod{p} \). But Legendre symbol \((3/p) = 1\) if and only if \( p \equiv 1, 11 \pmod{12} \). Since \( p \equiv 1 \pmod{4} \) we conclude that \( p \equiv 1 \pmod{12} \), and by much the same argument as the proof of Theorem 1, eq. (11) is impossible. So eq. (2) has no solution when \( \gcd(3, y) = 1 \).

Let \( p = 3 \), then of course 3 divides both \( x \) and \( y \), say \( x = 3x' \) and \( y = 3y' \) then dividing (2) by 27, we get

\[
x'^2 + 3^{2(m-1)} = y'^3,
\]  

(12)

If \( m = 1 \), eq. (12) has no solution [3]. So \( m > 1 \), and from Lemma 3, equation (12) has a unique solution given by \( m - 1 = 2 + 3M \), \( x' = 46.3^{3M} \) and \( y' = 13.3^{2M} \). So when \( p = 3 \), equation (2) has a solution only if \( m = 3 + 3M \) and the unique solution is given by \( x = 46.3^{3M+1} \), \( y = 13.3^{2M+1} \).

Now let \( p > 3 \), and suppose \( 3 | y \), so \( x = 3^u X \), \( y = 3^\nu Y \) where \( u > 0 \), \( \nu > 0 \) and \( \gcd(3, X) = \gcd(3, Y) = 1 \). Then (2) becomes

\[
p(3^u X)^2 + 3^{2m+1} = 3^{\nu p} Y^p
\]  

(13)

We have three cases:-

1- \( 2m + 1 = \min(\nu, 2u, 2m + 1) \). Then cancelling \( 3^{2m+1} \) in (13) we get

\[
3p(3^{u-m-1} X)^2 + 1 = 3^{\nu p - 2m - 1} Y^p
\]  

(14)

equation (14) modulo 3, implies \( \nu p - 2m - 1 = 0 \), so \( 3p(3^{u-m-1} X)^2 + 1 = Y^p \), and this equation is known to have no solution (Lemma 1).

2- \( 2u = \min(\nu, 2u, 2m + 1) \). Then cancelling \( 3^{2u} \) in equation (13) we get \( pX^2 + 3^{2(m-u)+1} = 3^{\nu p - 2u} Y^p \). Considering this equation modulo 3, we get \( \nu p - 2u = 0 \), then \( pX^2 + 3^{2(m-u)+1} = Y^p \), with \( \gcd(3, Y) = 1 \) and this equation has no solution from the first part of this proof.

3- \( \nu = \min(\nu, 2u, 2m + 1) \). Then

\[
p3^{2u-\nu} X^2 + 3^{2m+1-\nu} = Y^p,
\]

and this equation is possible only if \( 2u - \nu = 0 \) or \( 2m + 1 - \nu = 0 \), and these two cases have been discussed before.
References


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