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COINCIDENCE THEOREMS IN COMPLETE METRIC SPACES

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Abstract. The purpose of this paper is to introduce new classes of generalized contractive type set-valued mappings and weakly dissipative mappings and to prove some coincidence theorems for these mappings by using the concept of ω-distances.

1. Introduction

Recently, in [6] and [8], Kada-Suzuki-Takahashi introduced the new concept of ω-distances in metric spaces and improved Caristi's fixed point theorem, Ekeland's ε-variational principle and the nonconvex minimization theorem in metric spaces.

On the other hand, in [5], Husain-Latif introduced a class of generalized contractive type set-valued mappings in metric spaces and showed the existence of fixed points for these mappings ([3], [4], [9]).

In this paper, we introduce new classes of generalized contractive type set-valued mappings and weakly dissipative mappings in metric spaces and prove some coincidence theorems for these mappings by using the concept of ω-distances. Our main results extend, generalize and improve the results of Caristi ([1]), Nadler ([17]), Kada-Suzuki-Takahashi ([6]) and others ([2]-[5], [9]).

2. Preliminaries

Throughout this paper, let N and R denote the sets of positive integers and real numbers, respectively.

Definition 2.1. ([6]) Let (X,d) be a metric space. A function $p : X\times X\to[0,\infty)$ is called a ω-distance on X if the following conditions are satisfied:

(1) $p(x,z)\le p(x,y)+p(y,z)$ for all $x,y,z\in X$,
(2) for any $x\in X$, $p(x,\cdot) : X\to[0,\infty)$ is lower semicontinuous,

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(3) for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imple $d(x,y) \leq \varepsilon$.

Some examples of ω-distances are given as follows:

Example. ([6]) (1) If X is a metric space with the metric d, then $p = d$ is a ω-distance on X.

(2) If X is a normed linear space with the norm $||\cdot||$, then a function $p : X \times X \to [0,\infty)$ defined by $p(x,y) = ||x|| + ||y||$ for all $x,y \in X$ is a ω-distance on X.

(3) Let (X,d) be a metric space. If T is a continuous mapping from X into itself, then a function $p : X \times X \to [0,\infty)$ defined by $p(x,y) = \max\{d(Tx,y), D(Tx,Ty)\}$ for all, $x,y \in X$ is a ω-distance on X.

(4) Let (X,d) be a metric space and F be a bounded and closed subset of X. If F contains at least two points and C is a constant with $C \geq diam\ F$, where $diam\ F$ denotes the diameter of F, then a function $p : X \times X \to [0,\infty)$ defined by

$$p(x,y) = \begin{cases} d(x,y) & \text{for } x,y \in F, \\ C & \text{for } x \notin F \text{ or } y \notin F \end{cases}$$

is a ω-distance on X.

We need the following lemma for our main theorems:

Lemma 2.1. ([6]) *Let (X,d) be a metric space, p be a ω-distance and $\{x_n\}$, $\{y_n\}$ be sequences in X. Let $\{a_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to 0 and let $x,y,z \in X$. Then we have the following:*

(1) *If $p(x_n,y) \leq a_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$.*
(2) *If $p(x_n,y_n) \leq a_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, then $\{y_n\}$ converges to z.*
(3) *If $p(x_n,x_m) \leq a_n$ for any, n, $m \in N$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.*
(4) *If $p(y,x_n) \leq a_n$ for any $n \in N$, then $\{x_n\}$ is a Cauchy sequence in X.*

Let 2^X denote the family of all nonempty subsets of a metric space (X,d).

Definition 2.2. Let (X,d) be a metric space and M be a nonempty subset of X. Let p be a ω-distance on X and f be a function from M into X. $J : M \to 2^X$ is said to be a weakly f-contractive type set-valued mapping if, for all $x \in M$ and $u_x \in Jx$, there exists $v_y \in Jy$ for all $y \in M$ such that

$$p(u_x, v_y) \leq hp(fx, fy)$$

for some $h \in [0,1)$.

Definition 2.3. ([16]) Let (X,d) be a metric space and p be a ω-distance on X. A set-valued mapping $J : X \to 2^X$ is said to be weakly contractive (or p-contractive) if, for any $x_1, x_2 \in X$ and $y_1 \in Jx_1$, there exists $y_2 \in Jx_2$ such that

$$p(y_1, y_2) \leq rp(x_1, x_2)$$

for some $r \in [0, 1)$.

Definition 2.4. ([5]) Let (X,d) be a metric space and M be a nonempty subset of X. $J : M \to 2^X$ is called a contractive type set-valued mapping if, for all $x \in M$ and $u_x \in Jx$, there exists $v_y \in Jy$ for all $y \in M$ such that

$$d(u_x, v_y) \leq hd(x, y)$$

for some $h \in [0, 1)$.

Remark. It is obvious that a weakly f-contractive type set-valued mapping is more generalized than the weakly contractive type and contractive type set-valued mapppings.

Let $C(X)$ denote the family of all nonempty closed subsets of a metric space (X,d) and A be a set-valued mapping from X into $C(X)$.

Definition 2.5. Let p be a ω-distance on X and $f : X \to X$ be a mapping. A function $\phi : X \to [0, +\infty)$ is called a (f,p)-weak entropy of a set-valued mapping $A : X \to C(X)$ if, for all $x \in X$, there exists $y \in Ax$ such that

$$p(fx, fy) \leq \phi(x) - \phi(y). \tag{2.1}$$

Definition 2.6. A set-valued mapping $A : X \to C(X)$ is said to be (f,p)-weakly dissipative if there exists a (f,p)-weak entropy of A.

Definition 2.7. A set-valued mapping $A : X \to C(X)$ is said to be upper semicontinuous if

$$\lim_{n\to\infty} H_+(Ax_n, Ax) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} x_n = x \in X$, where

$$H_+(S,T) = \sup_{y\in T} \inf_{x\in S} d(x,y)$$

for all $S, T \in C(X)$.

3. Main results

Now, we are ready to give our main theorems.

Theorem 3.1. *Let (X, d) be a complete metric space, M be a nonempty closed subset of X and p be a ω-distance on X. Let $f : M \to M$ be a finction and $J : M \to 2^M$ be a weakly f-contractive type set-valued mapping with closed values. If $f(M) = M$, then there exists a point $z \in M$ such that $fz \in Jz$.*

Proof. Let x_0 be an arbitrary but fixed element of M and choose $y_1 \in J(x_0)$. Since $f(M) = M$, we can choose $x_1 \in M$ such that $y_1 = fx_1$. Moreover, since J is weakly f-contractive type, there exists $y_2 \in Jx_1$ such that, for some $h \in [0, 1)$,

$$p(y_1, y_2) \leq hp(fx_0, fx_1).$$

Since $f(M) = M$, we can choose again $x_2 \in M$ such that $fx_2 = y_2$. Inductively, we can obtain a sequence $\{fx_n\}$ in $M = f(M)$ such that

$$\begin{cases} fx_{n+1} \in Jx_n, \\ p(fx_n, fx_{n+1}) \leq hp(fx_{n-1}, fx_n) \end{cases} \tag{3.1}$$

for $n = 0, 1, 2, \ldots$. Thus we have

$$\begin{aligned} p(fx_n, fx_{n+1}) &\leq hp(fx_{n-1}, fx_n) \\ &\leq \cdots \\ &\leq h^n p(fx_0, fx_1). \end{aligned} \tag{3.2}$$

Since $h \in [0, 1)$, it follows from (3.2) and Lemma 2.1 that $\{fx_n\}$ is a Cauchy sequence in $M = f(M)$. Hence, by the completeness of M, $\{fx_n\}$ converges to a point $u \in M$. Since $f(M) = M$, there exists $z \in M$ such that $fz = u$.

Furthermore, since $Fx_{n+1} \in Jx_n$ for $n = 0, 1, 2, \ldots, J$ is weakly f-contractive type and $f(M) = M$, we can choose $fv_n \in Jz$ such that

$$p(fx_{n+1}, fv_n) \leq hp(fx_n, fz). \tag{3.3}$$

Since $fx_n \to fz = u$ as $n \to \infty$, by the lower semicontinuity of p, we have

$$\begin{aligned} p(fx_n, fz) &\leq \lim_{m \to \infty} p(fx_n, fx_m) \\ &\leq \lim_{m \to \infty} h^{m-n} p(fx_0, fx_1) \end{aligned} \tag{3.4}$$

for all $m, n \in N$ with $m > n$. Therefore, from (3.3), (3.4) and lemma 2.1, it follows that $fv_n \to fz$ as $n \to \infty$. Since Jz is closed, $fz \in Jz$. This completes the proof.

Remark. If $f = I_X$ (: the identity mapping on X), then we obtain Theorem 2 in [6]. Further, if $f = I_X$ and $p = d$, then we have Theorem 2.3 in [2] as a corollary.

Next, we need the following lemma for a generalization of Caristi's fixed point theorem. The proof of the following lemma is similar to that of Lemma 1 in [6]:

Lemma 3.2. *Let (X, d) be a metric space and p be a ω-distance on X. If $p(x_n, x_m) \leq a_{n,m}$ for any $n, m \in N$ with $m > n$ and $a_{n,m} \geq 0$ with $a_{n,m} \to 0$ as $n, m \to \infty$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. Note that, for sufficiently large $m, n \in N$ with $m > n$, $a_{m,n} \to 0^+$. Let $\varepsilon \geq 0$. By the definition of ω-distance, there exists $\delta > 0$ such that $p(u, v) \leq \delta$ and $p(u, z) \leq \delta$ imply $d(v, z) \leq \varepsilon$. Thus, since $a_{m,n} \to 0^+$ as $n, m \to \infty$, there exists a positive integer $n_0 \in N$ such that $a_{n_0,m} < \delta$ for $n, m \geq n_0$, which implies that $p(x_{n_0}, x_m) < \delta$ and $p(x_{n_0}, x_n) < \delta$. Therefore, $d(x_n, x_m) \leq \varepsilon$, i.e., $\{x_n\}$ is a Cauchy sequence in X. This completes the proof.

Theorem 3.3. *Let (X, d) be a complete metric space and p be a ω-distance on X. If a function $f : X \to X$ is surjective and a set-valued mapping $A : X \to C(X)$ is (f, p)-weakly dissipative and upper semicontinuous, then there exists $z \in X$ such that $fz \in Az$.*

Proof. Let ϕ be a (f, p)-weak entropy of a set-valued mapping $A : X \to C(X)$. By (2.1), we can construct a sequence $\{x_n\}$ in X such that

$$\begin{cases} fx_{n+1} \in Ax_n, \\ p(fx_n, fx_{n+1}) \leq \phi(x_n) - \phi(x_{n+1}) \end{cases} \tag{3.5}$$

for $n = 0, 1, 2, \ldots$. From (3.5), it follows that $\phi(x_n) \geq 0$ for $n = 0, 1, 2, \ldots$ and $\{\phi(x_n)\}$ is nonincreasing.

Suppose that $\phi(x_n) \to a \in [0, \infty)$ as $n \to \infty$. Then, since, for any i, $j \in N$ with $i < j$, we have

$$\begin{aligned} p(fx_i, fx_j) &\leq \sum_{n=i}^{j-1} p(fx_n, fx_{n+1}) \\ &\leq \sum_{n=i}^{j-1} (\phi(x_n) - \phi(x_{n+1})) \\ &= \phi(x_i) - \phi(x_j), \end{aligned} \tag{3.6}$$

from Lemma 3.2, it follows that $\{fx_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, $\{fx_n\}$ converges to a point $u \in X$. since f is surjective, there exists $z \in X$ such that $fz = u$. Furthermore, since $fx_{n+1} \in Ax_n$, we have

$$d(fx_{n+1}, Az) \leq H_+(Az, Ax_n). \tag{3.7}$$

Thus, since A is upper semicontinuous, $\lim_{n\to\infty} H_+(Az, Ax_n) = 0$. On the other hand, we have

$$\lim_{n\to\infty} d(fx_{n+1}, Az) = d(fz, Az).$$

Therefore, from (3.7), it follows that $d(fz, Az) = 0$ and so, since Az is closed, $fz \in Az$. This completes the proof.

Remark. From Theorem 3.3, we have Caristi's fixed point theorem [1] as a corollary.

The following example illustrates our theorem:

Example. Let $X = \{1,2,3,4\}$ and define $d, p : X \times X \to [0,\infty)$ as follows, respectively:

$$\begin{aligned} &d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0,\\ &d(1,2) = d(2,1) = 1, d(1,3) = d(3,1) = 2,\\ &d(1,4) = d(4,1) = 3, d(2,3) = d(3,2) = 2,\\ &d(1,1) = d(4,2) = 3, d(3,4) = d(4,3) = 1 \end{aligned}$$

and

$$\begin{aligned} &p(1,1) = 1, p(1,2) = 1, p(1,3) = 2, p(1,4) = 3,\\ &p(2,1) = 2, p(2,2) = 1, p(2,3) = 3, p(2,4) = 2,\\ &p(3,1) = 2, p(3,2) = 1, p(3,3) = 2, p(3,4) = 1,\\ &p(4,1) = 1, p(4,2) = 2, p(4,3) = 2, p(4,4) = 2. \end{aligned}$$

Clearly d is a metric and p is a ω-distance on X, respectively. Define $f : X \to X$ by

$$f(1) = 2, f(2) = 3, f(3) = 1, f(4) = 4$$

and $\phi : X \to [0,+\infty)$ by

$$\phi(1) = 8, \phi(2) = 4, \phi(3) = 7, \phi(4) = 6.$$

Next, define $A : X \to C(X)$ by

$$A(1) = \{2,3\}, A(2) = \{3,4\}, A(3) = \{1,2\}, A(4) = \{2,4\}.$$

By routine computation, it is easy to check that all the conditions of Theorem 3.3 are satisfied. Moreover, for all $z \in X$, we have $fz \in Az$.

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